Title: Encyclopaedia Britannica, 11th Edition, "Frost" to "Fyzabad"

Author: Various

Release date: August 13, 2011 [EBook #37064]
Most recently updated: January 8, 2021

Language: English

Credits: Produced by Marius Masi, Don Kretz and the Online
Distributed Proofreading Team at https://www.pgdp.net

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THE ENCYCLOPÆDIA BRITANNICA

A DICTIONARY OF ARTS, SCIENCES, LITERATURE AND GENERAL INFORMATION

ELEVENTH EDITION

VOLUME XI SLICE III

Frost to Fyzabad

Articles in This Slice

FROST (a common Teutonic word, cf. Dutch, *vorst*, Ger. *Frost*, from the common Teutonic verb meaning “to freeze,” Dutch, *vriezen*, Ger. *frieren*; the Indo-European root is seen in Lat. *pruina*, hoar-frost, cf. *prurire*, to itch, burn, *pruna*, burning coal, Sansk. *plush*, to burn), in meteorology, the act, or agent of the process, of freezing; hence the terms “hoar-frost” and “white-frost” applied to visible frozen vapour formed on exposed surfaces. A frost can only occur when the surface temperature falls below 32° F., the freezing-point of water; if the temperature be between 28° and 32° it is a “light frost,” if below 28° it is a “heavy,” “killing” or “black frost”; the term “black frost” is also used when no hoar-frost is present. The number of degrees below freezing-point is termed “degrees of frost.” As soon as a mass of air is cooled to its dew-point, water begins to be precipitated in the form of rain, dew, snow or hail. Hoarfrost is only formed at the immediate surface of the land if the latter be at a temperature below 32°, and this may occur even when the temperature of the air a few feet above the ground is 12°-16° above the freezing-point. The heaviest hoar-frosts are formed under weather conditions similar to those under which the heaviest summer dews occur, namely, clear and calm nights, when there is no cloud to impede the radiation of heat from the surface of the land, which thereby becomes rapidly and completely cooled. The danger of frost is minimized when the soil is very moist, as for example after 10-12 mm. of rain; and it is a practice in America to flood fields on the receipt of a frost 251 warning, radiation being checked by the light fog sheets which develop over moist soils, just as a cloud-layer in the upper atmosphere impedes radiation on a grand scale. A layer of smoke will also impede radiation locally, and to this end smoky fires are sometimes lit in such positions that the smoke may drift over planted ground which it is desirable to preserve from frost. Similarly, frost may occur in open country when a town, protected by its smoke-cloud above, is free of it. In a valley with fairly high and steep flanks frost sometimes occurs locally at the bottom, because the layer of air cooled by contact with the cold surface of the higher ground is heavier than that not so cooled, and therefore tends to flow or settle downwards along the slope of the land. When meteorological considerations point to a frost, an estimate of the night temperature may be obtained by multiplying the difference between the readings of the wet and dry bulb thermometer by 2.5 and subtracting the result from the dry bulb temperature. This rule applies when the evening air is at about 50° and 30.1 in. pressure, the sky being clear. An instrument has been devised in France for the prediction of frost. It consists of a wet bulb and a dry bulb thermometer, mounted on a board on which is also a scale of lines corresponding to degrees of the dry bulb, and a pointer traversing a scale graduated according to degrees of the wet bulb. Observations for the night are taken about half an hour before sunset. By means of the pointer and scale, the point may be found at which the line of the dry-bulb reading meets the pointer set to the reading of the wet bulb. The scale is further divided by colours so that the observed point may fall within one of three zones, indicating certain frost, probable frost or no probability of frost.

FROSTBITE, a form of MORTIFICATION (*q.v.*), due to the action of extreme cold in cutting off the blood-supply from the fingers, toes, nose, ears, &c. In comparatively trifling forms it occurs as “chaps” and “chilblains,” but the term frostbite is usually applied only to more severe cases, where the part affected becomes in danger of gangrene. An immediate application of snow, or ice-water, will restore the circulation; the application of heat would cause inflammation. But if the mortification has gone too far for the circulation to be restored, the part will be lost, and surgical treatment may be necessary.

FROSTBURG, a town of Allegany county, Maryland, U.S.A., 11 m. W. of Cumberland. Pop. (1890) 3804; (1900) 5274 (578 foreign-born and 236 negroes); (1910) 6028. It is served by the Cumberland & Pennsylvania railway and the Cumberland & Westernport electric railway. The town is about 2000 ft. above sea-level on a plateau between the Great Savage and Dans mountains, and its delightful scenery and air have made it attractive as a summer resort. It is the seat of the second state normal school, opened in 1904. Frostburg is in the midst of the coal region of the state, and is itself almost completely undermined; it has planing mills and manufactures large quantities of fire-brick. The municipality owns and operates its waterworks. Natural gas is piped to Frostburg from the West Virginia fields, 120 m. away. Frostburg was first settled in 1812; was called Mount Pleasant until about 1830, when the present name was substituted in honour of Meshech Frost, one of the town’s founders; and was incorporated in 1870.

FROTHINGHAM, OCTAVIUS BROOKS (1822-1895), American clergyman and author, was born in Boston on the 26th of November 1822, son of Nathaniel Langdon Frothingham (1793-1870), a prominent Unitarian preacher of Boston, and through his mother’s family related to Phillips Brooks. He graduated from Harvard College in 1843 and from the Divinity School in 1846. He was pastor of the North Unitarian church of Salem, Massachusetts, in 1847-1855. From 1855 to 1860 he was pastor of a new Unitarian society in Jersey City, where he gave up the Lord’s Supper, thinking that it ministered to self-satisfaction; and it was as a radical Unitarian that he became pastor of another young church in New York City in 1860. Indeed in 1864 he was recognized as leader of the radicals after his reply to Dr Hedge’s address to the graduating students of the Divinity School on *Anti-Supernaturalism in the Pulpit*. In 1865, when he had practically given up “transcendentalism,” his church building was sold and his congregation began to worship in Lyric Hall under the name of the Independent Liberal Church; in 1875 they removed to the Masonic Temple, but four years later ill-health compelled Frothingham’s resignation, and the church dissolved. Paralysis threatened him and he never fully recovered his health; in 1881 he returned to Boston, where he died on the 27th of November 1895. To this later period of his life belongs his best literary work. While he was in New York he was for a time art critic of the *Tribune*. Always himself on the unpopular side and an able but thoroughly fair critic of the majority, he habitually under-estimated his own worth; he was not only an anti-slavery leader when abolition was not popular even in New England, and a radical and rationalist when it was impossible for him to stay conveniently in the Unitarian Church, but he was the first president of the National Free Religious Association (1867) and an early and ardent disciple of Darwin and Spencer. To his radical views he was always faithful. It is a mistake to say that he grew more conservative in later years; but his judgment grew more generous and catholic. He was a greater orator than man of letters, and his sermons in New York were delivered to large audiences, averaging one thousand at the Masonic Temple, and were printed each week; in eloquence and in the charm of his spoken word he was probably surpassed in his day by none save George William Curtis. Personally he seemed cold and distant, partly because of his impressive appearance, and partly because of his own modesty, which made him backward in seeking friendships.

His principal published works are: *Stories from the Life of the Teacher* (1863), *A Child's Book of Religion* (1866), and other works of religious teaching for children; several volumes of sermons; *Beliefs of Unbelievers* (1876), *The Cradle of the Christ: a Study in Primitive Christianity* (1877), *The Spirit of New Faith* (1877), *The Rising and the Setting Faith* (1878), and other expositions of the "new faith" he preached; *Life of Theodore Parker* (1874), *Transcendentalism in New England* (1876), which is largely biographical, *Gerrit Smith, a Biography* (1878), *George Ripley* (1882), in the "American Men of Letters" series, *Memoir of William Henry Channing* (1886), *Boston Unitarianism, 1820-1850* (1890), really a biography of his father; and *Recollections and Impressions, 1822-1890* (1891).

FROUDE, JAMES ANTHONY (1818-1894), English historian, son of R.H. Froude, archdeacon of Totnes, was born at Dartington, Devon, on the 23rd of April 1818. He was educated at Westminster and Oriel College, Oxford, then the centre of the ecclesiastical revival. He obtained a second class and the chancellor's English essay prize, and was elected a fellow of Exeter College (1842). His elder brother, Richard Hurrell Froude (1803-1836), had been one of the leaders of the High Church movement at Oxford. Froude joined that party and helped J.H. Newman, afterwards cardinal, in his *Lives of the English Saints*. He was ordained deacon in 1845. By that time his religious opinions had begun to change, he grew dissatisfied with the views of the High Church party, and came under the influence of Carlyle's teaching. Signs of this change first appeared publicly in his *Shadows of the Clouds*, a volume containing two stories of a religious sort, which he published in 1847 under the pseudonym of "Zeta," and his complete desertion of his party was declared a year later in his *Nemesis of Faith*, an heretical and unpleasant book, of which the earlier part seems to be autobiographical.

On the demand of the college he resigned his fellowship at Oxford, and mainly at least supported himself by writing, contributing largely to *Fraser's Magazine* and the *Westminster Review*. The excellence of his style was soon generally recognized. The first two volumes of his *History of England from the Fall of Wolsey to the Defeat of the Spanish Armada* appeared in 1856, and the work was completed in 1870. As an historian he is chiefly remarkable for literary excellence, for the art with which he represents his conception of the past. He condemns a scientific treatment of history and disregards its philosophy. He held that its office was simply to record human actions and that it should be written as a drama. Accordingly he gives prominence to the personal element in history. His presentations of character and motives, whether truthful or not, are undeniably fine; but his doctrine that there should be "no theorizing" about history tended to narrow his survey, and consequently he sometimes, as in his remarks on the foreign policy of Elizabeth, seems to misapprehend the tendencies of a period on which he is writing.

Froude's work is often marred by prejudice and incorrect statements. He wrote with a purpose. The keynote of his *History* is contained in his assertion that the Reformation was "the root and source of the expansive force which has spread the Anglo-Saxon race over the globe." Hence he overpraises Henry VIII. and others who forwarded the movement, and speaks too harshly of some of its opponents. So too, in his *English in Ireland* (1872-1874), which was written to show the futility of attempts to conciliate the Irish, he aggravates all that can be said against the Irish, touches too lightly on English atrocities, and writes unjustly of the influence of Roman Catholicism. A strong anti-clerical prejudice is manifest in his historical work generally, and is doubtless the result of the change in his views on Church matters and his abandonment of the clerical profession. Carlyle's influence on him may be traced both in his admiration for strong rulers and strong government, which led him to write as though tyranny and brutality were excusable, and in his independent treatment of character. His rehabilitation of Henry VIII. was a useful protest against the idea that the king was a mere sanguinary profligate, but his representation of him as the self-denying minister of his people's will is erroneous, and is founded on the false theory that the preambles of the acts of Henry's parliaments represented the opinions of the educated laymen of England. As an advocate he occasionally forgets that sobriety of judgment and expression become an historian. He was not a judge of evidence, and seems to have been unwilling to admit the force of any argument or the authority of any statement which militated against his case. In his *Divorce of Catherine of Aragon* (1891) he made an unfortunate attempt to show that certain fresh evidence on the subject, brought forward by Dr Gairdner, Dr Friedmann and others, was not inconsistent with the views which he has expressed in his *History* nearly forty years before. He worked diligently at original manuscript authorities at Simancas, the Record Office and Hatfield House; but he used his materials carelessly, and evidently brought to his investigation of them a mind already made up as to their significance. His *Life of Caesar* (1879), a glorification of imperialism, betrays an imperfect acquaintance with Roman politics and the life of Cicero; and of his two pleasant books of travel, *The English in the West Indies* (1888) shows that he made little effort to master his subject, and *Oceana* (1886), the record of a tour in Australia and New Zealand, among a multitude of other blunders, notes the prosperity of the working-classes in Adelaide at the date of his visit, when, in fact, owing to a failure in the wheat-crop, hundreds were then living on charity. He was constitutionally inaccurate, and seems to have been unable to represent the exact sense of a document which lay before him, or even to copy from it correctly. Historical scholars ridiculed his mistakes, and Freeman, the most violent of his critics, never let slip a chance of hitting at him in the *Saturday Review*. Froude's temperament was sensitive, and he suffered from these attacks, which were often unjust and always too savage in tone. The literary quarrel between him and Freeman excited general interest when it blazed out in a series of articles which Freeman wrote in the *Contemporary Review* (1878-1879) on Froude's *Short Study* of Thomas Becket.

Notwithstanding its defects, Froude's *History* is a great achievement; it presents an important and powerful account of the Reformation period in England, and lays before us a picture of the past magnificently conceived, and painted in colours which will never lose their freshness and beauty. As with Froude's work generally, its literary merit is remarkable; it is a well-balanced and orderly narrative, coherent in design and symmetrical in execution. Though it is perhaps needlessly long, the thread of the story is never lost amid a crowd of details; every incident is made subordinate to the general idea, appears in its appropriate place, and

contributes its share to the perfection of the whole. The excellence of its form is matched by the beauty of its style, for Froude was a master of English prose. The most notable characteristic of his style is its graceful simplicity; it is never affected or laboured; his sentences are short and easy, and follow one another naturally. He is always lucid. He was never in doubt as to his own meaning, and never at a loss for the most appropriate words in which to express it. Simple as his language is, it is dignified and worthy of its subject. Nowhere perhaps does his style appear to more advantage than in his four series of essays entitled *Short Studies on Great Subjects* (1867-1882), for it is seen there unfettered by the obligations of narrative. Yet his narrative is admirably told. For the most part flowing easily along, it rises on fit occasions to splendour, picturesque beauty or pathos. Few more brilliant pieces of historical writing exist than his description of the coronation 253
procession of Anne Boleyn through the streets of London, few more full of picturesque power than that in which he relates how the spire of St Paul's was struck by lightning; and to have once read is to remember for ever the touching and stately words in which he compares the monks of the London Charterhouse preparing for death with the Spartans at Thermopylae. Proofs of his power in the sustained narration of stirring events are abundant; his treatment of the Pilgrimage of Grace, of the sea fight at St Helens and the repulse of the French invasion, and of the murder of Rizzio, are among the most conspicuous examples of it. Nor is he less successful when recording pathetic events, for his stories of certain martyrdoms, and of the execution of Mary queen of Scots, are told with exquisite feeling and in language of well-restrained emotion. And his characters are alive. We may not always agree with his portraiture, but the men and women whom he saw exist for us instinct with the life with which he endows them and animated by the motives which he attributes to them. His successes must be set against his failures. At the least he wrote a great history, one which can never be disregarded by future writers on his period, be their opinions what they may; which attracts and delights a multitude of readers, and is a splendid example of literary form and grace in historical composition.

The merits of his work met with full recognition. Each instalment of his *History*, in common with almost everything which he wrote, was widely read, and in spite of some adverse criticisms was received with eager applause. In 1868 he was elected rector of St Andrews University, defeating Disraeli by a majority of fourteen. He was warmly welcomed in the United States, which he visited in 1872, but the lectures on Ireland which he delivered there caused much dissatisfaction. On the death of his adversary Freeman in 1892, he was appointed, on the recommendation of Lord Salisbury, to succeed him as regius professor of modern history at Oxford. Except to a few Oxford men, who considered that historical scholarship should have been held to be a necessary qualification for the office, his appointment gave general satisfaction. His lectures on Erasmus and other 16th-century subjects were largely attended. With some allowance for the purpose for which they were originally written, they present much the same characteristics as his earlier historical books. His health gave way in the summer of 1894, and he died on the 20th of October.

His long life was full of literary work. Besides his labours as an author, he was for fourteen years editor of *Fraser's Magazine*. He was one of Carlyle's literary executors, and brought some sharp criticism upon himself by publishing Carlyle's *Reminiscences* and the *Memorials of Jane Welsh Carlyle*, for they exhibited the domestic life and character of his old friend in an unpleasant light. Carlyle had given the manuscripts to him, telling him that he might publish them if he thought it well to do so, and at the close of his life agreed to their publication. Froude therefore declared that in giving them to the world he was carrying out his friend's wish by enabling him to make a posthumous confession of his faults. Besides publishing these manuscripts he wrote a *Life of Carlyle*. His earlier study of Irish history afforded him suggestions for a historical novel entitled *The Two Chiefs of Dunboy* (1889). In spite of one or two stirring scenes it is a tedious book, and its personages are little more than machines for the enunciation of the author's opinions and sentiments. Though Froude had some intimate friends he was generally reserved. When he cared to please, his manners and conversation were charming. Those who knew him well formed a high estimate of his ability in practical affairs. In 1874 Lord Carnarvon, then colonial secretary, sent Froude to South Africa to report on the best means of promoting a confederation of its colonies and states, and in 1875 he was again sent to the Cape as a member of a proposed conference to further confederation. Froude's speeches in South Africa were rather injudicious, and his mission was a failure (see South Africa: *History*). He was twice married. His first wife, a daughter of Pascoe Grenfell and sister of Mrs Charles Kingsley, died in 1860; his second, a daughter of John Warre, M.P. for Taunton, died in 1874.

Froude's *Life*, by Herbert Paul, was published in 1905.

(W. Hu.)

FRUCTOSE, Laevulose, or Fruit-Sugar, a carbohydrate of the formula $C_6H_{12}O_6$. It is closely related to ordinary *d*-glucose, with which it occurs in many fruits, starches and also in honey. It is a hydrolytic product of inulin, from which it may be prepared; but it is more usual to obtain it from "invert sugar," the mixture obtained by hydrolysing cane sugar with sulphuric acid. Cane sugar then yields a syrupy mixture of glucose and fructose, which, having been freed from the acid and concentrated, is mixed with water, cooled in ice and calcium hydroxide added. The fructose is precipitated as a saccharate, which is filtered, suspended in water and decomposed by carbon dioxide. The liquid is filtered, the filtrate concentrated, and the syrup so obtained washed with cold alcohol. On cooling the fructose separates. It may be obtained as a syrup, as fine, silky needles, a white crystalline powder, or as a granular crystalline, somewhat hygroscopic mass. When anhydrous it melts at about 95° C. It is readily soluble in water and in dilute alcohol, but insoluble in absolute alcohol. It is sweeter than cane sugar and is more easily assimilated. It has been employed under the name diabetin as a sweetening agent for diabetics, since it does not increase the sugar-content of the urine; other medicinal applications are in phthisis (mixed with quassia or other bitter), and for children suffering from tuberculosis or scrofula in place of cane sugar or milk-sugar.

Chemically, fructose is an oxyketone or ketose, its structural formula being $CH_2OH{\cdot}(CH{\cdot}OH)_3{\cdot}CO{\cdot}CH_2OH$;

this result followed from its conversion by H. Kiliani into methylbutylacetic acid. The form described above is *laevo*-rotatory, but it is termed *d*-fructose, since it is related to *d*-glucose. Solutions exhibit mutarotation, fresh solutions having a specific rotation of −104.0°, which gradually diminishes to −92°. It was synthesized by Emil Fischer, who found the synthetic sugar which he named α-acrose to be (*d* + *l*)-fructose, and by splitting this mixture he obtained both the d and *l* forms. Fructose resembles d-glucose in being fermentable by yeast (it is the one ketose which exhibits this property), and also in its power of reducing alkaline copper and silver solutions; this latter property is assigned to the readiness with which hydroxyl and ketone groups in close proximity suffer oxidation. For the structural (stereochemical) relations of fructose see Sugar.

FRUGONI, CARLO INNOCENZIO MARIA (1692-1768), Italian poet, was born at Genoa on the 21st of November 1692. He was originally destined for the church and at the age of fifteen, in opposition to his strong wishes, was shut up in a convent; but although in the following year he was induced to pronounce monastic vows, he had no liking for this life. He acquired considerable reputation as an elegant writer both of Latin and Italian prose and verse; and from 1716 to 1724 he filled the chairs of rhetoric at Brescia, Rome, Genoa, Bologna and Modena successively, attracting by his brilliant fluency a large number of students at each university. Through Cardinal Bentivoglio he was recommended to Antonio Farnese, duke of Parma, who appointed him his poet laureate; and he remained at the court of Parma until the death of Antonio, after which he returned to Genoa. Shortly afterwards, through the intercession of Bentivoglio, he obtained from the pope the remission of his monastic vows, and ultimately succeeded in recovering a portion of his paternal inheritance. After the peace of Aix-la-Chapelle he returned to the court of Parma, and there devoted the later years of his life chiefly to poetical composition. He died on the 20th of December 1768. As a poet Frugoni was one of the best of the school of the Arcadian Academy, and his lyrics and pastorals had great facility and elegance.

His collected works were published at Parma in 10 vols. in 1799, and a more complete edition appeared at Lucca in the same year in 15 vols. A selection from his works was published at Brescia in 1782, in 4 vols.

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FRUIT (through the French from the Lat. *fructus*; *frui*, to enjoy), in its widest sense, any product of the soil that can be enjoyed by man or animals; the word is so used constantly in the Bible, and extended, as a Hebraism, to offspring or progeny of man and of animals, in such expressions as “the fruit of the body,” “of the womb,” “fruit of thy cattle” (Deut. xxviii. 4), &c., and generally to the product of any action or effort. Between this wide and frequently figurative use of the word and its application in the strict botanical sense treated below, there is a popular meaning, regarding the objects denoted by the word entirely from the standpoint of edibility, and differentiating them roughly from those other products of the soil, which, regarded similarly, are known as vegetables. In this sense “fruit” is applied to such seed-envelopes of plants as are edible, either raw or cooked, and are usually sweet, juicy or of a refreshing flavour. But applications of the word in this sense are apt to be loose and shifting according to the fashion of the time.

Fruit, in the botanical sense, is developed from the flower as the result of fertilization of the ovule. After fertilization various changes take place in the parts of the flower. Those more immediately concerned in the process, the anther and stigma, rapidly wither and decay, while the filaments and style often remain for some time; the floral envelopes become dry, the petals fall, and the sepals are either deciduous, or remain persistent in an altered form; the ovary becomes enlarged, forming the *pericarp*; and the ovules are developed as the seeds, containing the embryo-plant. The term fruit is strictly applied to the mature pistil or ovary, with the seeds in its interior; but it often includes other parts of the flower, such as the bracts and floral envelopes. Thus the fruit of the hazel and oak consists of the ovary enveloped by the bracts; that of the apple and pear, of the ovary and floral receptacle; and that of the pine-apple, of the whole inflorescence. Such fruits are sometimes distinguished as *pseudocarps*. In popular language, the fruit includes all those parts which exhibit a striking change as the result of fertilization. In general, the fruit is not ripened unless fertilization has been effected; but cases occur as the result of cultivation in which the fruit swells and becomes to all appearance perfect, while no seeds are produced. Thus, there are seedless oranges, grapes and pineapples. When the ovules are unfertilized, it is common to find that the ovary withers and does not come to maturity; but in the case of bananas, plantains and bread-fruit, the non-development of seeds seems to lead to a larger growth and a greater succulence of fruit.

The fruit, like the ovary, may be formed of a single carpel or of several. It may have one cell or cavity, being *unilocular*; or many, *multilocular*, &c. The number and nature of the divisions depend on the number of carpels and the extent to which their edges are folded inwards. The appearances presented by the ovary do not always remain permanent in the fruit. Great changes are observed to take place, not merely as regards the increased size of the ovary, its softening or hardening, but also in its internal structure, owing to the suppression, additional formation or enlargement of parts. Thus, in the ash (fig. 1) an ovary with two cells, each containing an ovule attached to a central placenta, is changed into a unilocular fruit with one seed; one ovule becomes abortive, while the other, *g*, gradually enlarging until the septum is pushed to one side, unites with the walls of the cell, and the placenta appears to be parietal. In the oak and hazel, an ovary with three and two cells respectively, and two ovules in each, produces a one-celled fruit with one seed. In the coco-nut, a trilocular and triovular ovary produces a one-celled, one-seeded fruit. This abortion may depend on the pressure caused by the development of certain ovules, or it may proceed from non-fertilization of all the ovules and consequent non-enlargement of the carpels. Again, by the growth of the placenta, or the folding inwards of parts of the carpels, divisions occur in the fruit which did not exist in the ovary. In *Cathartocarpus Fistula* a one-celled ovary is changed into a fruit having each of its seeds in a separate cell, in consequence of spurious dissepiments being produced horizontal from the inner wall of the ovary. In flax (*Linum*) by the folding inwards of the back of the carpels a five-celled ovary becomes a ten-celled fruit. In *Astragalus* the folding inwards of the dorsal suture converts a one-celled ovary into a two-celled fruit; and in *Oxytropis* the folding of the ventral suture gives rise to a similar change. The development of cellular or pulpy matter, and the enlargement of parts not forming whorls of the flower, frequently alter the appearance of the fruit, and render it difficult to discover its formation. In the gooseberry (fig. 29), grape, guava, tomato and pomegranate, the seeds nestle in pulp formed by the placentas. In the orange the pulpy matter surrounding the seeds is formed by succulent cells, which are produced from the inner partitioned lining of the pericarp. In the strawberry the receptacle becomes succulent, and bears the mature carpels on its convex surface (fig. 2); in the rose there is a fleshy hollow receptacle which bears the carpels on its concave surface (fig. 3). In the juniper the scaly bracts grow up round the seeds and become succulent, and in the fig (fig. 4) the receptacle becomes succulent and encloses an inflorescence.

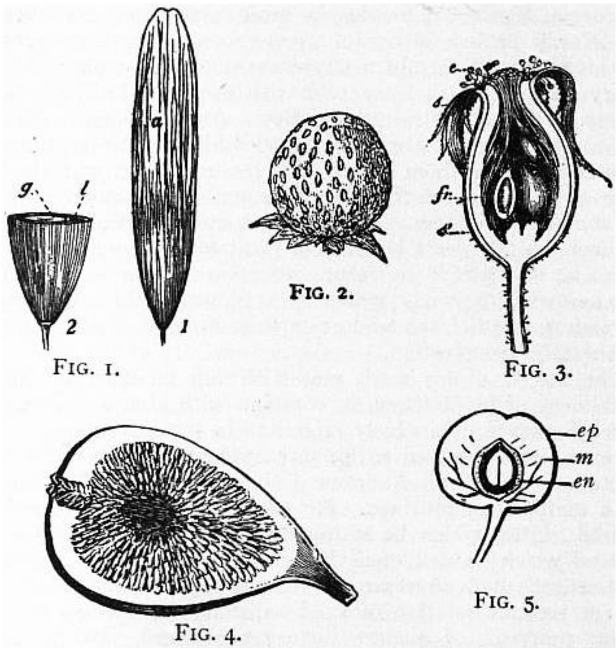


FIG. 1.—Samara or winged fruit of Ash (*Fraxinus*). 1, Entire, with its wing *a*; 2, lower portion cut transversely, to show that it consists of two cells; one of which, *l*, is abortive, and is reduced to a very small cavity, while the other is much enlarged and filled with a seed *g*.

FIG. 2.—Fruit of the Strawberry (*Fragaria vesca*), consisting of an enlarged succulent receptacle, bearing on its surface the small dry seed-like fruits (achenes). (After Duchartre.)

From Strasburger’s *Lehrbuch der Botanik*, by permission of Gustav Fischer.

FIG. 3.—Fruit of the Rose cut vertically. *s′*, Fleshy hollowed receptacle; *s*, persistent sepals; *fr*, ripe carpels; *e*, stamens, withered.

FIG. 4.—Peduncle of Fig (*Ficus Carica*), ending in a hollow receptacle enclosing numerous male and female flowers.

FIG. 5.—Fruit of Cherry (*Prunus Cerasus*) in longitudinal section. *ep*, Epicarp; *m*, mesocarp; *en*, endocarp.

From Strasburger’s *Lehrbuch der Botanik*, by permission of Gustav Fischer.

The pericarp consists usually of three layers, the external, or *epicarp* (fig. 5, *ep*); the middle, or *mesocarp*, *m*; and the internal, or *endocarp*, *en*. These layers are well seen in such a fruit as the peach, plum or cherry, where they are separable one from the other; in them the epicarp forms what is commonly called the skin; the mesocarp, much developed, forms the flesh or pulp, and hence has sometimes been called *sarcocarp*; while the endocarp, hardened by the production of woody cells, forms the *stone* or *putamen* immediately covering the kernel or seed. The pulpy matter found in the interior of fruits, such as the gooseberry, grape and others, is formed from the placentas, and must not be confounded with the sarcocarp. In some fruits, as in the nut, the three layers become blended together and are indistinguishable. In bladder senna (*Colutea arborescens*) the pericarp retains its leaf-like appearance, but in most cases it becomes altered both in consistence and in colour. Thus in the date the epicarp is the outer brownish skin, the pulpy matter is the mesocarp or sarcocarp, and the thin papery-like lining is the endocarp covering the hard seed. In the medlar the endocarp becomes of a stony hardness. In the melon the epicarp and endocarp are very thin, while the mesocarp forms the bulk of the fruit, differing in texture and taste in its external and internal parts. The rind of the orange consists of epicarp and mesocarp, while the endocarp forms partitions in the interior, filled with pulpy cells. The part of the pericarp attached to the peduncle is the base, and the point where the style or stigma existed is the apex. This latter is not always the apparent apex, as in the case of the ovary; it may be lateral or even basilar. The style sometimes remains in a hardened form, rendering the fruit *apiculate*; at other times it falls off, leaving only traces of its existence. The presence of the style or stigma serves to distinguish certain single-seeded pericarps from seeds.

When the fruit is mature and the seeds are ripe, the carpels usually give way either at the ventral or dorsal suture or at both, and so allow the seeds to escape. The fruit in this case is *dehiscent*. But some fruits are *indehiscent*, 255

Dehiscence of fruits. falling to the ground entire, and the seeds eventually reaching the soil by their decay. By dehiscence the pericarp becomes divided into different pieces, or *valves*, the fruit being univalvular, bivalvular or multivalvular, &c., according as there are one, two or many valves. The splitting extends the whole length of the fruit, or is partial, the valves forming teeth at the apex, as in the order Caryophyllaceae (fig. 6). Sometimes the valves are detached only at certain points, and thus dehiscence takes place by pores at the apex, as in poppy (fig. 7), or at the base, as in *Campanula*. Indehiscent fruits are either dry, as the nut, or fleshy, as the cherry and apple. They are formed of one or several carpels. In the former case they usually contain only a single seed, which may become so incorporated with the pericarp as to appear to be naked, as in the grain of wheat and generally in grasses. In such cases the presence of the remains of style or stigma determines their true nature.

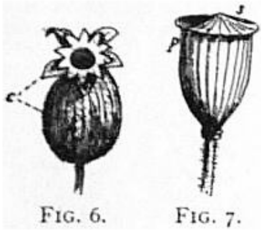
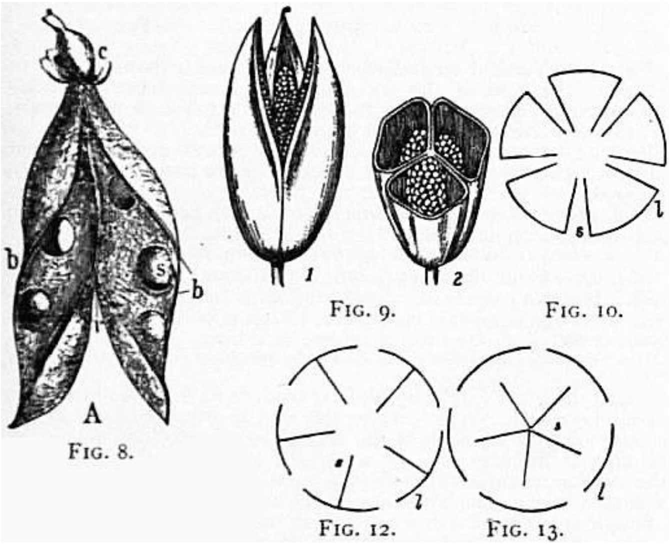


FIG. 6.—Seed-vessel or capsule of Campion, opening by ten teeth at the apex. The calyx *c* is seen surrounding the seed-vessel.

FIG. 7.—Capsule of Poppy, opening by pores *p*, under the radiating peltate stigma *s*.



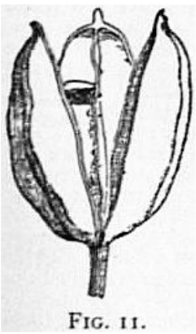


FIG. 8.—Dry dehiscent fruit. The pod (legume) of the Pea; *r*, the dorsal suture; *b*, the ventral; *c*, calyx; *s*, seeds.

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FIG. 9.—(1) Fruit or capsule of Meadow Saffron (*Colchicum autumnale*), dehiscing along the septa (septicidally); (2) same cut across, showing the three chambers with the seeds attached along the middle line (axile placentation).

FIG. 10.—Diagram to illustrate the septicidal dehiscence in a pentalocular capsule. The loculaments *l* correspond to the number of the carpels, which separate by splitting through the septa, *s*.

FIG. 11.—The seed vessel (capsule) of the Flower-de-Luce (*Iris*), opening in a loculicidal manner. The three valves bear the septa in the centre, and the opening takes place through the back of the loculaments. Each valve is formed by the halves of contiguous carpels.

FIG. 12.—Diagram to illustrate loculicidal dehiscence. The loculaments *l*, split at the back, and the valves separate, bearing the septa *s* on their centres.

FIG. 13.—Diagram to illustrate septifragal dehiscence, in which the dehiscence takes place through the back of the loculaments *l*, and the valves separate from the septa *s*, which are left attached to the placentas in the centre.

Dehiscent fruits, when composed of single carpels, may open by the ventral suture only, as in the paeony, hellebore, *Aquilegia* (fig. 28) and *Caltha*; by the dorsal suture only, as in magnolias and some *Proteaceae*, or by both together, as in the pea (fig. 8) and bean; in these cases the dehiscence is *sutural*. When composed of several united carpels, two types of dehiscence occur—a longitudinal and a transverse. In the longitudinal the separation may take place by the dissepiments throughout their length, so that the fruit is resolved into its original carpels, and each valve represents a carpel, as in rhododendron, *Colchicum*, &c.; this dehiscence, in consequence of taking place through the septum, is called *septicidal* (figs. 9, 10). The valves separate from their commissure, or central line of union, carrying the placentas with them, or they leave the latter in the centre, so as to form with the axis a column of a cylindrical, conical or prismatic shape. Dehiscence is *loculicidal* when the union between the edges of the carpels is persistent, and they dehisce by the dorsal suture, or through the back of the loculaments, as in the lily and iris (figs. 11, 12). In these cases each valve consists of a half of each of two contiguous carpels. The placentas either remain united to the axis, or they separate from it, being attached to the septa on the valves. When the outer walls of the carpels break off from the septa, leaving them attached to the central column, the dehiscence is said to be *septifragal* (fig. 13), and where, as in *Linum catharticum* and *Calluna*, the splitting takes place first of all in a septicidal manner, the fruit is described as *septicidally septifragal*; while in other cases, as in thorn apple (*Datura Stramonium*), where the splitting is at first loculicidal, the dehiscence is *loculicidally septifragal*. In all those forms the separation of the valves takes place either from above downwards or from below upwards. In *Saxifraga* a splitting for a short distance of the ventral sutures of the carpels takes place, so that a large apical pore is formed. In the fruit of Cruciferae, as wallflower (fig. 14), the valves separate from the base of the fruit, leaving a central *replum*, or frame, which supports the false septum formed by a prolongation from the parietal placentas on opposite sides of the fruit, extending between the ventral sutures of the carpels. In Orchidaceae (fig. 15) the pericarp, when ripe, separates into three valves in a loculicidal manner, but the midribs of the carpels, to which the placentas are attached, often remain adherent to the axis both at the apex and base after the valves bearing the seeds have fallen. The other type of dehiscence is transverse, or *circumscissile*, when the upper part of the united carpels falls off in the form of a lid or operculum, as in *Anagallis* and in henbane (*Hyoscyamus*) (fig. 16).

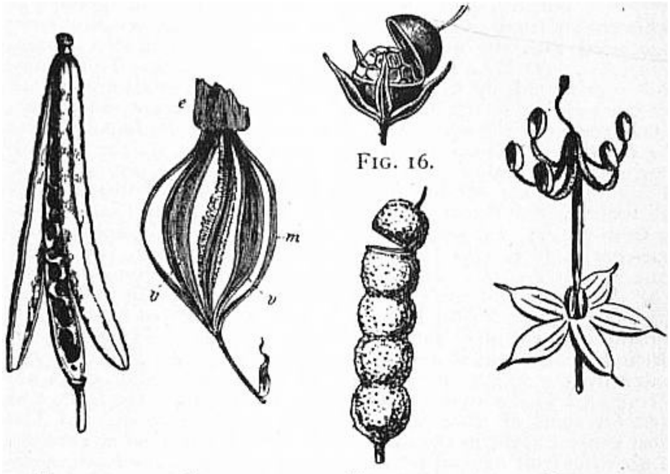

FIG. 16.

FIG. 14. FIG. 15. FIG. 17. FIG. 18.

FIG. 14.—Siliqua or seed-vessel of Wallflower (*Cheiranthus Cheiri*), opening by two valves, which separate from the base upwards, leaving the seeds attached to the dissepiment which is supported by the replum.

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FIG. 15.—Capsule of an Orchid (*Xylobium*). *v*, valve.

FIG. 16.—Seed-vessel of *Anagallisarvensis*, opening by circumscissile dehiscence.

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FIG. 17.—Lomentum of *Hedysarum* which, when ripe, separates transversely into single-seeded portions or mericarps.

FIG. 18.—Fruit of *Geranium pratense*, after splitting.

Sometimes the axis is prolonged beyond the base of the carpels, as in the mallow and castor-oil plant, the carpels being united to it throughout their length by their faces, and separating from it without opening. In the Umbelliferae the two carpels separate from the lower part of the axis, and remain attached by their apices to a prolongation of it, called a *carpophore* or *podocarp*, which splits into two (fig. 25) and suspends them; hence the fruit is termed a *cremocarp*, which divides into two *mericarps*. The general term *schizocarp* is applied to all dry fruits, which break up into two or more one-seeded indehiscent mericarps, as in *Hedysarum* (fig. 17). In the order Geraniaceae the styles remain attached to a central column, and the mericarps separate from below upwards, before dehiscing by their ventral suture (fig. 18). Carpels which separate one from another in this manner are called *cocci*. They are well seen in the order Euphorbiaceae, where there are 256
usually three such carpels, and the fruit is termed tricoccus. In many of them, as *Hura crepitans*, the cocci separate with great force and elasticity. In many leguminous plants, such as *Ornithopus*, *Hedysarum* (fig. 17), *Entada*, *Coronilla* and the gum-arabic plant (*Acacia arabica*), the fruit becomes a schizocarp by the formation of transverse partitions from the folding in of the sides of the pericarp, and distinct separations taking place at these partitions.

Fruits are formed by one flower, or are the product of several flowers combined. In the former case they are either *apocarpous*, of one mature carpel or of several separate free carpels; or *syncarpous*, of several carpels, more or less completely united. When the fruit is composed of the ovaries of several flowers united, it is usual to find the bracts and floral envelopes also joined with them, so as to form one mass; hence such fruits are known as multiple, confluent or *anthocarpous*. The term simple is applied to fruits which are formed by the ovary of a single flower, whether they are composed of one or several carpels, and whether these carpels are separate or combined.

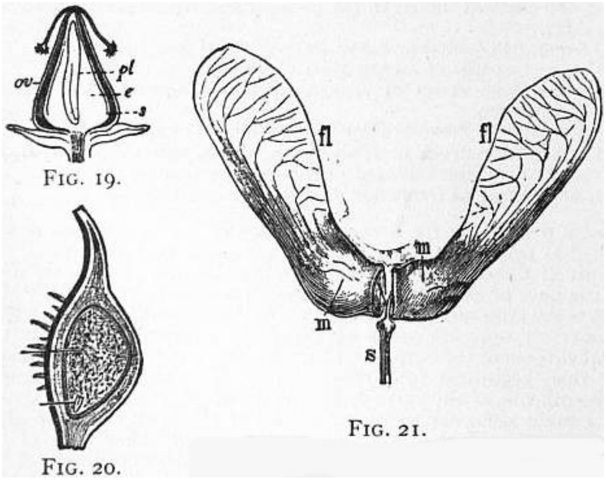

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FIG. 19.—Dry one-seeded fruit of dock (*Rumex*) cut vertically. ov, Pericarp formed from ovary wall; *s*, seed; *e*, endosperm; *pl*, embryo with radicle pointing upwards and cotyledons downwards—enlarged.

FIG. 20.—Achene of *Ranunculus arvensis* in longitudinal section; *e*, endosperm; *pl*, embryo. (After Baillon, enlarged.)

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FIG. 21.—Fruit of Common Sycamore (*Acer Pseudoplatanus*), dividing into two mericarps *m*; *s*, pedicel; *fl*, wings (nat. size).

Dispersal of fruit or seed.

The object of the fruit in the economy of the plant is the protection and nursing of the developing seed and the dispersion of the ripe seeds. Hence, generally, one-seeded fruits are indehiscent, while fruits containing more than one seed open to allow of the dispersal of the seeds over as wide an area as possible. The form, colour, structure and method of dehiscence of fruits and the form of the contained seeds are intimately associated with the means of dispersal, which fall into several categories. (1) By a mechanism residing in the fruit. Thus many fruits open suddenly when they are dry, and the seeds are ejected by the twisting or curving of the valves, or in some other way; *e.g.* in gorse, by the spiral curving of the valves; in *Impatiens*, by the twisting of the cocci; in squirting cucumber, by the pressure exerted on the pulpy contents by the walls of the pericarp. (2) By aid of various external agencies such as water. Fruits or seeds are sometimes sufficiently buoyant to float for a long time on sea- or fresh-water; *e.g.* coco-nut, by means of its thick, fibrous coat (mesocarp), is carried hundreds of miles in the sea, the tough, leathery outer coat (epicarp) preventing it from becoming water-soaked. Fruits and seeds of West Indian plants are thrown up on the coasts of north-west Europe, having been carried by the Gulf Stream, and will often germinate; many are rendered buoyant by air-containing cavities, and the embryo is protected from the seawater by the tough coat of fruit or seed. Water-lily seeds are surrounded with a spongy tissue when set free from the fruit, and float for some distance before dropping to the bottom. (3) The most general agent in the dispersal of seeds is the wind or currents of air—the fruit or seed being rendered buoyant by wing-developments as in fruits of ash (fig. 1) or maple (fig. 21), seeds of pines and firs, or many members of the order Bignoniaceae; or hair-developments as in fruits of clematis, where the style forms a feathery appendage, fruits of many Compositae (dandelion, thistle, &c.), which are crowned by a plumose pappus, or seeds of willow and poplar, or *Asclepias* (fig. 36), which bear tufts of silky hairs; to this category belong bladder-like fruits, such as bladder-senna, which are easily rolled by the wind, or cases like the so-called rose of Jericho, a small cruciferous plant (*Anastatica hierocuntica*), where the plant dries up after developing its fruits and becomes detached from the ground; the branches curl inwards, and the whole plant is rolled over the dry ground by the wind. The wind also aids the dispersal of the seeds in the case of fruits which open by small teeth (many Caryophyllaceae [fig. 6]) or pores (poppy [fig. 7], *Campanula*, &c.); the seeds are in these cases small and numerous, and are jerked through the pores when the capsules, which are generally borne on long, dry stems or stalks, are shaken by the wind. (4) In other cases members of the animal world aid in seed-dispersal. Fruits often bear stiff hairs or small hooks, which cling to the coat of an animal or the feathers of a bird; such are fruits of cleavers (*Galium Aparine*), a common hedge-row plant, *Ranunculus arvensis* (fig. 20), carrot, *Geum*, &c.; or the fruit or seed has an often bright-coloured, fleshy covering, which is sought by birds as food, as in stone-fruits such as plum, cherry (fig. 5), &c., where the seed is protected from injury in the mouth or stomach of the animal by the hard endocarp; or the hips of the rose (fig. 3), where the succulent scarlet “fruit” (the swollen receptacle) envelops a number of small dry true fruits (achenes), which cling by means of stiff hairs to the beak of the bird.

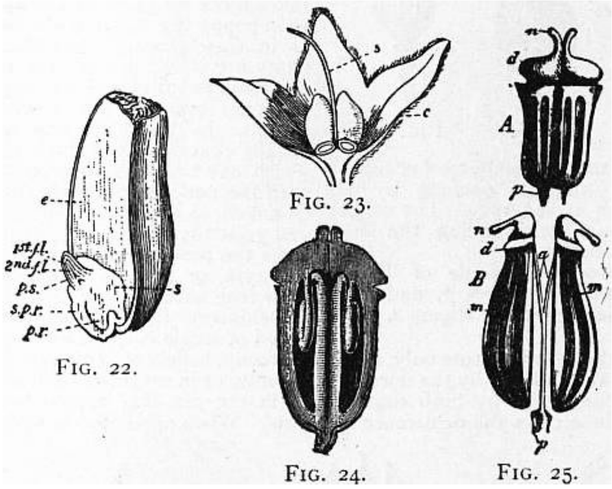

FIG. 22.—Vertical section of a grain of wheat, showing embryo below at the base of the endosperm *e*; *s*, scutellum separating embryo from endosperm; *f.l*, foliage leaf; *p.s*, sheath of plumule; *p.r*, primary root; *s.p.r*, sheath of primary root.

FIG. 23.—Fruit of Comfrey (*Symphytum*) surrounded by persistent calyx, *c*. The style s appears to arise from the base of the carpels, enlarged.

FIG. 24.—Ovary of *Foeniculum officinale* with pendulous ovules, in longitudinal section. (After Berg and Schmidt, magnified.)

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FIG. 25.—Fruit of *Carum Carui*. A, Ovary of the flower; B, ripe fruit. The two carpels have separated so as to form two mericarps (*m*). Part of the septum constitutes the carpophore (*a*). *p*, Top of flower-stalk; *d*, disk on top of ovary; *n*, stigma.

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Forms of fruit.

Simple fruits have either a *dry* or *succulent* pericarp. The *achene* is a dry, one-seeded, indehiscent fruit, the pericarp of which is closely applied to the seed, but separable from it. It is solitary, forming a single fruit, as in the dock (fig. 19) and in the cashew, where it is supported on a fleshy peduncle; or *aggregate*, as in *Ranunculus* (fig. 20), where several achenes are placed on a common elevated receptacle. In the strawberry the achenes (fig. 2) are aggregated on a convex succulent receptacle. In the rose they are supported on a concave receptacle (fig. 3), and in the fig the succulent receptacle completely encloses the achenes (fig. 4). In *Dorstenia* the achenes are situated on a flat or slightly concave receptacle. Hence what in common language are called the seeds of the strawberry, rose and fig, are in reality ripe carpels. The styles occasionally remain attached to the achenes in the form of feathery appendages, as in *Clematis*. In Compositae, the fruit is an inferior achene (*cypsela*), to which the pappus (modified calyx) remains adherent. Such is also the nature of the fruit in Dipsacaceae (*e.g.* scabious). When the pericarp is thin, and appears like a bladder surrounding the seed, the achene is termed a *utricle*, as in Amarantaceae. When the pericarp is extended in the form of a winged appendage, a *samara* or *samaroid achene* is produced, as in the ash (fig. 1) and common sycamore (fig. 21). In these cases there are usually two achenes united, one of which, however, as in *Fraxinus* (fig. 1), may be abortive. The wing surrounds the fruit longitudinally in the elm. When the pericarp becomes so incorporated with the seed as to be inseparable from it, as in grains of wheat (fig. 22), maize, oats and other grasses, then the name *caryopsis* is given. The one-seeded portions (mericarps) of schizocarps often take the form of achenes, *e.g.* the mericarps of the mallows or of umbellifers (figs. 24, 25). In Labiatae and Boraginaceae (*e.g.* comfrey, fig. 23), where the bicarpellary ovary becomes our one-seeded portions in the fruit, the partial fruits are of the nature of achenes or nutlets according to the texture (leathery or hard) of the pericarp.

The *nut* or *glans* is a dry one-celled indehiscent fruit with a hardened pericarp, often surrounded by bracts at the base, and, when mature, containing only one seed. In the young state the ovary often contains two or more ovules, but only one comes to maturity. It is illustrated by the fruits of the hazel and chestnut, which are covered by leafy bracts, in the form of a *husk*, and by the acorn, in which the bracts and receptacle form a *cupula* or *cup* (fig. 26). The parts of the pericarp of the nut are united so as to appear one. In common language the term nut is very vaguely applied both to fruit and seeds.

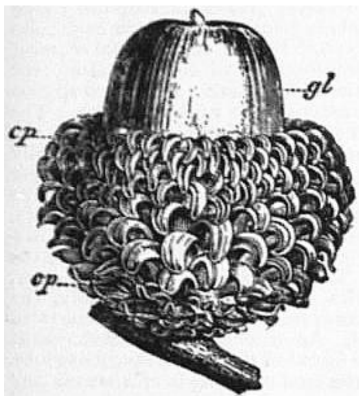

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FIG. 26.—Cupule of *Quercus Aegilops*. *cp*, cupule; *gl*, fruit. (After Duchartre.)

The *drupe* is a succulent usually one-seeded indehiscent fruit, with a pericarp easily distinguishable into epicarp, mesocarp and endocarp. This term is applied to such fruits as the cherry (fig. 5), peach, plum, apricot or mango. The endocarp is usually hard, forming the stone (putamen) of the fruit, which encloses the kernel or seed. The mesocarp is generally pulpy and succulent, so as to be truly a sarcocarp, as in the peach, but it is sometimes of a tough texture, as in the almond, and at other times is more or less fibrous, as in the coco-nut. In the almond there are often two ovules formed, only one of which comes to perfection. In the raspberry and bramble several small drupes or *drupels* are aggregated so as to constitute an *etaerio*.

The *follicle* is a dry unilocular many-seeded fruit, formed from one carpel and dehiscing by the ventral suture. It is rare to meet with a solitary follicle forming the fruit. There are usually several aggregated together, either in a whorl on a shortened receptacle, as in hellebore, aconite, larkspur, columbine (figs. 27, 28) or the order Crassulaceae, or in a spiral manner on an elongated receptacle, as in *Magnolia* and *Banksia*. Occasionally, follicles dehisce by the dorsal suture, as in *Magnolia grandiflora* and *Banksia*.

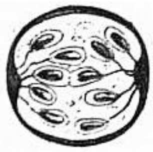


FIG. 29.

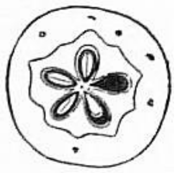


FIG. 27. FIG. 28. FIG. 30.

FIG. 27.—Fruit of Columbine (*Aquilegia*), formed of five follicles.

FIG. 28.—Single follicle, showing dehiscence by the ventral suture.

FIG. 29.—Transverse section of berry of Gooseberry, showing the seeds attached to the parietal placentas and immersed in pulp, which is formed partly from the endocarp, partly from the seed-coat.

FIG. 30.—Section of the fruit of the Apple (*Pyrus Malus*), or pome, consisting of a fleshy covering formed by the floral receptacle and the true fruit or core with five cavities with seeds.

The *legume* or *pod* is a dry monocarpellary unilocular many-seeded fruit, formed from one carpel, dehiscing both by the ventral and the dorsal suture. It characterizes leguminous plants, as the bean and pea (fig. 8). In the bladder-senna it forms an inflated legume. In some Leguminosae, as *Arachis*, *Cathartocarpus Fistula* and the tamarind, the fruit must be considered a legume, although it does not dehisce. The first of these plants produces its fruit underground, and is called earth-nut; the second has a partitioned legume and is schizocarpic; and both the second and third have pulpy matter surrounding the seeds. Some legumes are schizocarpic by the formation of constrictions externally. Such a form is the *lomentum* or *lomentaceous legume* of *Hedysarum* (fig. 17), *Coronilla*, *Ornithopus*, *Entada* and of some Acacias. In *Medicago* the legume is twisted like a snail, and in *Caesalpinia coriaria*, or Divi-divi, it is vermiform or curved like a worm. Sometimes the number of seeds is reduced, as in *Erythrina monosperma* and *Geoffroya superba*, which are one-seeded, and in *Pterocarpus* and *Dalbergia*, which are two-seeded.

The *berry* (*bacca*) is a term applied generally to all fruits with seeds immersed in pulp, and includes fruits of very various origin. In *Actaea* (baneberry) or *Berberis* (barberry) it is derived from a single free carpel; generally, however, it is the product of a syncarpous ovary, which is superior, as in grape or potato, or inferior, as in gooseberry (fig. 29) or currant. In the pomegranate there is a peculiar baccate many-celled inferior fruit, having a tough rind, enclosing two rows of carpels placed one above the other. The seeds are immersed in pulp, and are attached irregularly to the wall, base and centre of the loculi. In the baobab there is a multilocular syncarpous fruit, in which the seeds are immersed in pulp.

FIG. 31.—Transverse section of the fruit of the Melon (*Cucumis Melo*), showing the placentas with the seeds attached to them. The three carpels forming the pepo are separated by partitions. From the centre processes pass outwards, ending in the curved placenta.

The *pepo*, another indehiscent syncarpous fruit, is illustrated by the fruit of the gourd, melon (fig. 31) and other Cucurbitaceae. It is formed of three carpels, surmounted by the calyx; the rind is thick and fleshy, and there are three or more seed-bearing parietal placentas, either surrounding a central cavity or prolonged inwards into it. The fruit of the papaw resembles the pepo, but the calyx is not superior.

The *hesperidium* is the name given to such indehiscent fleshy syncarpous fruits as the orange, lemon and shaddock, in which the epicarp and mesocarp form a separable rind, and the endocarp sends prolongations inwards, forming triangular divisions, to the inner angle of which the seeds are attached, pulpy cells being developed around them from the wall. Both pepo and hesperidium may be considered as modifications of the berry.

The *pome* (fig. 30), seen in the apple, pear, quince, medlar and hawthorn, is a fleshy indehiscent syncarpous fruit, in the formation of which the receptacle takes part. The outer succulent part is the swollen receptacle, the horny core being the true fruit developed from the usually five carpels and enclosing the seeds. In the medlar the core (or true pericarp) is of a stony hardness, while the outer succulent covering is open at the summit. The pome somewhat resembles the fruit of the rose (fig. 3), where the succulent receptacle surrounds a number of separate achenes.

The name *capsule* is applied generally to all dry syncarpous fruits, which dehisce by valves. It may thus be unilocular or multilocular, one- or many-seeded. The true valvular capsule is observed in *Colchicum* (fig. 9), lily and iris (fig. 11). The *porose capsule* is seen in the poppy (fig. 7), *Antirrhinum* and *Campanula*. In *Campanula* the pores occur at the base of the capsule, which becomes inverted when ripe. When the capsule opens by a lid, or by circumscissile dehiscence, it is called a *pyxidium*, as in pimpernel (*Anagallis arvensis*) (fig. 16), henbane and monkey-pot (*Lecythis*). The capsule assumes a screw-like form in *Helicteres*, and a star-like form in star-anise (*Illicium anisatum*). In certain instances the cells of the capsule separate from each other, and open with elasticity to scatter the seeds. This kind of capsule is met with in the sandbox tree (*Hura crepitans*) and other Euphorbiaceae, where the cocci, containing each a single seed, burst asunder with force; and in Geraniaceae, where the cocci, each containing, when mature, usually one seed, separate from the carpophore, become curved upwards by their adherent styles, and open by the ventral suture (fig. 18).

The *siliqua* is a dry syncarpous bilocular many-seeded fruit, formed from two carpels, with a false septum, dehiscing by two valves from below upwards, the valves separating from the placentas and leaving them united by the septum (fig. 32). The seeds are attached on both sides of the septum, either in one row or in two. When the fruit is long and narrow it is a *siliqua* (fig. 14); when broad and short, *silicula* (fig. 33). It occurs in cruciferous plants, as wallflower, cabbage and cress. In *Glaucium* and *Eschscholtzia* (Papaveraceae) the dissepiment is of a spongy nature. It may become transversely constricted (*lomentaceous*), as in radish (*Raphanus*) and sea-kale, and it may be reduced, as in woad (*Isatis*), to a one-seeded condition.

It sometimes happens that the ovaries of two flowers unite so as to form a double fruit (*syncarp*). This may be seen in many species of honeysuckle. But the fruits which are now to be considered consist usually of the floral envelopes, as well as the ovaries of several flowers united into one, and are called *multiple* or *confluent*. The term *anthocarpous* has also been applied as indicating that the floral envelopes as well as the carpels are concerned in the formation of the fruit.

The *sorosis* is a succulent multiple fruit formed by the confluence of a spike of flowers, as in the fruit of the pine-apple (fig. 34), the bread-fruit and jack-fruit. Similarly the fruit of the mulberry represents a catkin-like inflorescence. 258

The *syconus* is an anthocarpous fruit, in which the receptacle completely encloses numerous flowers and becomes succulent. The fig (fig. 4) is of this nature, and what are called its seeds are the achenes of the numerous flowers scattered over the succulent hollowed receptacle. In *Dorstenia* the axis is less deeply hollowed, and of a harder texture, the fruit exhibiting often very anomalous forms.

The *strobilus*, or *cone*, is a seed-bearing spike, more or less elongated, covered with scales, each of which may be regarded as representing a separate flower, and has often two seeds at its base; the seeds are naked, no ovary being present. This fruit is seen in the cones of firs, spruces, larches and cedars, which have received the name of Coniferae, or cone-bearers, on this account. Cone-like fruit is also seen in most Cycadaceae. The scales of the strobilus are sometimes thick and closely united, so as to form a more or less angular and rounded mass, as in the cypress; while in the juniper they become fleshy, and are so incorporated as to form a globular fruit like a berry. The dry fruit of the cypress and the succulent fruit of the juniper have received the name of *galbulus*. In the hop the fruit is called also a strobilus, but in it the scales are thin and membranous, and the seeds are not naked but are contained in pericarps.

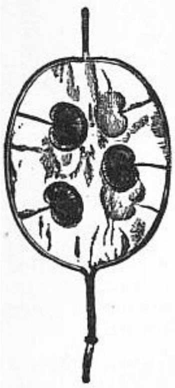


FIG. 32.

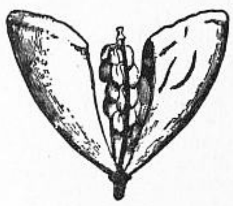


FIG. 33.

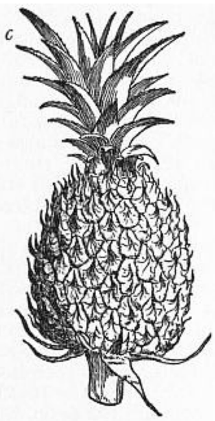


FIG. 34.

FIG. 32.—Honesty (*Lunaria biennis*), showing the septum after the carpels have fallen away.

From Strasburger’s *Lehrbuch der Botanik*, by permission of Gustav Fischer.

FIG. 33.—Silicula or pouch of shepherd’s purse (*Capsella*), opening by two folded valves, which separate from above downwards. The partition is narrow, hence the silicula is angustiseptal.

From Strasburger’s *Lehrbuch der Botanik*, by permission of Gustav Fischer.

FIG. 34.—Fruit of the pine-apple (*Ananassa sativa*), developed from a spike of numerous flowers with bracts, united so as to form a collective or anthocarpous fruit. The crown of the pine-apple, c, consists of a series of empty bracts prolonged beyond the fruit.

The same causes which produce alterations in the other parts of the flower give rise to anomalous appearances in the fruit. The carpels, in place of bearing seeds, are sometimes changed into leaves, with lobes at their margins. Leaves are sometimes produced from the upper part of the fruit. In the genus *Citrus*, to which the orange and lemon belong, it is very common to meet with a separation of the carpels, so as to produce what are called horned oranges and fingered citrons. In this case a syncarpous fruit has a tendency to become apocarpous. In the orange we occasionally find a supernumerary row of carpels produced, giving rise to the appearance of small and imperfect oranges enclosed within the original one; the navel orange is of this nature. It sometimes happens that, by the union of flowers, double fruits are produced. Occasionally a double fruit is produced, not by the incorporation of two flowers, but by the abnormal development of a second carpel in the flower.

Arrangement of Fruits.

A. True fruits—developed from the ovary alone.

1. Pericarp not fleshy or fibrous.

   i. Indehiscent—not opening to allow the escape of the seeds—generally one-seeded. Achene; caryopsis; cypsela; nut; schizocarp.

   ii. Dehiscent—the pericarp splits to allow the escape of the seeds—generally many-seeded. Follicle; legume; siliqua; capsule.

2. Pericarp generally differentiated into distinct layers, one of which is succulent or fibrous. Drupe; berry.

B. Pseudocarps—the development extends beyond the ovary. Pome; syconus; sorosis.

The Seed.—The *seed* is formed from the ovule as the result of fertilization. It is contained in a seed-vessel formed from the ovary in the plants called *angiospermous*; while in *gymnospermous* plants, such as Coniferae and Cycadaceae, it is naked, or, in other words, has no true pericarp. It sometimes happens in Angiosperms, that the seed-vessel is ruptured at an early period of growth, so that the seeds become more or less exposed during their development; this occurs in mignonette, where the capsule opens at the apex, and in *Cuphea*, where the placenta bursts through the ovary and floral envelopes, and appears as an erect process bearing the young seeds. After fertilization the ovule is greatly changed, in connexion with the formation of the embryo. In the embryo-sac of most Angiosperms (*q.v.*) there is a development of cellular tissue, the endosperm, more or less filling the embryo-sac. In Gymnosperms (*q.v.*) the endosperm is formed preparatory to fertilization. The fertilized egg enlarges and becomes multicellular, forming the embryo. The embryo-sac enlarges greatly, displacing gradually the surrounding nucellus, which eventually forms merely a thin layer around the sac, or completely disappears. The remainder of the nucellus and the integuments of the ovules form the seed-coats. In some cases (fig. 35) a delicate inner coat or *tegmen* can be distinguished from a tougher outer coat or *testa*; often, however, the layers are not thus separable. The consistency of the seed-coat, its thickness, the character of its surface, &c., vary widely, the variations being often closely associated with the environment or with the means of seed-dispersal. An account of the development of the seed from the ovule will be found in the article Angiosperms. When the pericarp is dehiscent the seed-covering is of a strong and often rough character; but when the pericarp is indehiscent and encloses the seed for a long period, the outer seed-coat is thin and soft. The cells of the testa are often coloured, and have projections and appendages of various kinds. Thus in *Abrus precatorius* and *Adenanthera pavonina* it is of a bright red colour; in French beans it is beautifully mottled; in the almond it is veined; in the tulip and primrose it is rough; in the snapdragon it is marked with depressions; in cotton and *Asclepias* (fig. 36) it has hairs attached to it; and in mahogany, *Bignonia*, and the pines and firs it is expanded in the form of wing-like appendages (fig. 37). In *Collomia*, *Acanthodium*, *Cobaea scandens* and other seeds, it contains spiral cells, from which, when moistened with water, the fibres uncoil in a beautiful manner; and in flax (*Linum*) and others the cells are converted into mucilage. These structural peculiarities of the testa in different plants have relation to the scattering of the seed and its germination upon a suitable nidus. But in some plants the pericarps assume structures which subserve the same purpose; this especially occurs in small pericarps enclosing single seeds, as achenes, caryopsides, &c. Thus in Compositae and valerian, the pappose limb of the calyx forms a parachute to the pericarp; in Labiatae and some Compositae spiral cells are formed in the epicarp; and the epicarp is prolonged as a wing in *Fraxinus* (fig. 1) and *Acer* (fig. 21).

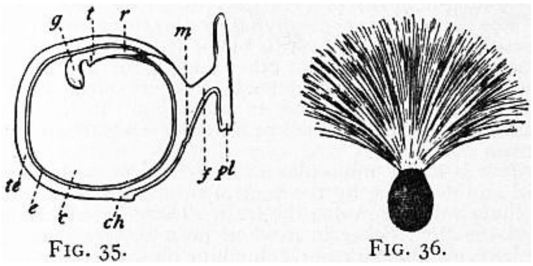


Fig. 35.—Seed of Pea (*Pisum*) with one cotyledon removed. *c*, Remaining cotyledon; *ch*, chalaza-point at which the nourishing vessels enter; *e*, tegmen or inner coat; *f*, funicle or stalk; *g*, plumule of embryo; *m*, micropyle; *pl*, placenta; *r*, radicle of embryo; *t*, tigellum or stalk between root and plumule; *te*, testa.

Fig. 36.—Seed of *Asclepias*, with a cluster of hairs arising from the edges of the micropyle.

Sometimes there is an additional covering to the seed, formed after fertilization, to which the name *arillus* has been given (fig. 38). This is seen in the passion-flower, where the covering arises from the placenta or extremity of the funicle at the base of the ovule and passes upwards towards the apex, leaving the micropyle uncovered. In the nutmeg and spindle tree this additional coat is formed from above downwards, constituting in the former case a laciniated scarlet covering called *mace*. In such instances it has been called an *arillode* (fig. 39). This arillode, after growing downwards, may be reflected upwards so as to cover the micropyle. The fleshy scarlet covering formed around the naked seed in the yew is by some considered of the nature of an aril. On the testa, at various points, there are produced at times other cellular bodies, to which the name of *strophioles*, or *caruncles*, has been given, the seeds being strophiolate or carunculate. These tumours may occur near the base of the seed, as in *Polygala*, or at the apex, as in Castor-oil plant (*Ricinus*); or they may occur in the course of the raphe, as in blood-root (*Sanguinaria*) and *Asarabacca*. The funicles of the ovules frequently attain a great length in the seed, and in some magnolias, when the fruit dehisces, they appear as long scarlet cords suspending the seeds outside. The hilum or umbilicus of the seed is usually well marked, as a scar of varying size; in the calabar bean and in some species of Mucuna and Dolichos it extends along a large portion of the edge of the seed; it frequently exhibits marked colours, being black in the bean, white in many species of Phaseolus, &c. The micropyle (fig. 35, *m*) of the seed may be recognizable by the naked eye, as in the pea and bean tribe, *Iris*, &c., or it may be very minute or microscopic. It indicates the true apex of the seed, and is important as marking the point to which the root of the embryo is directed. At the micropyle in the bean is observed a small process of integument, which, when the young plant sprouts, is pushed up like a lid; it is called the *embryotega*. The chalaza (fig. 38, *ch*) is often of a different colour from the rest of the seed. In the orange (fig. 40) it is of a reddish-brown colour, and is easily recognized at one end of the seed when the integuments are carefully removed. In anatropal seeds the raphe forms a distinct ridge along one side of the seed (fig. 41).

The position of the seed as regards the pericarp resembles that of the ovule in the ovary, and the same terms are applied—erect, ascending, pendulous, suspended, curved, &c. These terms have no reference to the mode in which the fruit is attached to the axis. Thus the seed may be erect while the fruit itself is pendent, in the ordinary meaning of that term. The part of the seed next the axis or the ventral suture is its face, the opposite side being the back. Seeds exhibit great varieties of form. They may be flattened laterally (*compressed*), or from above downwards (*depressed*). They may be round, oval, triangular, polygonal, rolled up like a snail, as in *Physostemon*, or coiled up like a snake, as in *Ophiocaryon paradoxum*.

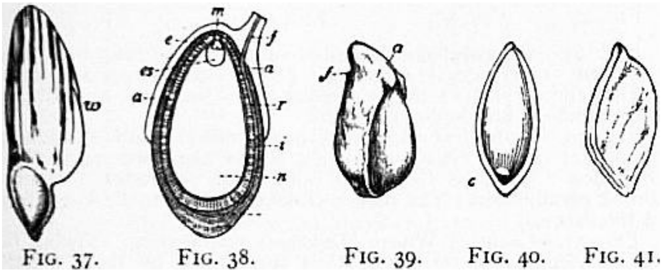


FIG. 37.—Seed of Pine (*Pinus*), with a membranous appendage *w* to the testa, called a wing.

FIG. 38.—Young anatropal seed of the white Water-lily (*Nymphaea alba*), cut vertically. It is attached to the placenta by the funicle f, cellular prolongations from which form an aril *a a*. The vessels of the cord are prolonged to the base of the nucellus n by means of the raphe *r*. The base of the nucellus is indicated by the chalaza ch, while the apex is at the micropyle *m*. The covering of the seed is marked *i*. *n* is the nucellus or perisperm, enclosing the embryo-sac es, in which the endosperm is formed. The embryo *e*, with its suspensor, is contained in the sac, the radicle pointing to the micropyle *m*.

FIG. 39.—Arillode *a*, or false aril, of the Spindle-tree (*Euonymus*), arising from the micropyle *f*.

FIG. 40.—Anatropal seed of the Orange (*Citrus Aurantium*) opened to show the chalaza *c*, which forms a brown spot at one end.

FIG. 41.—Entire anatropal seed of the Orange (*Citrus Aurantium*), with its rugose or wrinkled testa, and the raphe *r* ramifying in the thickness of the testa on one side.

The endosperm formed in the embryo-sac of angiosperms after fertilization, and found previous to it in gymnosperms, consists of cells containing nitrogenous and starchy or fatty matter, destined for the nutriment of the embryo. It occupies the whole cavity of the embryo-sac, or is formed only at certain portions of it, at the apex, as in *Rhinanthus*, at the base, as in *Vaccinium*, or in the middle, as in *Veronica*. As the endosperm increases in size along with the embryo-sac and the embryo, the substance of the original nucellus of the ovule is gradually absorbed. Sometimes, however, as in Musaceae, Cannaceae, Zingiberaceae, no endosperm is formed; the cells of the original nucellus, becoming filled with food-materials for the embryo, are not absorbed, but remain surrounding the embryo-sac with the embryo, and constitute the *perisperm*. Again, in other plants, as Nymphaeaceae (fig. 38) and Piperaceae, both endosperm and perisperm are present. It was from observations on cases such as these that old authors, imagining a resemblance betwixt the plant-ovule and the animal ovum, applied the name *albumen* to the outer nutrient mass or perisperm, and designated the endosperm as *vitellus*. The term albumen is very generally used as including all the nutrient matter stored up in the seed, but it would be advisable to discard the name as implying a definite chemical substance. There is a large class of plants in which although at first after fertilization a mass of endosperm is formed, yet, as the embryo increases in size, the nutrient matter from the endospermic cells passes out from them, and is absorbed by the cells of the embryo plant. In the mature seed, in such cases, there is no separate mass of tissue containing nutrient food-material apart from the embryo itself. Such a seed is said to be *exalbuminous*, as in Compositae, Cruciferae and most Leguminosae (*e.g.* pea, fig. 35). When either endosperm or perisperm or both are present the seed is said to be *albuminous*.

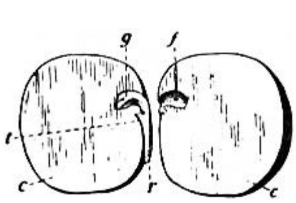


FIG. 42.—The dicotyledonous embryo of the Pea laid open. *c*, *c*, The two fleshy cotyledons, or seed-lobes, which remain under ground when the plant sprouts; *r*, the radicular extremity of the axis whence the root arises; *t*, the axis (hypocotyl) bearing the young stalk and leaves *g* (plumule), which lie in a depression of the cotyledons *f*.

The albumen varies much in its nature and consistence, and furnishes important characters. It may be farinaceous or mealy, consisting chiefly of cells filled with starch, as in cereal grains, where it is abundant; fleshy or cartilaginous, consisting of thicker cells which are still soft, as in the coco-nut, and which sometimes contain oil, as in the oily albumen of *Croton*, *Ricinus* and poppy; horny, when the cell-walls are slightly thickened and capable of distension, as in date and coffee; the cell-walls sometimes become greatly thickened, filling up the testa as a hard mass, as in vegetable ivory (*Phytelephas*). The albumen may be uniform throughout, or it may present a mottled appearance, as in the nutmeg, the seeds of Anonaceae and some Palms, where it is called *ruminated*. This mottled appearance is due to a protrusion of a dark lamella of the integument between folded protuberances of albumen. A cavity is sometimes left in the centre which is usually filled with fluid, as in the coco-nut. The relative size of the embryo and of the endosperm varies much. In Monocotyledons the embryo is usually small, and the endosperm large, and the same is true in the case of coffee and many other plants amongst Dicotyledons. The opposite is the case in other plants, as in the Labiatae, Plumbaginaceae, &c.

The embryo consists of an axis bearing the *cotyledons* (fig. 42, *c*), or the first leaves of the plant. To that part of this axis immediately beneath the cotyledons the terms *hypocotyl*, *caulicle* or *tigellum* (*t*) have been applied, and continuous backwards with it is the young root or *radicle* (*r*), the descending axis, their point of union being the collar or neck. The terminal growing bud of the axis is called the *plumule* or *gemmule* (*g*), and represents the ascending axis. The radicular extremity points towards the micropyle, while the cotyledonary extremity is pointed towards the base of the ovule or the chalaza. Hence, by ascertaining the position of the micropyle and chalaza, the two extremities of the embryo can in general be discovered. It is in many cases difficult to recognize the parts in an embryo; thus in *Cuscuta*, the embryo appears as an elongated axis without divisions; and in *Caryocar* the mass of the embryo is made up by the radicular extremity and hypocotyl, in a groove of which the cotyledonary extremity lies embedded (fig. 52). In some monocotyledonous embryos, as in Orchidaceae, the embryo is a cellular mass showing no parts. In parasitic plants also which form no chlorophyll, as *Orobanche*, *Monotropa*, &c., the embryo remains without differentiation, consisting merely of a mass of cells until the ripening of the seed. When the embryo is surrounded by the endosperm on all sides except its radicular extremity it is internal (see figs. 19, 20); when lying outside the endosperm, and only coming into contact with it at certain points, it is external, as in grasses (*e.g.* wheat, fig. 22). When the embryo follows the direction of the axis of the seed, it is axile or axial (fig. 43); when it is not in the direction of the axis, it becomes abaxile or abaxial. In campylotropal seeds the embryo is curved, and in place of being embedded in endosperm, is frequently external to it, following the concavity of the seed (fig. 44), and becoming peripherical, with the chalaza situated in the curvature of the embryo, as in Caryophyllaceae.

It has been already stated that the radicle of the embryo is directed to the micropyle, and the cotyledons to the chalaza. In some cases, by the growth of the integuments, the former is turned round so as not to correspond with the apex of the nucellus, and then the embryo has the radicle directed to one side, and is called excentric, as is seen in Primulaceae, Plantaginaceae and many palms, especially the date. The position of the embryo in different kinds of seeds varies. In an orthotropal seed the embryo is inverted or *antitropal*, the radicle pointing to the apex of the seed, or to the part opposite the hilum. Again, in an anatropal seed the embryo is erect or *homotropal* (fig. 43), the radicle being directed to the base of the seed. In curved or campylotropal seeds the embryo is folded so that its radicular and cotyledonary extremities are approximated, and it becomes *amphitropal* (fig. 44). In this instance the seed may be exalbuminous, and the embryo may be folded on itself; or albuminous, the embryo surrounding more or less completely the endosperm and being peripherical. According to the mode in which the seed is attached to the pericarp, the radicle may be directed upwards or downwards, or laterally, as regards the ovary. In an orthotropal seed attached to the base of the pericarp it is superior, as also in a suspended anatropal seed. In other anatropal seeds the radicle is inferior. When the seed is horizontal as regards the pericarp, the radicle is either centrifugal, when it points to the outer wall of the ovary; or centripetal, when it points to the axis or inner wall of the ovary. These characters are of value for purposes of classification, as they are often constant in large groups of genera.

Plants in which there are two cotyledons produced in the embryo are *dicotyledonous*. The two cotyledons 260
thus formed are opposite to each other (figs. 42 and 45), but are not always of the same size. Thus, in Abronia and other members of the order Nyctaginaceae, one of them is smaller than the other (often very small), and in *Carapa guianensis* there appears to be only one, in consequence of the intimate union which takes place between the two. The union between the cotyledonary leaves may continue after the young plant begins to germinate. Such embryos have been called *pseudomonocotyledonous*. The texture of the cotyledons varies. They may be thick, as in the pea (fig. 42), exhibiting no traces of venation, with their flat internal surfaces in contact, and their backs more or less convex; or they may be in the form of thin and delicate laminae, flattened on both sides, and having distinct venation, as in *Ricinus*, *Jatropha*, *Euonymus*, &c. The cotyledons usually form the greater part of the mature embryo, and this is remarkably well seen in such exalbuminous seeds as the bean and pea.

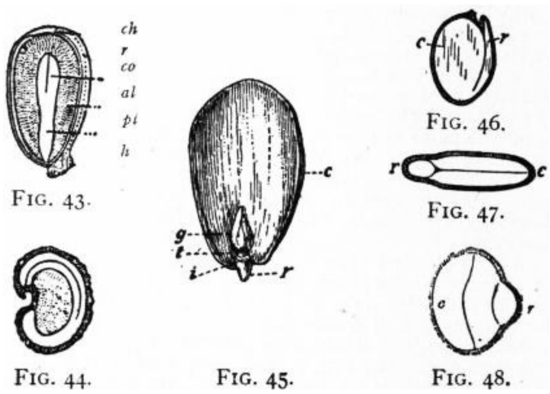


Fig. 43.—Seed of Pansy (*Viola tricolor*) cut vertically. The embryo *pl* is axial, in the midst of fleshy endosperm al. The seed is anatropal, and the embryo is homotropal; the cotyledons co point to the base of the nucellus or chalaza *ch*, while the radicle, or the other extremity of the embryo, points to the micropyle, close to the hilum *h*. The hilum or base of the seed, and the chalaza or base of the nucellus are united by means of the raphe *r*.

Fig. 44.—Seed of the Red Campion (*Lychnis*), cut vertically, showing the peripheral embryo, with its two cotyledons and its radicle. The embryo is curved round the albumen, so that its cotyledons and radicle both come near the hilum (*amphitropal*).

Fig. 45.—Mature dicotyledonous embryo of the Almond, with one of the cotyledons removed. *r*, Radicle; *t*, young stem or caulicle; *c*, one of the cotyledons left; *i*, line of insertion of the cotyledon which has been removed; *g*, plumule.

Fig. 46.—Exalbuminous seed of Wallflower (Cheiranthus) cut vertically. The radicle *r* is folded on the edges of the cotyledons *c* which are accumbent.

Fig. 47.—Transverse section of the seed of the Wallflower (*Cheiranthus*), showing the radicle *r* folded on the

edges of the accumbent cotyledons *c*.

FIG. 48.—Transverse section of the seed of the Dame's Violet (*Hesperis*). The radicle *r* is folded on the back of the cotyledons *c*, which are said to be incumbent.

Cotyledons are usually entire and sessile. But they occasionally become lobed, as in the walnut and the lime; or petiolate, as in *Geranium molle*; or auriculate, as in the ash. Like leaves in the bud, cotyledons may be either applied directly to each other, or may be folded in various ways. In geranium the cotyledons are twisted and doubled; in convolvulus they are corrugated; and in the potato and in *Bunias*, they are spiral,—the same terms being applied as to the foliage leaves. The radicle and cotyledons are either straight or variously curved. Thus, in some cruciferous plants, as the wallflower, the cotyledons are applied by their faces, and the radicle (figs. 46, 47) is folded on their edges, so as to be lateral; the cotyledons are here *accumbent*. In others, as *Hesperis*, the cotyledons (fig. 48) are applied to each other by their faces, and the radicle, *r*, is folded on their back, so as to be dorsal, and the cotyledons are *incumbent*. Again, the cotyledons are *conduplicate* when the radicle is dorsal, and enclosed between their folds. In other divisions the radicle is folded in a spiral manner, and the cotyledons follow the same course.

In many gymnosperms more than two cotyledons are present, and they are arranged in a whorl. This occurs in Coniferae, especially in the pine, fir (fig. 49), spruce and larch, in which six, nine, twelve and even fifteen have been observed. They are linear, and resemble in their form and mode of development the clustered or fasciculated leaves of the larch. Plants having numerous cotyledons are termed *polycotyledonous*. In species of *Streptocarpus* the cotyledons are permanent, and act the part of leaves. One of them is frequently largely developed, while the other is small or abortive.

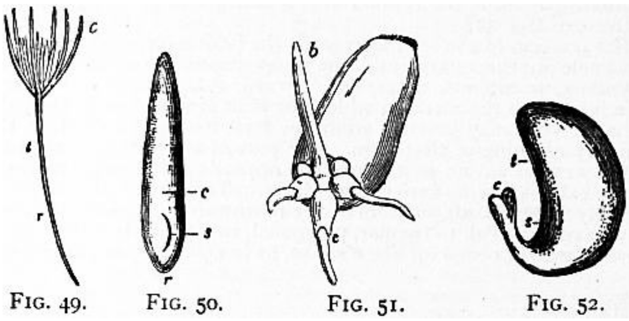


FIG. 49.—Polycotylodonous embryo of the Pine (*Pinus*) beginning to sprout. *t*, Hypocotyl; *r*, radicle. The cotyledons *c* are numerous. Within the cotyledons the primordial leaves are seen, constituting the plumule or first bud of the plant.

FIG. 50.—Embryo of a species of Arrow-grass (*Triglochin*), showing a uniform conical mass, with a slit *s* near the lower part. The cotyledon *c* envelops the young bud, which protrudes at the slit during germination. The radicle is developed from the lower part of the axis *r*.

FIG. 51.—Grain of wheat (*Triticum*) germinating, showing (*b*) the cotyledon and (*c*) the rootlets surrounded by their sheaths (*coleorrhizae*).

FIG. 52.—Embryo of *Caryocar*. *t*, Thick hypocotyl, forming nearly the whole mass, becoming narrowed and curved at its extremity, and applied to the groove *s*. In the figure this narrowed portion is slightly separated from the groove; *c*, two rudimentary cotyledons.

In those plants in which there is only a single cotyledon in the embryo, hence called *monocotyledonous*, the embryo usually has a cylindrical form more or less rounded at the extremities, or elongated and fusiform, often oblique. The axis is usually very short compared with the cotyledon, which in general encloses the plumule by its lower portion, and exhibits on one side a small slit which indicates the union of the edges of the vaginal or sheathing portion of the leaf (fig. 50). In grasses, by the enlargement of the embryo in a particular direction, the endosperm is pushed on one side, and thus the embryo comes to lie outside at the base of the endosperm (figs. 22, 51). The lamina of the cotyledon is not developed. Upon the side of the embryo next the endosperm and enveloping it is a large shield-shaped body, termed the *scutellum*. This is an outgrowth from the base of the cotyledon, enveloping more or less the cotyledon and plumule, in some cases, as in maize, completely investing it; in other cases, as in rice, merely sending small prolongations over its anterior face at the apex. By others this scutellum is considered as the true cotyledon, and the sheathing structure covering the plumule is regarded as a ligule or axillary stipule (see GRASSES). In many aquatic monocotyledons (*e.g. Potamogeton*, *Ruppia* and others) there is a much-developed hypocotyl, which forms the greater part of the embryo and acts as a store of nutriment in germination; these are known as *macropodous* embryos. A similar case is that of *Caryocar* among Dicotyledons, where the swollen hypocotyl occupies most of the embryo (fig. 52). In some grasses, as oats and rice, a projection of cellular tissue is seen upon the side of the embryo opposite to the scutellum, that is, on the anterior side. This has been termed the *epiblast*. It is very large in rice. This by some was considered the rudimentary second cotyledon; but is now generally regarded as an outgrowth of the sheath of the true cotyledon.

(A. B. R.)

FRUIT AND FLOWER FARMING. The different sorts of fruits and flowers are dealt with in articles under their own headings, to which reference may be made; and these give the substantial facts as to their cultivation. See also the article Horticulture.

Great Britain

Table I.—*Extent of Orchards in Great Britain in each Year, 1887 to 1901.*

Year.	Acres.	Year.	Acres.	Year.	Acres.
1887	202,234	1892	208,950	1897	224,116
1888	199,178	1893	211,664	1898	226,059
1889	199,897	1894	214,187	1899	228,603
1890	202,305	1895	218,428	1900	232,129
1891	209,996	1896	221,254	1901	234,660

Table II.—*Areas under Orchards in England, Wales and Scotland—Acres.*

Year.	England.	Wales.	Scotland.	Great Britain.
1896	215,642	3677	1935	221,254
1897	218,261	3707	2148	224,116
1898	220,220	3690	2149	226,059
1899	222,712	3666	2225	228,603
1900	226,164	3695	2270	232,129
1901	228,580	3767	2313	234,660
1908	244,430	3577	2290	250,297

The extent of the fruit industry may be gathered from the figures for the acreage of land under cultivation in orchards and small fruit plantations. The Board of Agriculture returns concerning the orchard areas of Great Britain showed a continuous expansion year by year from 199,178 acres in 1888 to 234,660 acres in 1901, as will be learnt from Table I. There was, it is true, an exception in 1892, but the decline in that year is explained by the circumstance that since 1891 the agricultural returns have been collected only from holdings of more than one acre, whereas they were previously obtained from all holdings of a quarter of an acre or more. As there are many holdings of less than an acre in extent upon which fruit is grown, and as fruit is largely raised also in suburban and other gardens which do not come into the returns, it may be taken for granted that the actual extent of land devoted to fruit culture exceeds that which is indicated by the official figures. In the Board of Agriculture returns up to June 1908, 308,000 acres are stated to be devoted to fruit cultivation of all kinds in Great Britain. Table II. shows that the expansion of the orchard area of Great Britain is mainly confined to England, for it has slightly decreased in Wales and Scotland. The acreage officially returned as under orchards is that of arable or grass land which is also used for fruit trees of any kind. Conditions of soil and climate determine the irregular distribution of orchards in Great Britain. The dozen counties which possess the largest extent of orchard land all lie in the south or west of the island. According to the returns for 1908 (excluding small fruit areas) they were the following:— 261

County.	Acres.	County.	Acres.	County.	Acres.
Kent	32,751	Worcester	23,653	Salop	4685
Devon	27,200	Gloucester	20,424	Dorset	4464
Hereford	28,316	Cornwall	5,415	Monmouth	3914
Somerset	25,279	Middlesex	5,300	Wilts	3630

Leaving out of consideration the county of Kent, which grows a greater variety of fruit than any of the others, the counties of Devon, Hereford, Somerset, Worcester and Gloucester have an aggregate orchard area of 124,872 acres. These five counties of the west and south-west of England—constituting in one continuous area what is essentially the cider country of Great Britain—embrace therefore rather less than half of the entire orchard area of the island, while Salop, Monmouth and Wilts have about 300 less than they had a few years ago. Five English counties have less than 1000 acres each of orchards, namely, the county of London, and the northern counties of Cumberland, Westmorland, Northumberland and Durham. Rutland has just over 100 acres. The largest orchard areas in Wales are in the two counties adjoining Hereford—Brecon with 1136 acres and Radnor with 727 acres; at the other extreme is Anglesey, with a decreasing orchard area of only 22 acres. Of the Scottish counties, Lanark takes the lead with 1285 acres, Perth, Stirling and Haddington following with 684 and 129 acres respectively. Ayr and Midlothian are the only other counties possessing 100 acres or more of orchards, whilst Kincardine, Orkney and Shetland return no orchard area, and Banff, Bute, Kinross, Nairn, Peebles, Sutherland and Wigtown return less than 10 acres each. It may be added that in 1908 Jersey returned 1090 acres of orchards, Guernsey, &c., 144 acres, and the Isle of Man, 121 acres; the two last-named places showing a decline as compared with eight years previously.

Outside the cider counties proper of England, the counties in which orchards for commercial fruit-growing have increased considerably in recent years include Berks, Buckingham, Cambridge, Essex, Lincoln, Middlesex, Monmouth, Norfolk, Oxford, Salop, Sussex, Warwick and Wilts. Apples are the principal fruit grown in the western and south-western counties, pears also being fairly common. In parts of Gloucestershire, however, and in the Evesham and Pershore districts of Worcestershire, plum orchards exist. Plums are almost as largely grown as apples in Cambridgeshire. Large quantities of apples, plums, damsons, cherries, and a fair quantity of pears are grown for the market in Kent, whilst apples, plums and pears predominate in Middlesex. In many counties damsons are cultivated around fruit plantations to shelter the latter from the wind.

Of small fruit (currants, gooseberries, strawberries, raspberries, &c.) no return was made of the acreage previous to 1888, in which year it was given as 36,724 acres for Great Britain. In 1889 it rose to 41,933 acres.

Later figures are shown in Table III. It will be observed that, owing to corrections made in the enumeration in 1897, a considerable reduction in the area is recorded for that year, and presumably the error then discovered existed in all the preceding returns. The returns for 1907 gave the acreage of small fruit as 82,175 acres, and in 1908 at 84,880 acres—an area more than double that of 1889.

Table III.—*Areas of Small Fruit in Great Britain.*

Year.	Acres.	Year.	Acres.	Year.	Acres.
1890	46,234	1894	68,415	1898	69,753
1891	58,704	1895	74,547	1899	71,526
1892	62,148	1896	76,245	1900	73,780
1893	65,487	1897	69,792	1901	74,999

Table IV.—*Areas under Small Fruit in England, Wales and Scotland—Acres.*

Year.	England.	Wales.	Scotland.	Great Britain.
1898	63,438	1044	5271	69,753
1899	64,867	1106	5553	71,526
1900	66,749	1109	5922	73,780
1901	67,828	1092	6079	74,999
1908	75,750	1200	7930	84,880

There has undoubtedly been a considerable expansion, rather than a contraction, of small fruit plantations since 1896. The acreage of small fruit in Great Britain is about one-third that of the orchards. As may be seen in Table IV., it is mainly confined to England, though Scotland has over 4000 more acres of small fruit than of orchards. About one-third of the area of small fruit in England belongs to Kent alone, that county having returned 24,137 acres in 1908. Cambridge now ranks next with 6878 acres, followed by Norfolk with 5876 acres, Worcestershire with 4852 acres, Middlesex with 4163 acres, Hants with 3320 acres and Essex with 2150 acres. It should be remarked that between 1900 and 1908 Cambridgeshire had almost doubled its area of small fruits, from 3740 to 6878 acres; whilst both Norfolk and Worcestershire in 1908 had larger areas devoted to small fruits than Middlesex—in which county there had been a decrease of about 400 acres during the same period. The largest county area of small fruit in Wales is 806 acres in Denbighshire, and in Scotland 2791 acres in Perthshire, 2259 acres in Lanarkshire, followed by 412 acres in Forfarshire. The only counties in Great Britain which make no return under the head of small fruit are Orkney and Shetland; and Sutherland only gives 2½ acres. It is hardly necessary to say that considerable areas of small fruit, in kitchen gardens and elsewhere, find no place in the official returns, which, however, include small fruit grown between and under orchard trees.

Gooseberries are largely grown in most small fruit districts. Currants are less widely cultivated, but the red currant is more extensively grown than the black, the latter having suffered seriously from the ravages of the black currant mite. Kent is the great centre for raspberries and for strawberries, though, in addition, the latter fruit is largely grown in Cambridgeshire (2411 acres), Hampshire (2327 acres), Norfolk (2067 acres) and Worcestershire (1273 acres). Essex, Lincolnshire, Cheshire, Cornwall and Middlesex each has more than 262
500 acres devoted to strawberry cultivation.

The following statement from returns for 1908 shows the area under different kinds of fruit in 1907 and 1908 in Great Britain, and also whether there had been an increase or decrease:

	1907.	1908.	Increase or Decrease.
	Acres.	Acres.	Acres.
Small Fruit—			
Strawberries	27,827	28,815	+ 988
Raspberries	8,878	9,323	+ 445
Currants and Gooseberries	25,590	26,241	+ 651
Other kinds	19,880	20,501	+ 621
	82,175	84,880	+2705
Orchards—			
Apples	172,643	172,751	+ 108
Pears	8,911	9,604	+ 693
Cherries	12,027	11,868	− 159
Plums	14,901	15,683	+ 782
Other kinds	41,694	40,391	−1303
	250,176	250,297	+ 121

It appears from the Board of Agriculture returns that 27,433 acres of small fruit was grown in orchards, so that the total extent of land under fruit cultivation in Great Britain at the end of 1908 was about 308,000 acres.

There are no official returns as to the acreage devoted to orchard cultivation in Ireland. The figures relating to small fruit, moreover, extend back only to 1899, when the area under this head was returned as 4809 acres, which became 4359 acres in 1900 and 4877 acres in 1901. In most parts of the country there are districts favourable to the culture of small fruits, such as strawberries, raspberries, gooseberries and currants, and of top fruits, such as apples, pears, plums and damsons. The only localities largely identified with fruit culture as an industry are the Drogheda district and the Armagh district. In the former all the kinds named are grown except strawberries, the speciality being raspberries, which are marketed in Dublin, Belfast and Liverpool. In the Armagh district, again, all the kinds named are grown, but in this case strawberries are the speciality, the markets utilized being Richhill, Belfast, and those in Scotland. In the Drogheda district the grower bears the cost of picking, packing and shipping, but he cannot estimate his net returns until his fruit is on the market. Around Armagh the Scottish system prevails—that is, the fruit is sold while growing, the buyer being responsible for the picking and marketing.

The amount of fruit imported into the United Kingdom has such an important bearing on the possibilities of the industry that the following figures also may be useful:

> The quantities of apples, pears, plums, cherries and grapes imported in the raw condition into the United Kingdom in each year, 1892 to 1901, are shown in Table V. Previous to 1892 apples only were separately enumerated. Up to 1899 inclusive the quantities were given in bushels, but in 1900 a change was made to hundred-weights. This renders the quantities in that and subsequent years not directly comparable with those in earlier years, but the comparison of the values, which are also given in the table, continues to hold good. The figures for 1908 have been added to show the increase that had taken place. In some years the value of imported apples exceeds the aggregate value of the pears, plums, cherries and grapes imported. The extreme values for apples shown in the table are £844,000 in 1893 and £2,079,000 in 1908. Grapes rank next to apples in point of value, and over the seventeen years the amount ranged between £394,000 in 1892 and £728,000 in 1908. On the average, the annual outlay on imported pears is slightly in excess of that on plums. The extremes shown are £167,000 in 1895 and £515,000 in 1908. In the case of plums, the smallest outlay tabulated is £166,000 in 1895, whilst the largest is £498,000 in 1897. The amounts expended upon imported cherries varied between £96,000 in 1895 and £308,000 in 1900. In 1900 apricots and peaches, imported raw, previously included with raw plums, were for the first time separately enumerated, the import into the United Kingdom for that year amounting to 13,689 cwt., valued at £25,846; in 1901 the quantity was 13,463 cwt. and the value £32,350. The latter rose in 1908 to £60,000. In 1900, also, currants, gooseberries and strawberries, hitherto included in unenumerated raw fruit, were likewise for the first time separately returned. Of raw currants the import was 64,462 cwt., valued at £87,170 (1908, £121,850); of raw gooseberries 26,045 cwt., valued at £14,626 (1908, £25,520); and of raw strawberries, 52,225 cwt., valued at £85,949. In 1907 only 44,000 cwt. of strawberries were imported. In 1901 the quantities and values were respectively—currants, 70,402 cwt., £75,308; gooseberries, 21,735 cwt., £11,420; strawberries, 38,604 cwt., £51,290. Up to 1899 the imports of tomatoes were included amongst unenumerated raw vegetables, so that the quantity was not separately ascertainable. For 1900 the import of tomatoes was 833,032 cwt., valued at £792,339, which is equivalent to a fraction under 2½d. per ℔. For 1901 the quantity was 793,991 cwt., and the value £734,051; for 1906, there were 1,124,700 cwt., valued at £953,475; for 1907, 1,135,499 cwt., valued at £1,020,805; and for 1908, 1,160,283 cwt., valued at £955,983.

Table V.—*Imports of Raw Apples, Pears, Plums, Cherries and Grapes into the United Kingdom, 1892 to 1901. Quantities in Thousands of Bushels (thousands of cwt. in 1900 and 1901). Values in Thousands of Pounds Sterling.*

Year.	Quantities.				
	Apples.	Pears.	Plums.	Cherries.	Grapes.
1892	4515	637	413	217	762
1893	3460	915	777	346	979
1894	4969	1310	777	311	833
1895	3292	407	401	196	865
1896	6177	483	560	219	883
1897	4200	1052	1044	312	994
1898	3459	492	922	402	1136
1899	3861	572	558	281	1158
1900	2129*	477*	423*	243*	593*
1901	1830*	349*	264*	213*	680*
	Values.				
1892	1354	297	200	135	394
1893	844	347	332	195	530
1894	1389	411	302	167	470
1895	960	167	166	96	487
1896	1582	207	242	106	443
1897	1187	378	498	178	495
1898	1108	222	435	231	550
1899	1186	266	294	154	588
1900	1225	367	393	308	595
1901	1183	296	244	214	695
1908	2079	515	428	235	728

* Thousands of cwts.

> In 1908 the outlay of the United Kingdom upon imported raw fruits, such as can easily be produced at home, was £4,195,654, made up as follows:

Apples	£2,079,703	Plums	£428,966
Grapes	728,026	Currants	121,852

Pears	515,914	Apricots and peaches	60,141
Cherries	235,523	Gooseberries	25,529

In addition about £280,000 was spent upon “unenumerated” raw fruit, and £560,000 on nuts other than almonds “used as fruit,” which would include walnuts and filberts, both produced at home. It is certain, therefore, that the expenditure on imported fruits, such as are grown within the limits of the United Kingdom, exceeds four millions sterling per annum. The remainder of the outlay on imported fruit in 1908, amounting to over £5,000,000, was made up of £2,269,651 for oranges, £471,713 for lemons, £1,769,249 for bananas, and £560,301 for almond-nuts; these cannot be grown on an industrial scale in the British Isles.

It may be interesting to note the source of some of these imported fruits. The United States and Canada send most of the apples, the quantity for 1907 being 1,413,000 cwt. and 1,588,000 cwt. respectively, while Australia contributes 280,000 cwt. Plums come chiefly from France (200,000 cwt.), followed with 38,000 cwt. from Germany and 28,000 cwt. from the Netherlands. Pears are imported chiefly from France (204,000 cwt.) and Belgium (176,000); but the Netherlands send 52,000 cwt., and the United States 24,000 cwt. The great bulk of imported tomatoes comes from the Canary Islands, the quantity in 1907 being 604,692 cwt. The Channel Islands also sent 223,800 cwt., France 115,500 cwt., Spain 169,000 cwt., and Portugal a long way behind with 11,700 cwt. Most of the strawberries imported come from France (33,800 cwt.) and the Netherlands (10,300 cwt.).

Fruit-growing in Kent.—Kent is by far the largest fruit-growing county in England. For centuries that county has been famous for its fruit, and appears to have been the centre for the distribution of trees and grafts throughout the country. The cultivation of fruit land upon farms in many parts of Kent has always been an 263 important feature in its agriculture. An excellent description of this noteworthy characteristic of Kentish farming is contained in a comprehensive paper on the agriculture of Kent by Mr Charles Whitehead,[1] whose remarks, with various additions and modifications, are here reproduced.

Where the conditions are favourable, especially in East and Mid Kent, there is a considerable acreage of fruit land attached to each farm, planted with cherry, apple, pear, plum and damson trees, and with bush fruits, or soft fruits as they are sometimes called, including gooseberries, currants, raspberries, either with or without standard trees, and strawberries, and filberts and cob-nuts in Mid Kent. This acreage has largely increased, and will no doubt continue to increase, as, on the whole, fruit-growing has been profitable and has materially benefited those fortunate enough to have fruit land on their farms. There are also cultivators who grow nothing but fruit. These are principally in the district of East Kent, between Rochester and Canterbury, and in the district of Mid Kent near London, and they manage their fruit land, as a rule, better than farmers, as they give their undivided attention to it and have more technical knowledge. But there has been great improvement of late in the management of fruit land, especially of cherry and apple orchards, the grass of which is fed off by animals having corn or cake, or the land is well manured. Apple trees are grease-banded and sprayed systematically by advanced fruit-growers to prevent or check the attacks of destructive insects. Far more attention is being paid to the selection of varieties of apples and pears having colour, size, flavour, keeping qualities, and other attributes to meet the tastes of the public, and to compete with the beautiful fruit that comes from the United States and Canada.

Of the various kinds of apples at present grown in Kent mention should be made of Mr Gladstone, Beauty of Bath, Devonshire Quarrenden, Lady Sudely, Yellow Ingestre and Worcester Pearmain. These are dessert apples ready to pick in August and September, and are not stored. For storing, King of the Pippins, Cox’s Orange Pippin (the best dessert apple in existence), Cox’s Pomona, Duchess, Favourite, Gascoyne’s Scarlet Seedling, Court Pendu Plat, Baumann’s Red Reinette, Allington Pippin, Duke of Devonshire and Blenheim Orange. Among kitchen apples for selling straight from the trees the most usually planted are Lord Grosvenor, Lord Suffield, Keswick Codlin, Early Julian, Eclinville Seedling, Pott’s Seedling, Early Rivers, Grenadier, Golden Spire, Stirling Castle and Domino. For storing, the cooking sorts favoured now are Stone’s or Loddington, Warner’s King, Wellington, Lord Derby, Queen Caroline, Tower of Glamis, Winter Queening, Lucombe’s Seedling, Bismarck, Bramley’s Seedling, Golden Noble and Lane’s Prince Albert. Almost all these will flourish equally as standards, pyramids and bushes. Among pears are Hessle, Clapp’s Favourite, William’s Bon Chrétien, Beurré de Capiaumont, Fertility, Beurré Riche, Chissel, Beurré Clairgeau, Louise Bonne of Jersey, Doyenne du Comice and Vicar of Winkfield. Among plums, Rivers’s Early Prolific, Tsar, Belgian Purple, Black Diamond, Kentish Bush Plum, Pond’s Seedling, Magnum Bonum and Victoria are mainly cultivated. The damson known as Farleigh Prolific, or Crittenden’s, is most extensively grown throughout the county, and usually yields large crops, which make good prices. As a case in point, purchasers were offering to contract for quantities of this damson at £20 per ton in May of 1899, as the prospects of the yield were unsatisfactory. On the other hand, in one year recently when the crop was abnormally abundant, some of the fruit barely paid the expenses of sending to market. The varieties of cherries most frequently grown are Governor Wood, Knight’s Early Black, Frogmore Blackheart, Black Eagle, Waterloo, Amberheart, Bigarreau, Napoleon Bigarreau and Turk. A variety of cherry known as the Kentish cherry, of a light red colour and fine subacid flavour, is much grown in Kent for drying and cooking purposes. Another cherry, similar in colour and quality, which comes rather late, known as the Flemish, is also extensively cultivated, as well as the very dark red large Morello, used for making cherry brandy. These three varieties are grown extensively as pyramids, and the last-named also on walls and sides of buildings. Sometimes the cherry crop is sold by auction to dealers, who pick, pack and consign the fruit to market. Large prices are often made, as much as £80 per acre being not uncommon. The crop on a large cherry orchard in Mid Kent has been sold for more than £100 per acre.

Where old standard trees have been long neglected and have become overgrown by mosses and lichens, the attempts made to improve them seldom succeed. The introduction of bush fruit trees dwarfed by grafting on the Paradise stock has been of much advantage to fruit cultivators, as they come into bearing in two or three years, and are more easily cultivated, pruned, sprayed and picked than standards. Many plantations of these bush trees have been formed in Kent of apples, pears and plums. Half standards and pyramids have also been planted of these fruits, as well as of cherries. Bushes of gooseberries and currants, and clumps or stools of raspberry canes, have been planted to a great extent in many parts of the East and Mid divisions of Kent, but not much in the Weald, where apples are principally grown. Sometimes fruit bushes are put in alternate rows with bush of standard trees of apple, pear, plum or damson, or they are planted by themselves. The distances apart for planting are generally for cherry and apple trees on grass 30 ft. by 30 ft.; for standard apples and pear trees from 20 ft. to 24 ft. upon arable land, with bush fruit, as gooseberries and currants, under them.

These are set 6 ft. by 6 ft. apart, and 5 ft. by 2 ft. for raspberries, and strawberries 2 ft. 6 in. to 3 ft. by 1 ft. 6 in. to 1 ft. 3 in. apart. On some fruit farms bush or dwarf trees—apples, pears, plums—are planted alone, at distances varying from 8 ft. to 10 ft. apart, giving from 485 to 680 bush trees per acre, nothing being grown between them except perhaps strawberries or vegetables during the first two or three years. It is believed that this is the best way of ensuring fruit of high quality and colour. Another arrangement consists in putting standard apple or pear trees 30 ft. apart (48 trees per acre), and setting bush trees of apples or pears 15 ft. apart between them; these latter come quickly into bearing, and are removed when the standards are fully grown. Occasionally gooseberry or currant bushes, or raspberry canes or strawberry plants, are set between the bush trees, and taken away directly they interfere with the growth of these. Half standard apple or plum trees are set triangularly 15 ft. apart, and strawberry plants at a distance of 1½ ft. from plant to plant and 2½ ft. from row to row. Or currant or gooseberry bushes are set between the half standards, and strawberry plants between these.

These systems involve high farming. The manures used are London manure, where hops are not grown, and bone meal, super-phosphate, rags, shoddy, wool-waste, fish refuse, nitrate of soda, kainit and sulphate of ammonia. Where hops are grown the London manure is wanted for them. Fruit plantations are always dug by hand with the Kent spud. Fruit land is never ploughed, as in the United States and Canada. The soil is levelled down with the “Canterbury” hoe, and then the plantations are kept free from weeds with the ordinary draw or “plate” hoe. The best fruit farmers spray fruit trees regularly in the early spring, and continue until the blossoms come out, with quassia and soft soap and paraffin emulsions, and a very few with Paris green only, where there is no under fruit, in order to prevent and check the constant attacks of the various caterpillars and other insect pests. This is a costly and laborious process, but it pays well, as a rule. The fallacy that fruit trees on grass land require no manure, and that the grass may be allowed to grow up to their trunks without any harm, is exploding, and many fruit farmers are well manuring their grass orchards and removing the grass for some distance round the stems, particularly where the trees are young.

Strawberries are produced in enormous quantities in the northern part of the Mid Kent district round the Crays, and from thence to Orpington; also near Sandwich, and to some extent near Maidstone. Raspberry canes have been extensively put in during the last few years, and in some seasons yield good profits. There is a very great and growing demand for all soft fruits for jam-making, and prices are fairly good, taking an average of years, notwithstanding the heavy importations from France, Belgium, Holland, Spain and Italy. The extraordinary increase in the national demand for jam and other fruit preserves has been of great benefit to Kent fruit producers. The cheapness of duty-free sugar, as compared with sugar paying duty in the United States and other large fruit-producing countries, afforded one of the very few advantages possessed by British cultivators, but the reimposition of the sugar duty in the United Kingdom in 1901 has modified the position in this respect. Jam factories were established in several parts of Kent about 1889 or 1890, but most of them collapsed either from want of capital or from bad management. There are still a few remaining, principally in connexion with large fruit farms. One of these is at Swanley, whose energetic owners farm nearly 2000 acres of fruit land in Kent. The fruit grown by them that will not make satisfactory prices in a fresh raw state is made into jam, or if time presses it is first made into pulp, and kept until the opportunity comes for making it into jam. In this factory there are fifteen steam-jacketed vats in one row, and six others for candied peel. A season’s output on a recent occasion comprised about 3500 tons of jam, 850 tons of candied peel and 750 gross (108,000 bottles) of bottled fruit. A great deal of the fruit preserved is purchased, whilst much of that grown on the farms is sold. A strigging machine is employed, which does as much work as fifty women in taking currants off their strigs or stalks. Black currant pulp is stored in casks till winter, when there is time to convert it into jam. Strawberries cannot be pulped to advantage, but it is otherwise with raspberries, the pulp of which is largely made. Apricots for jam are obtained chiefly from France and Spain. There is another flourishing factory near Sittingbourne worked on the same lines. It is very advantageous to fruit farmers to have jam factories in connexion with their farms or to have them near, as they can thoroughly grade their fruit, and send only the best to market, thus ensuring a high reputation for its quality. Carriage is saved, which is a serious charge, though railway rates from Kent to the great manufacturing towns and to Scotland are very much less proportionally than those to London, and consequently Kent growers send increasing quantities to these distant markets, where prices are better, not being so directly interfered with by imported fruit, which generally finds its way to London.

Kentish fruit-growers are becoming more particular in picking, grading, packing and storing fruit, as well as 264
in marketing it. A larger quantity of fruit is now carefully stored, and sent to selected markets as it ripens, or when there is an ascertained demand, as it is found that if it is consigned to market direct from the trees there must frequently be forced sales and competition with foreign fruit that is fully matured and in good order. It was customary formerly for Kentish growers to consign all their fruit to the London markets; now a good deal of it is sent to Manchester, Birmingham, Liverpool, Sheffield, Newcastle and other large cities. Some is sent even to Edinburgh and Glasgow. Many large growers send no fruit to London now. It is by no means uncommon for growers to sell their fruit crops on the trees or bushes by auction or private treaty, or to contract to supply a stipulated quantity of specified fruit, say of currants, raspberries or strawberries, to jam manufacturers. There is a considerable quantity of fruit, such as grapes, peaches, nectarines, grown under glass, and this kind of culture tends to increase.

Filberts and cob-nuts are a special product of Kent, in the neighbourhood of Maidstone principally, and upon the Ragstone soils, certain conditions of soil and situation being essential for their profitable production. A part of the filbert and cob-nut crop is picked green in September, as they do well for dessert, though their kernels are not large or firm, and it pays to sell them green, as they weigh more heavily. One grower in Mid Kent has 100 acres of nuts, and has grown 100 tons in a good year. The average price of late years has been about 5d. per ℔, which would make the gross return of the 100 acres amount to £4660. Kentish filberts have long been proverbial for their excellence. Cobs are larger and look better for dessert, though their flavour is not so fine. They are better croppers, and are now usually planted. This cultivation is not much extending, as it is very long before the trees come into full bearing. The London market is supplied entirely with these nuts from Kent, and there is some demand in America for them. Filbert and cob trees are most closely pruned. All the year’s growth is cut away except the very finest young wood, which the trained eye of the tree-cutter sees at a glance is blossom-bearing. The trees are kept from 5½ to 7 ft. high upon stems from 1½ to 2 ft. high, and are trained so as to form a cup of from 7 to 8 ft. in diameter.

There seems no reason to expect any decrease in the acreage of fruit land in Kent, and if the improvement in the selection of varieties and in the general management continues it will yet pay. A hundred years ago every one was grubbing fruit land in order that hops might be planted, and for this many acres of splendid cherry orchards were sacrificed. Now the disposition is to grub hop plants and substitute apples, plums, or small fruit or cherry trees.

Fruit-growing in other Districts.—The large fruit plantations in the vicinity of London are to be found mostly in the valley of the Thames, around such centres as Brentford, Isleworth, Twickenham, Heston, Hounslow, Cranford and Southall. All varieties of orchard trees, but mostly apples, pears, and plums and small fruit, are grown in these districts, the nearness of which to the metropolitan fruit market at Covent Garden is of course an advantage. Some of the orchards are old, and are not managed on modern principles. They contain, moreover, varieties of fruit many of which are out of date and would not be employed in establishing new plantations. In the better-managed grounds the antiquated varieties have been removed, and their places taken by newer and more approved types. In addition to apples, pears, plums, damsons, cherries and quinces as top fruit, currants, gooseberries and raspberries are grown as bottom fruit. Strawberries are extensively grown in some of the localities, and in favourable seasons outdoor tomatoes are ripened and marketed.

Fruit is extensively grown in Cambridgeshire and adjacent counties in the east of England. A leading centre is Cottenham, where the Lower Greensand crops out and furnishes one of the best of soils for fruit-culture. In Cottenham about a thousand acres are devoted to fruit, and nearly the same acreage to asparagus, which is, however, giving place to fruit. Currants, gooseberries and strawberries are the most largely grown, apples, plums and raspberries following. Of varieties of plums the Victoria is first in favour, and then Rivers's Early Prolific, Tsar and Gisborne. London is the chief market, as it receives about half the fruit sent away, whilst a considerable quantity goes to Manchester, and some is sent to a neighbouring jam factory at Histon, where also a moderate acreage of fruit is grown. Another fruit-growing centre in Cambridgeshire is at Willingham, where—besides plums, gooseberries and raspberries—outdoor tomatoes are a feature. Greengages are largely grown near Cambridge. Wisbech is the centre of an extensive fruit district, situated partly in Cambridgeshire and partly in Norfolk. Gooseberries, strawberries and raspberries are largely grown, and as many as 80 tons of the first-named fruit have been sent away from Wisbech station in a single day. In the fruit-growing localities of Huntingdonshire apples, plums and gooseberries are the most extensively grown, but pears, greengages, cherries, currants, strawberries and raspberries are also cultivated. As illustrating variations in price, it may be mentioned that about the year 1880 the lowest price for gooseberries was £10 per ton, whereas it has since been down to £4. Huntingdonshire fruit is sent chiefly to Yorkshire, Scotland and South Wales, but railway freights are high.

Essex affords a good example of successful fruit-farming at Tiptree Heath, near Kelvedon, where under one management about 260 acres out of a total of 360 are under fruit. The soil, a stiff loam, grows strawberries to perfection, and 165 acres are allotted to this fruit. The other principal crops are 43 acres of raspberries and 30 acres of black currants, besides which there are small areas of red currants, gooseberries, plums, damsons, greengages, cherries, apples, quinces and blackberries. The variety of strawberry known as the Small Scarlet is a speciality here, and it occupies 55 acres, as it makes the best of jam. The Paxton, Royal Sovereign and Noble varieties are also grown. Strawberries stand for six or seven years on this farm, and begin to yield well when two years old. A jam factory is worked in conjunction with the fruit farm. Pulp is not made except when there is a glut of fruit. Perishable fruit intended for whole-fruit preserves is never held over after it is gathered. The picking of strawberries begins at 4 A.M., and the first lot is made into jam by 6 A.M.

Hampshire, like Cambridgeshire and Norfolk, are the only counties in which the area of small fruit exceeds that of orchards. The returns for 1908 show that Hampshire had 3320 acres of small fruit to 2236 acres of orchards; Cambridge had 6878 acres of small fruit to 5221 of orchards; and Norfolk had 5876 acres of small fruit against 5188 acres of orchards. Compared with twenty years previously, the acreage of small fruit had trebled. This is largely due in Hampshire to the extension of strawberry culture in the Southampton district, where the industry is in the hands of many small growers, few of whom cultivate more than 20 acres each. Sarisbury and Botley are the leading parishes in which the business is carried on. Most of the strawberry holdings are from half an acre to 5 acres in extent, a few are from 5 to 10 acres, fewer still from 10 to 20 acres and only half-a-dozen over that limit. Runners from one-year plants are used for planting, being found more fruitful than those from older plants. Peat-moss manure from London stables is much used, but artificial manures are also employed with good results. Shortly after flowering the plants are bedded down with straw at the rate of about 25 cwt. per acre. Picking begins some ten days earlier than in Kent, at a date between 1st June and 15th June. The first week's gathering is sent mostly to London, but subsequently the greater part of the fruit goes to the Midlands and to Scotland and Ireland.

In recent years fruit-growing has much increased in South Worcestershire, in the vicinity of Evesham and Pershore. Hand-lights are freely used in the market gardens of this district for the protection of cucumbers and vegetable marrows, besides which tomatoes are extensively grown out of doors. At one time the egg plum and the Worcester damson were the chief fruit crops, apples and cherries ranking next, pears being grown to only a moderate extent. According to the 1908 returns, however, apples come first, plums second, pears third and cherries fourth. In a prolific season a single tree of the Damascene or Worcester damson will yield from 400 to 500 ℔ of fruit. There is a tendency to grow plum trees in the bush shape, as they are less liable than standards to injury from wind. The manures used include soot, fish guano, blood manure and phosphates—basic slag amongst the last-named. In the Pershore district, where there is a jam factory, plums are the chief tree fruit, whilst most of the orchard apples and pears are grown for cider and perry. Gooseberries are a feature, as are also strawberries, red and black currants and a few white, but raspberries are little grown. The soil, a strong or medium loam of fair depth, resting on clay, is so well adapted to plums that trees live for fifty years. In order to check the ravages of the winter moth, plum and apple trees are grease-banded at the beginning of October and again at the end of March. The trees are also sprayed when necessary with insecticidal solutions. Pruning is done in the autumn. An approved distance apart at which to grow plum trees is 12 ft. by 12 ft. In the Earl of Coventry's fruit plantation, 40 acres in extent, at Croome Court, plums and apples are planted alternately, the bottom fruit being black currants, which are less liable to injury from birds than are red currants or gooseberries. Details concerning the methods of cultivation of fruit and flowers in various parts of England, the varieties commonly grown, the expenditure involved, and allied matters, will be found in Mr W.E. Bear's papers in the *Journal of the Royal Agricultural Society* in 1898 and 1899.

Apart altogether from market gardening and commercial fruit-growing, it must be borne in mind that an enormous business is done in the raising of young fruit-trees every year. Hundreds of thousands of apples, pears, plums, cherries, peaches, nectarines and apricots are budded or grafted each year on suitable stocks. They are trained in various ways, and are usually fit for sale the third year. These young trees replace old ones in private and commercial gardens, and are also used to establish new plantations in different parts of the kingdom.

The Woburn Experimental Fruit Farm.—The establishment in 1894 of the experimental fruit farm at Ridgmont, near Woburn, Beds, has exercised a healthy influence upon the progress and development of fruit-farming in England. The farm was founded and carried on by the public-spirited enterprise of the Duke of Bedford and Mr Spencer U. Pickering, the latter acting as director. The main object of the experimental station was "to ascertain facts relative to the culture of fruit, and to increase our knowledge of, and to improve our practice in, this industry." The farm is 20 acres in extent, and occupies a field which up to June 1894 had been used as arable land for the ordinary rotation of farm crops. The soil is a sandy loam 9 or 10 in. deep, resting on a bed of Oxford Clay. Although it contains a large proportion of sand, the land would generally be termed very heavy, and the water often used to stand on it in places for weeks together in a wet season. The tillage to which the ground was subjected for the purposes of the fruit farm much improved its character, and in dry weather it presents as good a tilth as could be desired. Chemical analyses of the soil from different parts of the field show such wide differences that it is admitted to be by no means an ideal one for experimental purposes. Without entering upon further details, it may be useful to give a summary of the chief results obtained. 265

Apples have been grown and treated in a variety of ways, but of the different methods of treatment careless planting, coupled with subsequent neglect, has given the most adverse results, the crop of fruit being not 5% of that from trees grown normally. Of the separate deleterious items constituting total neglect, by far the most effective was the growth of weeds on the surface; careless planting, absence of manure, and the omission of trenching all had comparatively little influence on the results. A set of trees that had been carelessly planted and neglected, but subsequently tended in the early part of 1896, were in the autumn of that year only 10% behind their normally-treated neighbours, thus demonstrating that the response to proper attention is prompt. The growth of grass around young apple trees produced a very striking effect, the injury being much greater than that due to weeds. It is possible, however, that in wet years the ill-effects of both grass and weeds would be less than in dry seasons. Nevertheless, the grass-grown trees, after five years, were scarcely bigger than when planted, and the actual increase in weight which they showed during that time was about eighteen times smaller than in the case of similar trees in tilled ground. It is believed that one of the main causes of the ill-effects is the large increase in the evaporation of water from the soil which is known to be produced by grass, the trees being thereby made to suffer from drought, with constant deprivation of other nourishment as well. That grass growing round young apple trees is deleterious was a circumstance known to many horticulturists, but the extent to which it interferes with the development of the trees had never before been realized. Thousands of pounds are annually thrown away in England through want of knowledge of this fact. Yet trees will flourish in grass under certain conditions. Whether the dominant factor is the age (or size) of the tree has been investigated by grassing over trees which have hitherto been in the open ground, and the results appear to indicate that the grass is as deleterious to the older trees as it was to the younger ones. Again, it appears to have been demonstrated that young apple trees, at all events in certain soils, require but little or no manure in the early stages of their existence, so that in this case also large sums must be annually wasted upon manurial dressings which produce no effects. The experiments have dealt with dwarf trees of Bramley, Cox and Potts, six trees of each variety constituting one investigation. Some of the experiments were repeated with Stirling Castle, and others with standard trees of Bramley, Cox and Lane's Prince Albert. All were planted in 1894-1895, the dwarfs being then three years old and the standards four. In each experiment the "normal" treatment is altered in some one particular, this normal treatment consisting of planting the trees carefully in trenched ground, and subsequently keeping the surface clean; cutting back after planting, pruning moderately in autumn, and shortening the growths when it appeared necessary in summer; giving in autumn a dressing of mixed mineral manures, and in February one of nitrate of soda, this dressing being probably equivalent to one of 12 tons of dung per acre. In the experiments on branch treatment, the bad effects of omitting to cut the trees back on planting, or to prune them subsequently, is evident chiefly in the straggling and bad shape of the resulting trees, but such trees also are not so vigorous as they should be. The quantity of fruit borne, however, is in excess of the average. The check on the vigour and growth of a tree by cutting or injuring its roots is in marked contrast with the effects of a similar interference with the branches. Trees which had been root-pruned each year were in 1898 little more than half as big as the normal trees, whilst those root-pruned every second year were about two-thirds as big as the normal. The crops borne by these trees were nevertheless heavy in proportion to the size of the trees. Such frequent root-pruning is not, of course, a practice which should be adopted. It was found that trees which had been carefully lifted every other year and replanted at once experienced no ill-effects from the operation; but in a case where the trees after being lifted had been left in a shed for three days before replanting—which would reproduce to a certain extent the conditions experienced when trees are sent out from a nursery—material injury was suffered, these trees after four years being 28% smaller than similar ones which had not been replanted. Sets of trees planted respectively in November, January and March have, on the whole, shown nothing in favour of any of these different times for planting purposes. Some doubt is thrown on the accepted view that there is a tendency, at any rate with young apple and pear trees, to fruit in alternate seasons.

Strawberries of eighty-five different varieties have been experimented with, each variety being represented in 1900 by plants of five different ages, from one to five years. In 1896 and 1898 the crops of fruit were about twice as heavy as in 1897 and 1899, but it has not been found possible to correlate these variations with the meteorological records of the several seasons. Taking the average of all the varieties, the relative weights of crop per plant, when these are compared with the two-year-old plants in the same season, are, for the five ages of one to five years, 31, 100, 122, 121 and 134, apparently showing that the bearing power increases rapidly up to two years, less rapidly up to three years, after which age it remains practically constant. The relative average size of the berries shows a deterioration with the age of the plant. The comparative sizes from plants of one to five years old were 115, 100, 96, 91 and 82 respectively. If the money value of the crop is taken to be directly dependent on its total weight, and also on the size of the fruits, the relative values of the crop for the different ages would be 34, 100, 117, 111 and 110, so that, on the Ridgmont ground, strawberry plants could be profitably retained up to five years and probably longer. As regards what may be termed the order of merit of different varieties of strawberries, it appears that even small differences in position and treatment cause large variations, not only in the features of the crop generally, but also in the relative behaviour of the different varieties. The relative cropping power of the varieties under apparently similar conditions may often be expressed by a number five or tenfold as great in one case as in the other. A comparison of the relative behaviour of the same varieties in different seasons is attended by similar variations. The varying sensitiveness of different varieties of strawberry plants to small and undefinable differences in circumstances is indeed one of the most important facts brought to light in the experiments.

Fruit Culture in Ireland.—The following figures have been kindly supplied by the Irish Board of Agriculture, and deal with the acreage under fruit culture in Ireland up to the end of the year 1907.

		Statute Acres.
1. *Orchard Fruit*—		
Apples		5829
Pears		224
Plums		223
Damsons		138
Other kinds		129
		——
	Total	6543
2. *Small Fruit*—		
Currants, black		234
Currants, red and white		159
Gooseberries		675
Raspberries		374
Strawberries		994
Mixed fruit		2470
		——
	Total	4906

It therefore appears that while Ireland grows only about one-thirty-third the quantity of apples that England does, it is nevertheless nearly 5000 acres ahead of Scotland and about 2000 acres ahead of Wales. It grows 41 times fewer pears than England, but still is ahead of Scotland and a long way ahead of Wales in this fruit. There are 70 times fewer plums grown in Ireland than in England, and about the same in Scotland, while Wales does very little indeed. In small fruit Ireland is a long way behind Scotland in the culture of strawberries and raspberries, although with currants and gooseberries it is very close. Considering the climate, and the fact that there are, according to the latest available returns, over 62,000 holdings above 1 acre but not exceeding 5 acres (having a total of 224,000 acres), it is possible fruit culture may become more prevalent than it has been in the past.

The Flower-growing Industry.—During the last two or three decades of the 19th century a very marked increase in flower production occurred in England. Notably was this the case in the neighbourhood of London, where, within a radius of 15 or 20 m., the fruit crops, which had largely taken the place of garden vegetables, were themselves ousted in turn to satisfy the increasing demand for land for flower cultivation. No flower has entered more largely into the development of the industry than the narcissus or daffodil, of which there are now some 600 varieties. Comparatively few of these, however, are grown for market purposes, although all are charming from the amateur point of view. On some flower farms a dozen or more acres are devoted to narcissi alone, the production of bulbs for sale as well as of flowers for market being the object of the growers.

In the London district the country in the Thames valley west of the metropolis is as largely occupied by flower farms as it is by fruit farms—in fact, the cultivation of flowers is commonly associated with that of fruit. In the vicinity of Richmond narcissi are extensively grown, as they also are more to the west in the Long Ditton district, and likewise around Twickenham, Isleworth, Hounslow, Feltham and Hampton. Roses come more into evidence in the neighbourhood of Hounslow, Cranford, Hillingdon and Uxbridge, and in some 266
gardens daffodils and roses occupy alternate rows. In this district also such flowers as herbaceous paeonies, Spanish irises, German irises, Christmas roses, lilies of the valley, chrysanthemums, foxgloves, hollyhocks, wallflowers, carnations, &c., are extensively grown in many market gardens. South of London is the Mitcham country, long noted for its production of lavender. The incessant growth of the lavender plant upon the same land, however, has led to the decline of this industry, which has been largely transferred to districts in the counties of Bedford, Essex and Hertford. At Mitcham, nevertheless, mixed flowers are very largely grown for the supply of the metropolis, and one farm alone has nearly 100 acres under flowers and glass-houses. Chrysanthemums, asters, Iceland poppies, gaillardias, pansies, bedding calceolarias, zonal pelargoniums and other plants are cultivated in immense quantities. At Swanley and Eynsford, in Kent, flowers are extensively cultivated in association with fruit and vegetables. Narcissi, chrysanthemums, violets, carnations, campanulas, roses, pansies, irises, sweet peas, and many other flowers are here raised, and disposed of in the form both of cut flowers and of plants.

The Scilly Isles are important as providing the main source of supply of narcissi to the English markets in the early months of the year. This trade arose almost by accident, for it was about the year 1865 that a box of narcissi sent to Covent Garden Market, London, realized £1; and the knowledge of this fact getting abroad, the farmers of the isles began collecting wild bulbs from the fields in order to cultivate them and increase their stocks. Some ten years, however, elapsed before the industry promised to become remunerative. In 1885 a Bulb and Flower Association was established to promote the industrial growth of flowers. The exports of flowers in that year reached 65 tons, and they steadily increased until 1893, when they amounted to 450 tons. A slight decline followed, but in 1896 the quantity exported was no less than 514 tons. This would represent upwards of 3½ million bunches of flowers, chiefly narcissi and anemones. Rather more than 500 acres are devoted to flower-growing in the isles, by far the greater part of this area being assigned to narcissi, whilst anemones, gladioli, marguerites, arum lilies, Spanish irises, pinks and wallflowers are cultivated on a much smaller scale. The great advantage enjoyed by the Scilly flower-growers is earliness of production, due to climatic causes; the soil, moreover, is well suited to flower culture and there is an abundance of sunshine. The long journey to London is somewhat of a drawback, in regard to both time and freight, but the earliness of the flowers more than compensates for this. Open-air narcissi are usually ready at the beginning of January, and the supply is maintained in different varieties up to the middle or end of May. The narcissus bulbs are usually planted in October, 4 in. by 3 in. apart for the smaller sorts and 6 in. by 4 to 6 in. for the larger. A compost of farmyard manure, seaweed, earth and road scrapings is the usual dressing, but nitrate of soda, guano and bones are also occasionally employed. A better plan, perhaps, is to manure heavily the previous crop, frequently potatoes, no direct manuring then being needed for the bulbs, these not being left in the ground more than two or three years. The expenses of cultivation are heavy, the cost of bulbs alone—of which it requires nearly a quarter of a million of the smaller varieties, or half as many of the largest, to plant an acre—being considerable. The polyanthus varieties of narcissus are likely to continue the most remunerative to the flower-growers of Scilly, as they flourish better in these isles than on the mainland.

In the district around the Wash, in the vicinity of such towns as Wisbech, Spalding and Boston, the industrial culture of bulbs and flowers underwent great expansion in the period between 1880 and 1909. At Wisbech one concern alone has a farm of some 900 acres, devoted chiefly to flowers and fruit, the soil being a deep fine alluvium. Roses are grown here, one field containing upwards of 100,000 trees. Nearly 20 acres are devoted to narcissi, which are grown for the bulbs and also, together with tulips, for cut flowers. Carnations are cultivated both in the field and in pots. Cut flowers are sent out in large quantities, neatly and effectively packed, the parcel post being mainly employed as a means of distribution. In the neighbourhood of Spalding crocuses and snowdrops are less extensively grown than used to be the case. On one farm, however, upwards of 20 acres are devoted to narcissi alone, whilst gladioli, lilies and irises are grown on a smaller scale. Around Boston narcissi are also extensively grown for the market, both bulbs and cut blooms being sold. The bulbs are planted 3 in. apart in rows, the latter being 9 in. apart, and are allowed to stand from two to four years.

The imports of fresh flowers into the United Kingdom were not separately shown prior to 1900. In that year, however, their value amounted to £200,585, in 1901 to £225,011, in 1906 to £233,884, in 1907 to £233,641, and in 1908 to £229,802, so that the trade showed a fairly steady condition. From the monthly totals quoted in Table VI. it would appear that the trade sinks to its minimum dimensions in the four months July to October inclusive, and that after September the business continually expands up to April, subsequent to which contraction again sets in. About one-half of the trade belongs practically to the three months of February, March and April.

Table VI.—*Values of Fresh Flowers imported into the United Kingdom.*

Month.	1906.	1907.	1908.
January	£31,035	£18,545	£29,180
February	34,647	25,541	30,541
March	50,232	42,611	35,185
April	30,809	50,418	42,681
May	22,980	21,767	23,129
June	17,641	18,358	16,904
July	3,386	4,509	3,467
August	1,646	1,539	1,081
September	852	736	953
October	4,481	3,180	4,504
November	17,506	15,763	15,097
December	18,669	30,674	27,080
Total	£233,884	£233,641	£229,802

Hothouse Culture of Fruit and Flowers.—The cultivation of fruit and flowers under glass has increased enormously since about the year 1880, especially in the neighbourhood of London, where large sums of money have been sunk in the erection and equipment of hothouses. In the parish of Cheshunt, Herts, alone there are upwards of 130 acres covered with glass, and between that place on the north and London on the south extensive areas of land are similarly utilized. In Middlesex, in the north, in the districts of Edmonton, Enfield, Ponders End and Finchley, and in the west from Isleworth to Hampton, Feltham, Hillingdon, Sipson and Uxbridge, many crops are now cultivated under glass. At Erith, Swanley, and other places in Kent, as also at Worthing, in Sussex, glass-house culture has much extended. A careful estimate puts the area of industrial hothouses in England at about 1200 acres, but it is probably much more than this. Most of the greenhouses are fixtures, but in some parts of the kingdom structures that move on rails and wheels are used, to enable the ground to be prepared in the open for one crop while another is maturing under glass. The leading products are grapes, tomatoes and cucumbers, the last-named two being true fruits from the botanist’s point of view, though commercially included with vegetables. To these may be added on the same ground dwarf or French beans, and runner or climbing beans. Peaches, nectarines and strawberries are largely grown under glass, and, in private hothouses—from which the produce is used mainly for household consumption, and which are not taken into consideration here—pineapples, figs and other fruit. Conservative estimates indicate the average annual yield of hothouse grapes to be about 12 tons per acre and of tomatoes 20 tons. The greater part of the space in the hothouses is assigned to fruit, but whilst some houses are devoted exclusively to flowers, in others, where fruit is the main object, flowers are forced in considerable quantities in winter and early spring. The flowers grown under glass include tulips, hyacinths, primulas, cyclamens, spiraeas, mignonettes, fuchsias, calceolarias, roses, chrysanthemums, daffodils, arum lilies or callas, liliums, azaleas, eucharises, camellias, stephanotis, tuberoses, bouvardias, gardenias, heaths or ericas, poinsettias, lilies of the valley, zonal pelargoniums, tuberous and fibrous rooted begonias, and many others. There is an increasing demand for foliage hothouse plants, such as ferns, palms, crotons, aspidistras, araucarias, dracaenas, India-rubber plants, aralias, grevilleas, &c. Berried plants like solanums and aucubas also find a ready sale, while the ornamental kinds of asparagus such as *sprengeri* and *plumosus* nanus, are ever in demand for trailing decorations, as well as myrsiphyilum. Special mention must be made of the winter or perpetual flowering carnations which are now grown by hundreds of thousands in all parts of the kingdom for decorative work during the winter season. The converse of forcing plants into early blossom is adopted with such an important crop as lily of the valley. During the summer season the crowns are placed in refrigerators with about 2 degrees of frost, and quantities are taken out as required every week and transferred to the greenhouse to develop. Tomatoes are grown largely in houses exclusively occupied by them, in which case two and sometimes three crops can be gathered in the year. In the Channel Islands, where potatoes grown under glass are lifted in April and May, in order to secure the high prices of the early markets, tomato seedlings are planted out from boxes into the ground as quickly as the potatoes are removed, the tomato planter working only a few rows behind the potato digger. The trade in imported tomatoes is so considerable that home growers are well justified in their endeavours to meet the demand more fully with native produce, whether raised under glass or in the open. Tomatoes were not separately enumerated in the imports previous to 1900. It has already been stated that in 1900 the raw tomatoes imported amounted to 833,032 cwt., valued at £792,339, and in 1901 to 793,991 cwt., valued at £734,051. From the monthly quantities given in Table VII., it would appear that the imports are largest in June, July and August, about one-half of the year’s total arriving during those three months. It is too early in June and July for home-grown outdoor tomatoes to enter into competition with the imported product, but home-grown hothouse tomatoes should be qualified to challenge this trade.

Table VII.—*Quantities of Tomatoes imported into the United Kingdom.*

Month.	1906.	1907.	1908.
January	61,940	56,022	73,409
February	58,187	58,289	69,350
March	106,458	98,028	86,928
April	103,273	109,057	74,917
May	67,933	114,041	88,901
June	62,906	144,379	127,793
July	238,362	150,907	171,978
August	180,046	102,600	124,757
September	114,860	101,198	119,224
October	52,678	67,860	75,722
November	41,513	66,522	74,292
December	36,316	66,591	73,012
Total	1,124,472	1,135,494	1,160,283
Value	£953,475	£1,135,499	£1,160,283

An important feature of modern flower growing is the production and cultivation of what are known as “hardy herbaceous perennials.” Some 2000 or 3000 different species and varieties of these are now raised in special nurseries; and during the spring, summer and autumn seasons magnificent displays are to be seen not only in the markets but at the exhibitions in London and at the great provincial shows held throughout the kingdom. The production of many of these perennials is so easy that amateurs in several instances have taken it up as a business hobby; and in some cases, chiefly through advertising in the horticultural press, very lucrative concerns have been established.

Ornamental flowering trees and shrubs constitute another feature of modern gardening. These are grown and imported by thousands chiefly for their sprays of blossom or foliage, and for planting in large or small gardens, public parks, &c., for landscape effect. Indeed there is scarcely an easily grown plant from the northern or southern temperate zones that does not now find a place in the nursery or garden, provided it is sufficiently attractive to sell for its flowers, foliage or appearance.

Conditions of the Fruit and Flower growing Industries.—As regards open-air fruit-growing, the outlook for new ventures is perhaps brighter than in the hothouse industry, not—as Mr Bear has pointed out—because the area of fruit land in England is too small, but because the level of efficiency, from the selection of varieties to the packing and marketing of the produce, is very much lower in the former than in the latter branch of enterprise. In other words, whereas the practice of the majority of hothouse nurserymen is so skilled, so up-to-date, and so entirely under high pressure that a new competitor, however well trained, will find it difficult to rise above mediocrity, the converse is true of open-air fruit-growers. Many, and an increasing proportion, of the latter are thoroughly efficient in all branches of their business, and are in possession of plantations of the best market varieties of fruit, well cultivated, pruned and otherwise managed. But the extent of fruit plantations completely up to the mark in relation to varieties and treatment of trees and bushes, and in connexion with which the packing and marketing of the produce are equally satisfactory, is small in proportion to the total fruit area of the country. Information concerning the best treatment of fruit trees has spread widely in recent years, and old plantations, as a rule, suffer from the neglect or errors of the past, however skilful their present holders may be. Although the majority of professional market fruit-growers may be well up to the standard in skill, there are numerous contributors to the fruit supply who are either ignorant of the best methods of cultivation and marketing or careless in their application. The bad condition of the great majority of farm orchards is notorious, and many landowners, farmers and amateur gardeners who have planted fruit on a more or less extensive scale have mismanaged their undertakings. For these reasons new growers of open-air fruit for market have opportunities of succeeding by means of superiority to the majority of those with whom they will compete, provided that they possess the requisite knowledge, energy and capital. It has been asserted on sound authority that there is no chance of success for fruit-growers except in districts favourable as regards soil, climate and nearness to a railway or a good market; and, even under these conditions, only for men who have had experience in the industry and are prepared to devote their unremitting attention to it. Most important is it to a beginner that he should ascertain the varieties of fruit that flourish best in his particular district. Certain kinds seem to do well or fairly well in all parts of the country; others, whilst heavy croppers in some localities, are often unsatisfactory in others.

As has been intimated, there is probably in England less room for expansion of fruit culture under glass than in the open. The large increase of glass-houses in modern times appears to have brought the supply of hothouse produce, even at greatly reduced prices, at least up to the level of the demand; and as most nurserymen continue to extend their expanse of glass, the prospect for new competitors is not a bright one. Moreover, the vast scale upon which some of the growers conduct the hothouse industry puts small producers at a great disadvantage, not only because the extensive producers can grow grapes and other fruit more economically than small growers—with the possible exception of those who do all or nearly all their own work—but also, and still more, because the former have greater advantages in transporting and marketing their fruit. There has, in recent years, been a much greater fall in the prices of hothouse than of open-air fruit, especially under the existing system of distribution, which involves the payment by consumers of 50 to 100% more in prices than growers receive. The best openings for new nurseries are probably not where they are now to be found in large groups, and especially not in the neighbourhood of London, but in suitable spots 268
near the great centres of population in the Midlands and the North, or big towns elsewhere not already well supplied with nurseries. By such a selection of a locality the beginner may build up a retail trade in hothouse fruit, or at least a trade with local fruiterers and grocers, thus avoiding railway charges and salesmen's commissions to a great extent, though it may often be advantageous to send certain kinds of produce to a distant market. Above all, a man who has no knowledge of the hothouse industry should avoid embarking his capital in it, trusting himself in the hands of a foreman, as experience shows that such a venture usually leads to disaster. Some years of training in different nurseries are desirable for any young man who is desirous of becoming a grower of hothouse fruits or flowers.

There can be no doubt that flower-growing is greatly extending in England, and that competition among home growers is becoming more severe. Foreign supplies of flowers have increased, but not nearly as greatly in proportion as home supplies, and it seems clear that home growers have gained ground in relation to their foreign rivals, except with respect to flowers for the growth of which foreigners have extraordinary natural advantages. There seems some danger of the home culture of the narcissus being over-done, and the florists' chrysanthemum appears to be produced in excess of the demand. Again, in the production of violets the warm and sunny South of France has an advantage not possessed by England, whilst Holland, likewise for climatic reasons, maintains her hold upon the hyacinth and tulip trade. Whether the production of flowers as a whole is gaining ground upon the demand or not is a difficult question to answer. It is true that the prices of flowers have fallen generally; but production, at any rate under glass, has been cheapened, and if a fair profit can be obtained, the fall in prices, without which the existing consumption of flowers would be impossible, does not necessarily imply over-production. There is some difference of opinion among growers upon this point; but nearly all agree that profits are now so small that production on a large scale is necessary to provide a fair income. Industrial flower-growing affords such a wide scope for the exercise of superior skill, industry and alertness, that it is not surprising to find some who are engaged in it doing remarkably well to all appearance, while others are struggling on and hardly paying their way. That a man with only a little capital, starting in a small way, has many disadvantages is certain; also, that his chance of saving money and extending his business quickly is much smaller than it was. To the casual looker-on, who knows nothing of the drudgery of the industry, flower-growing seems a delightful method of getting a living. That it is an entrancing pursuit there is no doubt; but it is equally true that it is a very arduous one, requiring careful forethought, ceaseless attention and abundant energy. Fortunately for those who might be tempted, without any knowledge of the industry, to embark capital in it, flower-growing, if at all comprehensive in scope, so obviously requires a varied and extensive technical knowledge, combined with good commercial ability, that any one can see that a thorough training is necessary to a man who intends to adopt it as a business, especially if hothouse flowers are to be produced.

The market for fruit, and more especially for flowers, is a fickle one, and there is nearly always some uncertainty as to the course of prices. The perishable nature of soft fruit and cut flowers renders the markets very sensitive to anything in the nature of a glut, the occurrence of which is usually attended with disastrous results to producers. Foreign competition, moreover, has constantly to be faced, and it is likely to increase rather than diminish. French growers have a great advantage over the open-air cultivators of England, for the climate enables them to get their produce into the markets early in the season, when the highest prices are obtainable. The geographical advantage which France enjoys in being so near to England is, however, considerably discounted by the increasing facilities for cold storage in transit, both by rail and sea. The development of such facilities permits of the retail sale in England of luscious fruit as fresh and attractive as when it was gathered beneath the sunny skies of California. In the case of flowers, fashion is an element not to be ignored. Flowers much in request in one season may meet with very little demand in another, and it is difficult for the producer to anticipate the changes which caprice may dictate. Even for the same kind of flower the requirements are very uncertain, and the white blossom which is all the rage in one season may be discarded in favour of one of another colour in the next. The sale of fresh flowers for church decoration at Christmas and Easter has reached enormous dimensions. The irregularity in the date of the festival, however, causes some inconvenience to growers. If it falls very early the great bulk of suitable flowers may not be sufficiently forward for sale, whilst a late Easter may find the season too far advanced. The trade in cut flowers, therefore, is generally attended by uncertainty, and often by anxiety.

(W. Fr.; J. Ws.)

United States

In the United States horticulture and market gardening have now assumed immense proportions. In a country of over 3,000,000 sq. m., stretching from the Atlantic to the Pacific on the one hand, and from the Gulf of Mexico to the great northern lakes and the Dominion of Canada on the other, a great variation of climatic conditions is not unnatural. From a horticultural point of view there are practically two well-defined regions: (1) that to the east of the Rocky Mountains across to the Atlantic, where the climate is more like that of eastern Asia than of western Europe so far as rainfall, temperature and seasonable conditions are concerned; (2) that to the west of the Rockies, known as the Pacific coast region, where the climate is somewhat similar to that of western Europe. It may be added that in the northern states—in Washington, Montana, North Dakota, Minnesota, Wisconsin, &c.—the winters are often very severe, while the southern states practically enjoy a temperature somewhat similar to that of the Riviera. Indeed the range of temperature between the extreme northern states and the extreme southern may vary as much as 120° F. The great aim of American gardeners, therefore, has been to find out or to produce the kinds of fruits, flowers and vegetables that are likely to flourish in different parts of this immense country.

Fruit Culture.—There is probably no country in the world where so many different kinds of fruit can be grown with advantage to the nation as in the United States. In the temperate regions apples, pears and plums are largely grown, and orchards of these are chiefly to be found in the states of New York, Massachusetts, Pennsylvania, Michigan, Missouri, Colorado, and also in northern Texas, Arkansas and N. California. To these may be added cranberries and quinces, which are chiefly grown in the New England states. The quinces are not a crop of first-rate importance, but as much as 800,000 bushels of cranberries are grown each year. The peach orchards are assuming great proportions, and are chiefly to be found in Georgia and Texas, while grapes are grown throughout the Republic from east to west in all favourable localities. Oranges, lemons and citrons are more or less extensively grown in Florida and California, and in these regions what are known as Japanese or "Kelsey" plums (forms of *Prunus triflora*) are also grown as marketable crops. Pomegranates are not yet largely grown, but it is possible their culture will develop in southern Texas and Louisiana, where the climate is tempered by the waters of the Gulf of Mexico. Tomatoes are grown in most parts of the country so easily that there is frequently a glut; while the strawberry region extends from Florida to Virginia, Pennsylvania and other states—thus securing a natural succession from south to north for the various great market centres.

Of the fruits mentioned apples are undoubtedly the most important. Not only are the American people themselves supplied with fresh fruit, but immense quantities are exported to Europe—Great Britain alone absorbing as much as 1,430,000 cwt. in 1908. The varieties originally grown were of course those taken or introduced from Europe by the early settlers. Since the middle of the 19th century great changes have been brought about, and the varieties mostly cultivated now are distinctly American. They have been raised by crossing and intercrossing the most suitable European forms with others since imported from Russia. In the extreme northern states indeed, where it is essential to have apple trees that will stand the severest winters, the Russian varieties crossed with the berry crab of eastern Europe (*Pyrus baccata*) have produced a race 269
eminently suited to that particular region. The individual fruits are not very large, but the trees are remarkably hardy. Farther south larger fruited varieties are grown, and among these may be noted Baldwins, Newton pippins, Spitzenbergs and Rhode Island greening. Apple orchards are numerous in the State of New York, where it is estimated that over 100,000 acres are devoted to them. In the hilly regions of Missouri, Arkansas and Colorado there are also great plantations of apples. The trees, however, are grown on different principles from those in New York State. In the latter state apple trees with ordinary care live to more than 100 years of age and produce great crops; in the other states, however, an apple tree is said to be middle-aged at 20, decrepit at 30 and practically useless at 40 years of age. They possess the advantage, however, of bearing early and heavily.

Until the introduction of the cold-storage system, about the year 1880, America could hardly be regarded as a commercial fruit-growing country. Since then, however, owing to the great improvements made in railway refrigerating vans and storage houses, immense quantities of fruit can be despatched in good condition to any part of the world; or they can be kept at home in safety until such time as the markets of Chicago, New York, Boston, Baltimore, Philadelphia, &c., are considered favourable for their reception.

Apple trees are planted at distances varying from 25 ft. to 30 ft. apart in the middle western states, to 40 ft. to 50 ft. apart in New York State. Here and there, however, in some of the very best orchards the trees are planted 60 ft. apart every way. Each tree thus has a chance to develop to its utmost limits, and as air and light reach it better, a far larger fruit-bearing surface is secured. Actual experience has shown that trees planted at 60 ft. apart—about 28 to the acre—produce more fruit by 43 bushels than trees at 30 ft. apart—*i.e.* about 48 to the acre.

Until recent years pruning as known to English and French gardeners was practically unknown. There was indeed no great necessity for it, as the trees, not being cramped for space, threw their branches outwards and upwards, and thus rarely become overcrowded. When practised, however, the operation could scarcely be called pruning; lopping or trimming would be more accurate descriptions.

Apple orchards are not immune from insect pests and fungoid diseases, and an enormous business is now done in spraying machines and various insecticides. It pays to spray the trees, and figures have been given to show that orchards that have been sprayed four times have produced an average income of £211 per acre against £103 per acre from unsprayed orchards.

The spring frosts are also troublesome, and in the Colorado and other orchards the process known as "smudging" is now adopted to save the crops. This consists in placing 20 or 30, or even more, iron or tin pots to an acre, each pot containing wooden chips soaked in tar (or pitch) mixed with kerosene. Whenever the thermometer shows 3 or 4 degrees of frost the smudge-pots are lighted. A dense white smoke then arises and is diffused throughout the orchards, enveloping the blossoming heads of the trees in a dense cloud. This prevents the frost from killing the tender pistils in the blossoms, and when several smudge-pots are alight at the same time the temperature of the orchard is raised two or three degrees. This work has generally to be done between 3 and 5 A.M., and the growers naturally have an anxious time until all danger is over. The failure to attend to smudging, even on one occasion, may result in the loss of the entire crop of plums, apples or pears.

Next to apples perhaps peaches are the most important fruit crop. The industry is chiefly carried on in Georgia, Texas and S. Carolina, and on a smaller scale in some of the adjoining states. Peaches thus flourish in regions that are quite unsuitable for apples or pears. In many orchards in Georgia, where over 3,000,000 acres have been planted, there are as many as 100,000 peach trees; while some of the large fruit companies grow as many as 365,000. In one place in West Virginia there is, however, a peach orchard containing 175,000 trees, and in Missouri another company has 3 sq. m. devoted to peach culture. As a rule the crops do well. Sometimes, however, a disease known as the "yellows" makes sad havoc amongst them, and scarcely a fruit is picked in an orchard which early in the season gave promise of a magnificent crop.

Plums are an important crop in many states. Besides the European varieties and those that have been raised by crossing with American forms, there is now a growing trade done in Japanese plums. The largest of these is popularly known as "Kelseys," named after John Kelsey, who raised the first fruit in 1876 from trees brought to California in 1870. Sometimes the fruits are 3 in. in diameter, and like most of the Japanese varieties are more heart-shaped and pointed than plums of European origin. One apparent drawback to the Kelsey plum is its irregularity in ripening. It has been known in some years to be quite ripe in June, while in others the fruits are still green in October.

Pears are much grown in such states as Massachusetts, New York, Pennsylvania, Missouri and California; while bush fruits like currants, gooseberries and raspberries find large spaces devoted in most of the middle and northern states. Naturally a good deal of crossing and intercrossing has taken place amongst the European and American forms of these fruits, but so far as gooseberries are concerned no great advance seems to have been made in securing varieties capable of resisting the devastating gooseberry mildew.

Other fruits of more or less commercial value are oranges, lemons and citrons, chiefly in Florida. Lemons are practically a necessity to the American people, owing to the heat of the summers, when cool and refreshing drinks with an agreeable acidulous taste are in great demand. The pomelo (grape-fruit) is a kind of lemon with a thicker rind and a more acid flavour. At one time its culture was confined to Florida, but of recent years it has found its way into Californian orchards. Notwithstanding the prevailing mildness of the climate in both California and Florida, the crops of oranges, lemons, citrons, &c., are sometimes severely injured by frosts when in blossom.

Other fruits likely to be heard of in the future are the kaki or persimmon, the loquat, which is already grown in Louisiana, as well as the pomegranate.

Great aid and encouragement are given by the government to the progress of American fruit-growing, and by the experiments that are being constantly carried out and tabulated at Cornell University and by the U.S.A. department of agriculture.

Flower Culture.—So far as flowers are concerned there appears to be little difference between the kinds of plants grown in the United States and in England, France, Belgium, Germany, Holland, &c. Indeed there is a great interchange of new varieties of plants between Europe and America, and modifications in systems of culture are being gradually introduced from one side of the Atlantic to the other. The building of greenhouses for commercial purposes is perhaps on a somewhat different scale from that in England, but there are probably no extensive areas of glass such as are to be seen north of London from Enfield Highway to Broxburne. Hot water apparatus differs merely in detail, although most of the boilers used resemble those on the continent of Europe rather than in England. Great business is done in bulbs—mostly imported from Holland—stove and greenhouse plants, hardy perennials, orchids, ferns of the "fancy" and "dagger" types of Nephrolepis, and in carnations and roses. Amongst the latter thousands of such varieties as Beauty, Liberty, Killarney, Richmond and Bride are grown, and realize good prices as a rule in the markets. Carnations of the winter-flowering or "perpetual" type have long been grown in America, and enormous prices have been given for individual plants on certain occasions, rivalling the fancy prices paid in England for certain orchids. The American system of carnation-growing has quite captivated English cultivators, and new varieties are being constantly raised in both countries. Chrysanthemums are another great feature of American florists, and sometimes during the winter season a speculative grower will send a living specimen to one of the London exhibitions in the hope of booking large orders for cuttings of it later on. Sweet peas, dahlias, lilies of the valley, arum lilies and indeed every flower that is popular in England is equally popular in America, and consequently is largely grown. 270

Vegetables.—So far as these are concerned, potatoes, cabbages, cauliflowers, beans of all kinds, cucumbers, tomatoes (already referred to under fruits), musk-melons, lettuces, radishes, endives, carrots, &c.; are naturally grown in great quantities, not only in the open air, but also under glass. The French system of intensive cultivation as practised on hot beds of manure round Paris is practically unknown at present. In the southern states there would be no necessity to practise it, but in the northern ones it is likely to attract attention.

(J. Ws.)

1 *Jour. Roy. Agric. Soc.*, 1899.

FRUMENTIUS (*c.* 300-*c.* 360), the founder of the Abyssinian church, traditionally identified in Abyssinian literature with Abba Salama or Father of Peace (but see Ethiopia), was a native of Phoenicia. According to the 4th-century historian Rufinus (x. 9), who gives Aedesius himself as his authority, a certain Tyrian, Meropius, accompanied by his kinsmen Frumentius and Aedesius, set out on an expedition to "India," but fell into the hands of Ethiopians on the shore of the Red Sea and, with his ship's crew, was put to death. The two young men were taken to the king at Axum, where they were well treated and in time obtained great influence. With the help of Christian merchants who visited the country Frumentius gave Christianity a firm footing, which was strengthened when in 326 he was consecrated bishop by Athanasius of Alexandria, who in his *Epistola ad Constantinum* mentions the consecration, and gives some details of the history of Frumentius's mission. Later witnesses speak of his fidelity to the homoousian during the Arian controversies. Aedesius returned to Tyre, where he was ordained presbyter.

FRUNDSBERG, GEORG VON (1473-1528), German soldier, was born at Mindelheim on the 24th of September 1473. He fought for the German king Maximilian I. against the Swiss in 1499, and in the same year was among the imperial troops sent to assist Ludovico Sforza, duke of Milan, against the French. Still serving Maximilian, he took part in 1504 in the war over the succession to the duchy of Bavaria-Landshut, and afterwards fought in the Netherlands. Convinced of the necessity of a native body of trained infantry Frundsberg assisted Maximilian to organize the *Landsknechte* (*q.v.*), and subsequently at the head of bands of these formidable troops he was of great service to the Empire and the Habsburgs. In 1509 he shared in the war against Venice, winning fame for himself and his men; and after a short visit to Germany returned to Italy, where in 1513 and 1514 he gained fresh laurels by his enterprises against the Venetians and the French. Peace being made, he returned to Germany, and at the head of the infantry of the Swabian league assisted to drive Ulrich of Württemberg from his duchy in 1519. At the diet of Worms in 1521 he spoke words of encouragement to Luther, and when the struggle between France and the Empire was renewed he took part in the invasion of Picardy, and then proceeding to Italy brought the greater part of Lombardy under the influence of Charles V. through his victory at Bicocca in April 1522. He was partly responsible for the great victory over the French at Pavia in February 1525, and, returning to Germany, he assisted to suppress the Peasant revolt, using on this occasion, however, diplomacy as well as force. When the war in Italy was renewed Frundsberg raised an army at his own expense, and skilfully surmounting many difficulties, joined the constable de Bourbon near Piacenza and marched towards Rome. Before he reached the city, however, his unpaid troops showed signs of mutiny, and their leader, stricken with illness and unable to pacify them, gave up his command. Returning to Germany, he died at Mindelheim on the 20th of August 1528. He was a capable and chivalrous soldier, and a devoted servant of the Habsburgs. His son Caspar (1500-1536) and his grandson Georg (d. 1586) were both soldiers of some distinction. With the latter's death the family became extinct.

See Adam Reissner, *Historia Herrn Georgs und Herrn Kaspars von Frundsberg* (Frankfort, 1568). A German translation of this work was published at Frankfort in 1572. F.W. Barthold, *Georg von Frundsberg* (Hamburg, 1833); J. Heilmann, *Kriegsgeschichte von Bayern, Franken, Pfalz und Schwaben* (Munich, 1868).

FRUSTUM (Latin for a "piece broken off"), a term in geometry for the part of a solid figure, such as a cone or pyramid, cut off by a plane parallel to the base, or lying between two parallel planes; and hence in architecture a name given to the drum of a column.

FRUYTIERS, PHILIP (1627-1666), Flemish painter and engraver, was a pupil of the Jesuits' college at Antwerp in 1627, and entered the Antwerp gild of painters without a fee in 1631. He is described in the register of that institution as "illuminator, painter and engraver." The current account of his life is "that he worked exclusively in water colours, yet was so remarkable in this branch of his art for arrangement, drawing, and especially for force and clearness of colour, as to excite the admiration of Rubens, whom he portrayed with all his family." The truth is that he was an artist of the most versatile talents, as may be judged from the fact that in 1646 he executed an Assumption with figures of life size, and four smaller pictures in oil, for the church of St Jacques at Antwerp, for which he received the considerable sum of 1150 florins. Unhappily no undoubted production of his hand has been preserved. All that we can point to with certainty is a series of etched plates, chiefly portraits, which are acknowledged to have been powerfully and skilfully handled. If, however, we search the portfolios of art collections on the European continent, we sometimes stumble upon miniatures on vellum, drawn with great talent and coloured with extraordinary brilliancy. In form they quite recall the works of Rubens, and these, it may be, are the work of Philip Fruytiers.

FRY, the name of a well-known English Quaker family, originally living in Wiltshire. About the middle of the 18th century Joseph Fry (1728-1787), a doctor, settled in Bristol, where he acquired a large practice, but eventually abandoned medicine for commerce. He became interested in china-making, soap-boiling and type-founding businesses in Bristol, and in a chemical works at Battersea, all of which ventures proved very profitable. The type-founding business was subsequently removed to London and conducted by his son Edmund. Joseph Fry, however, is best remembered as the founder of the great Bristol firm of J.S. Fry & Sons, chocolate manufacturers. He purchased the chocolate-making patent of William Churchman and on it laid the foundations of the present large business. After his death the Bristol chocolate factory was carried on with increasing success by his widow and by his son, Joseph Storrs Fry (1767-1835).

In 1795 a new and larger factory was built in Union Street, Bristol, which still forms the centre of the firm's premises, and in 1798 a Watt's steam-engine was purchased and the cocoa-beans ground by steam. On the death of Joseph Storrs Fry his three sons, Joseph (1795-1879), Francis, and Richard (1807-1878) became partners in the firm, the control being mainly in the hands of Francis Fry (1803-1886). Francis Fry was in every way a remarkable character. The development of the business to its modern enormous proportion was chiefly his work, but this did not exhaust his activities. He took a principal part in the introduction of railways to the west of England, and in 1852 drew up a scheme for a general English railway parcel service. He was an ardent bibliographer, taking a special interest in early English Bibles, of which he made in the course of a long life a large and striking collection, and of the most celebrated of which he published facsimiles with bibliographical notes. Francis Fry died in 1886, and his son Francis J. Fry and nephew Joseph Storrs Fry carried on the business, which in 1896 was for family reasons converted into a private limited company, Joseph Storrs Fry being chairman and all the directors members of the Fry family.

FRY, SIR EDWARD (1827-), English judge, second son of Joseph Fry (1795-1879), was born at Bristol on the 4th of November 1827, and educated at University College, London, and London University. He was called to the bar in 1854 and was made a Q.C. in 1869, practising in the rolls court and becoming recognized as a leading equity lawyer. In 1877 he was raised to the bench and knighted. As chancery judge he will be 271
remembered for his careful interpretations and elucidations of the Judicature Acts, then first coming into operation. In 1883 he was made a lord justice of appeal, but resigned in 1892; and subsequently his knowledge of equity and talents for arbitration were utilized by the British government from time to time in various special directions, particularly as chairman of many commissions. He was also one of the British representatives at the Paris North Sea Inquiry Commission (1905), and was appointed a member of the Hague Permanent Arbitration Court. He wrote *A Treatise on the Specific Performance of Public Contracts* (London, 1858, and many subsequent editions).

FRY, ELIZABETH (1780-1845), English philanthropist, and, after Howard, the chief promoter of prison reform in Europe, was born in Norwich on the 21st of May 1780. Her father, John Gurney, afterwards of Earlham Hall, a wealthy merchant and banker, represented an old family which for some generations had belonged to the Society of Friends. While still a girl she gave many indications of the benevolence of disposition, clearness and independence of judgment, and strength of purpose, for which she was afterwards so distinguished; but it was not until after she had entered her eighteenth year that her religion assumed a decided character, and that she was induced, under the preaching of the American Quaker, William Savery, to become an earnest and enthusiastic though never fanatical “Friend.” In August 1800 she became the wife of Joseph Fry, a London merchant.

Amid increasing family cares she was unwearied in her attention to the poor and the neglected of her neighbourhood; and in 1811 she was acknowledged by her co-religionists as a “minister,” an honour and responsibility for which she was undoubtedly qualified, not only by vigour of intelligence and warmth of heart, but also by an altogether unusual faculty of clear, fluent and persuasive speech. Although she had made several visits to Newgate prison as early as February 1813, it was not until nearly four years afterwards that the great public work of her life may be said to have begun. The association for the Improvement of the Female Prisoners in Newgate was formed in April 1817. Its aim was the much-needed establishment of some of what are now regarded as the first principles of prison discipline, such as entire separation of the sexes, classification of criminals, female supervision for the women, and adequate provision for their religious and secular instruction, as also for their useful employment. The ameliorations effected by this association, and largely by the personal exertions of Mrs Fry, soon became obvious, and led to a rapid extension of similar methods to other places. In 1818 she, along with her brother, visited the prisons of Scotland and the north of England; and the publication (1819) of the notes of this tour, as also the cordial recognition of the value of her work by the House of Commons committee on the prisons of the metropolis, led to a great increase of her correspondence, which now extended to Italy, Denmark and Russia, as well as to all parts of the United Kingdom. Through a visit to Ireland, which she made in 1827, she was led to direct her attention to other houses of detention besides prisons; and her observations resulted in many important improvements in the British hospital system, and in the treatment of the insane. In 1838 she visited France, and besides conferring with many of the leading prison officials, she personally visited most of the houses of detention in Paris, as well as in Rouen, Caen and some other places. In the following year she obtained an official permission to visit all the prisons in that country; and her tour, which extended from Boulogne and Abbeville to Toulouse and Marseilles, resulted in a report which was presented to the minister of the interior and the prefect of police. Before returning to England she had included Geneva, Zürich, Stuttgart and Frankfort-on-Main in her inspection. The summer of 1840 found her travelling through Belgium, Holland and Prussia on the same mission; and in 1841 she also visited Copenhagen. In 1842, through failing health, Mrs Fry was compelled to forgo her plans for a still more widely extended activity, but had the satisfaction of hearing from almost every quarter of Europe that the authorities were giving increased practical effect to her suggestions. In 1844 she was seized with a lingering illness, of which she died on the 12th of October 1845. She was survived by a numerous family, the youngest of whom was born in 1822.

Two interesting volumes of *Memoirs, with Extracts from her Journals and Letters*, edited by two of her daughters, were published in 1847. See also *Elizabeth Fry*, by G. King Lewis (1910).

FRYXELL, ANDERS (1795-1881), Swedish historian, was born at Hesselskog, Dalsland, Sweden, on the 7th of February 1795. He was educated at Upsala, took holy orders in 1820, was made a doctor of philosophy in 1821, and in 1823 began to publish the great work of his life, the *Stories from Swedish History*. He did not bring this labour to a close until, fifty-six years later, he published the forty-sixth and crowning volume of his vast enterprise. Fryxell, as a historian, appealed to every class by the picturesqueness of his style and the breadth of his research; he had the gift of awakening to an extraordinary degree the national sense in his readers. In 1824 he published his *Swedish Grammar*, which was long without a rival. In 1833 he received the title of professor, and in 1835 he was appointed to the incumbency of Sunne, in the diocese of Karlstad, where he resided for the remainder of his life. In 1840 he was elected to the Swedish Academy in succession to the poet Wallin (1779-1839). In 1847 Fryxell received from his bishop permission to withdraw from all the services of the Church, that he might devote himself without interruption to historical investigation. Among his numerous minor writings are prominent his *Characteristics of Sweden between 1592 and 1600* (1830), his *Origins of the Inaccuracy with which the History of Sweden in Catholic Times has been Treated* (1847), and his *Contributions to the Literary History of Sweden*. It is now beginning to be seen that the abundant labours of Fryxell were rather of a popular than of a scientific order, and although their influence during his lifetime was unbounded, it is only fair to later and exacter historians to admit that they threaten to become obsolete in more than one direction. On the 21st of March 1881 Anders Fryxell died at Stockholm, and in 1884 his daughter Eva Fryxell (born 1829) published from his MS. an interesting *History of My History*, which was really a literary autobiography and displays the persistency and tirelessness of his industry.

(E. G.)

FUAD PASHA (1815-1869), Turkish statesman, was the son of the distinguished poet Kechéji-zadé Izzet Molla. He was educated at the medical school and was at first an army surgeon. About 1836 he entered the civil service as an official of the foreign ministry. He became secretary of the embassy in London; was employed on special missions in the principalities and at St Petersburg (1848), and was sent to Egypt as special commissioner in 1851. In that year he became minister for foreign affairs, a post to which he was appointed also on four subsequent occasions and which he held at the time of his death. During the Crimean War he commanded the troops on the Greek frontier and distinguished himself by his bravery. He was Turkish delegate at the Paris conference of 1856; was charged with a mission to Syria in 1860; grand vizier in 1860 and 1861, and also minister of war. He accompanied the sultan Abd-ul-Aziz on his journey to Egypt and Europe, when the freedom of the city of London was conferred on him. He died at Nice (whither he had been ordered for his health) in 1869. Fuad was renowned for his boldness and promptness of decision, as well as for his ready wit and his many bons mots. Generally regarded as the partisan of a pro-English policy, he rendered most valuable service to his country by his able management of the foreign relations of Turkey, and not least by his efficacious settlement of affairs in Syria after the massacres of 1860.

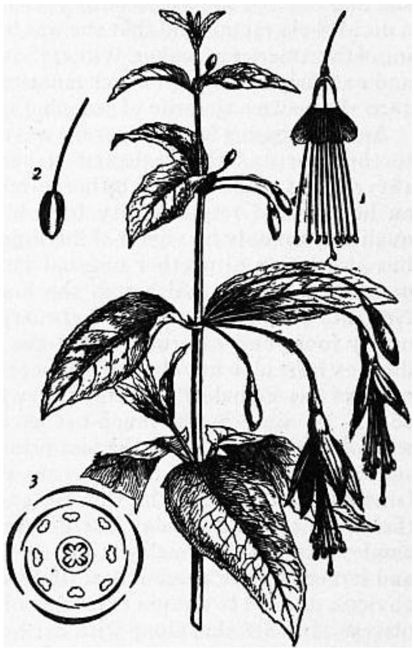
FUCHOW, Fu-Chau, Foochow, a city of China, capital of the province of Fu-kien, and one of the principal ports open to foreign commerce. In the local dialect it is called Hokchiu. It is situated on the river Min, about 35 m. from the sea, in 26° 5′ N. and 119º 20′ E., 140 m. N. of Amoy and 280 S. of Hang-chow. The city proper, lying nearly 3 m. from the north bank of the river, is surrounded by a wall about 30 ft. high and 12 ft. thick, which makes a circuit of upwards of 5 m. and is pierced by seven gateways surrounded by tall fantastic watch-towers. The whole district between the city and the river, the island of Nantai, and the southern banks 272
of the Min are occupied by extensive suburbs; and the river itself bears a large floating population. Communication from bank to bank is afforded by a long stone bridge supported by forty solid stone piers in its northern section and by nine in its southern. The most remarkable establishment of Fuchow is the arsenal situated about 3 m. down the stream at Pagoda Island, where the sea-going vessels usually anchor. It was founded in 1867, and is conducted under the direction of French engineers according to European methods. In 1870 it employed about 1000 workmen besides fifty European superintendents, and between that date and 1880 it turned out about 20 or 30 small gunboats. In 1884 it was partially destroyed by the French fleet, and for a number of years the workshops and machinery were allowed to stand idle and go to decay. On the 1st of August 1895 an attack was made on the English mission near the city of Ku-chang, 120 m. west of Fuchow, on which occasion nine missionaries, of whom eight were ladies, were massacred. The port was opened to European commerce in 1842; and in 1853 the firm of Russell and Co. shipped the first cargoes of tea from Fuchow to Europe and America. The total trade in foreign vessels in 1876 was imports to the value of £1,531,617, and exports to the value of £3,330,489. In 1904 the imports amounted to £1,440,351, and the exports to £1,034,436. The number of vessels that entered in 1876 was 275, and of these 211 were British, 27 German, 11 Danish and 9 American. While in 1904, 480 vessels entered the port, 216 of which were British. A large trade is carried on by the native merchants in timber, paper, woollen and cotton goods, oranges and olives; but the foreign houses mainly confine themselves to opium and tea. Commercial intercourse with Australia and New Zealand is on the increase. The principal imports, besides opium, are shirtings, T-cloths, lead and tin, medicines, rice, tobacco, and beans and peas. Two steamboat lines afford regular communication with Hong-Kong twice a month. The town is the seat of several important missions, of which the first was founded in 1846. That supported by the American board had in 1876 issued 1,3000,000 copies of Chinese books and tracts.

FUCHS, JOHANN NEPOMUK VON (1774-1856), German chemist and mineralogist, was born at Mattenzell, near Brennberg in the Bavarian Forest, on the 15th of May 1774. In 1807 he became professor of chemistry and mineralogy at the university of Landshut, and in 1823 conservator of the mineralogical collections at Munich, where he was appointed professor of mineralogy three years later, on the removal thither of the university of Landshut. He retired in 1852, was ennobled by the king of Bavaria in 1854, and died at Munich on the 5th of March 1856. His name is chiefly known for his mineralogical observations and for his work on soluble glass.

His collected works, including *Über den Einfluss der Chemie und Mineralogie* (1824), *Die Naturgeschichte des Mineralreichs* (1842), *Über die Theorien der Erde* (1844), were published at Munich in 1856.

FUCHS, LEONHARD (1501-1566), German physician and botanist, was born at Wembdingen in Bavaria on the 17th of January 1501. He attended school at Heilbronn and Erfurt, and in 1521 graduated at the university of Ingolstadt. About the same time he espoused the doctrines of the Reformation. Having in 1524 received his diploma as doctor of medicine, he practised for two years in Munich. He became in 1526 professor of medicine at Ingolstadt, and in 1528 physician to the margrave of Anspach. In Anspach he was the means of saving the lives of many during the epidemic locally known as the “English sweating-sickness.” By the duke of Württemberg he was, in 1535, appointed to the professorship of medicine at the university of Tübingen, a post held by him till his death on the 10th of May 1566. Fuchs was an advocate of the Galenic school of medicine, and published several Latin translations of treatises by its founder and by Hippocrates. But his most important publication was *De historia stirpium commentarii insignes* (Basel, 1542), a work illustrated with more than five hundred excellent outline illustrations, including figures of the common foxglove and of another species of the genus *Digitalis*, which was so named by him.

FUCHSIA, so named by Plumier in honour of the botanist Leonhard Fuchs, a genus of plants of the natural order Onagraceae, characterized by entire, usually opposite leaves, pendent flowers, a funnel-shaped, brightly coloured, quadripartite, deciduous calyx, 4 petals, alternating with the calycine segments, 8, rarely 10, exserted stamens, a long filiform style, an inferior ovary, and fruit, a fleshy ovoid many-seeded berry. All the members of the genus, with the exception of the New Zealand species, *F. excorticata, F. Colensoi* and *F. procumbens*, are natives of Central and South America—occurring in the interior of forests or in damp and shady mountainous situations. The various species differ not a little in size as well as in other characters; some, as *F. verrucosa*, being dwarf shrubs; others, as *F. arborescens* and *F. apetala*, attaining a height of 12 to 16 ft., and having stems several inches in diameter. Plumier, in his *Nova plantarum Americanarum genera* (p. 14, tab. 14, Paris, 1703), gave a description of a species of fuchsia, the first known, under the name of *Fuchsia triphylla, flore coccineo*, and a somewhat conventional outline figure of the same plant was published at Amsterdam in 1757 by Burmann. In the *Histoire des plantes médicinales* of the South American traveller Feuillée (p. 64, pl. XLVII.), written in 1709-1711, and published by him with his *Journal*, Paris, 1725, the name *Thilco* is applied to a species of fuchsia from Chile, which is described, though not evidently so figured, as having a pentamerous calyx. The *F. coccinea* of Alton (fig.) (see J.D. Hooker, in *Journal Linnean Soc.*, Botany, vol. x. p. 458, 1867), the first species of fuchsia cultivated in England, where it was long confined to the greenhouse, was brought from South America by Captain Firth in 1788 and placed in Kew Gardens. Of this species Mr Lee, a nurseryman at Hammersmith, soon afterwards obtained an example, and procured from it by means of cuttings several hundred plants, which he sold at a guinea each. In 1823 *F. macrostemma* and *F. gracilis*, and during the next two or three years several other species, were introduced into England; but it was not until about 1837, or soon after florists had acquired *F. fulgens*, that varieties of interest began to make their appearance. The numerous hybrid forms now existing are the result chiefly of the intercrossing of that or other long-flowered with globose-flowered plants. *F. Venus-victrix*, raised by Mr Gulliver, gardener to the Rev. S. Marriott of Horsemonden, Kent, and sold in 1822 to Messrs Cripps, was the earliest white-sepalled fuchsia. The first fuchsia with a white corolla was produced about 1853 by Mr Storey. In some varieties the blossoms are variegated, and in others they are double. There appears to be very little limit to the number of forms to be obtained by careful cultivation and selection. To hybridize, the flower as soon as it opens is emasculated, and it is then fertilized with pollen from some different flower.



Fuchsia coccinea.

1, Flower cut open after removal of sepals; 2, fruit; 3, floral diagram.

Ripe seed is sown either in autumn or about February or March in light, rich, well-drained mould, and is thinly covered with sandy soil and watered. A temperature of 70° to 75° Fahr. has been found suitable for raising. The seedlings are pricked off into shallow pots or pans, and when 3 in. in height are transferred to 3-in. pots, and are then treated the same as plants from cuttings. Fuchsias may be grafted as readily as camellias, preferably by the splice or whip method, the apex of a young shoot being employed as a scion; but the easiest and most usual method of propagation is by cuttings. The most expeditious way to procure these is to put plants in heat in January, and to take their shoots when 3 in. in length. For summer flowering in England they are best made about the end of August, and should be selected from the shortest-jointed young wood. They root readily in a compost of loam and silver-sand if kept close and sprinkled for a short time. In from two to three weeks they may be put into 3-in. pots containing a compost of equal parts of rich loam, silver-sand and leaf-mould. They are subsequently moved from the frame or bed, first to a warm and shady, and then to a more airy part of the greenhouse. In January a little artificial heat may be given, to be gradually increased as the days lengthen. The side-shoots are generally pruned when they have made three or four joints, and for bushy plants the leader is stopped soon after the first potting. Care is taken to keep the plants as near the glass as possible, and shaded from bright sunshine, also to provide them plentifully with water, except at the time of shifting, when the roots should be tolerably dry. For the second potting a suitable soil is a mixture of well-rotted cow-dung or old hotbed mould with leaf-mould and sandy peat, and to promote drainage a little peat-moss may be placed immediately over the crocks in the lower part of the pot. Weak liquid manure greatly promotes the advance of the plants, and should be regularly supplied twice or thrice a week during the flowering season. After this, water is gradually withheld from them, and they may be placed in the open air to ripen their wood.

Among the more hardy or half-hardy plants for inside borders are varieties of the Chilean species, *F. macrostemma* (or *F. magellanica*), a shrub 6 to 12 ft. high with a scarlet calyx, such as *F. m. globosa, F. m. gracilis*; one of the most graceful and hardy of these, a hybrid *F. riccartoni*, was raised at Riccarton, near Edinburgh, in 1830. For inside culture may be mentioned *F. boliviana* (Bolivia), 2 to 4 ft. high, with rich crimson flowers with a trumpet-shaped tube; *F. corymbiflora* (Peru), 4 to 6 ft. high, with scarlet flowers nearly 2 in. long in long terminal clusters; F. fulgens (Mexico), 4 to 6 ft., with drooping apical clusters of scarlet flowers; *F. microphylla* (Central America), with small leaves and small scarlet funnel-shaped flowers, the petals deep red; *F. procumbens* (New Zealand), a pretty little creeper, the small flowers of which are succeeded by oval magenta-crimson berries which remain on for months; and *F. splendens* (Mexico), 6 ft. high, with very showy scarlet and green flowers. But these cannot compare in beauty or freedom of blossom with the numerous varieties raised by gardeners. The nectar of fuchsia flowers has been shown to contain nearly 78% of cane sugar, the remainder being fruit sugar. The berries of some fuchsias are subacid or sweet and edible. From certain species a dye is obtainable. The so-called “native fuchsias” of southern and eastern Australia are plants of the genus *Correa*, natural order Rutaceae.

FUCHSINE, or Magenta, a red dye-stuff consisting of a mixture of the hydrochlorides or acetates of pararosaniline and rosaniline. It was obtained in 1856 by J. Natanson (*Ann.*, 1856, 98, p. 297) by the action of ethylene chloride on aniline, and by A.W. Hofmann in 1858 from aniline and carbon tetrachloride. It is prepared by oxidizing “aniline for red” (a mixture of aniline and ortho- and para-toluidine) with arsenic acid (H. Medlock, *Dingler’s Poly. Jour.*, 1860, 158, p. 146); by heating aniline for red with nitrobenzene, concentrated hydrochloric acid and iron (Coupier, *Ber.*, 1873, 6, p. 423); or by condensing formaldehyde with aniline and ortho-toluidine and oxidizing the mixture. It forms small crystals, showing a brilliant green reflex, and is soluble in water and alcohol with formation of a deep red solution. It dyes silk, wool and leather direct, and cotton after mordanting with tannin and tartar emetic (see Dyeing). An aqueous solution of fuchsine is decolorized on the addition of sulphurous acid, the easily soluble fuchsine sulphurous acid being formed. This solution is frequently used as a test reagent for the detection of aldehydes, giving, in most cases, a red coloration on the addition of a small quantity of the aldehyde.

The constitution of the fuchsine bases (pararosaniline and rosaniline) was determined by E. and O. Fischer in 1878 (*Ann.*, 1878, 194, p. 242); A.W. Hofmann having previously shown that oxidation of pure aniline alone or of pure toluidine yielded no fuchsine, whilst oxidation of a mixture of aniline and para-toluidine gave rise to the fine red dye-stuff para-fuchsine (pararosaniline hydrochloride)

$$CH_3{\cdot}C_6H_4NH_2 + 2C_6H_5NH_2 + 3O = \underset{\text{Colour base (pararosaniline).}}{HO{\cdot}C(C_6H_4NH_2)_3} + 2H_2O.$$

$$HO{\cdot}C(C_6H_4NH_2)_3{\cdot}HCl = H_2O + \underset{\text{Pararosaniline hydrochloride.}}{(H_2N{\cdot}C_6H_4)_2C : C_6H_4 : NH_2Cl.}$$

A. Rosenstiehl (*Jahres.*, 1869, p. 693) found also that different rosanilines were obtained according to whether ortho- or para-toluidine was oxidized with aniline; and he gave the name rosaniline to the one obtained from aniline and ortho-toluidine, reserving the term pararosaniline for the other. E. and O. Fischer showed that these compounds were derivatives of triphenylmethane and tolyldiphenylmethane respectively. Pararosaniline was reduced to the corresponding leuco compound (paraleucaniline), from which by diazotization and boiling with alcohol, the parent hydrocarbon was obtained

$$\underset{\text{Pararosaniline hydrochloride.}}{(H_2N{\cdot}C_5H_4)_2C : C_6H_4{:}NH_2Cl} \rightarrow \underset{\text{Paraleucaniline.}}{HC(C_6H_4NH_2{\cdot}HCl)_3} \rightarrow HC(C_6H_4N_2Cl_3) \rightarrow \underset{\text{Triphenylmethane.}}{HC(C_6H_5)_3.}$$

The reverse series of operations was also carried out by the Fischers, triphenylmethane being nitrated, and the nitro compound then reduced to triaminotriphenylmethane or paraleucaniline, which on careful oxidation is converted into the dye-stuff. A similar series of reactions was carried out with rosaniline, which was shown to be the corresponding derivative of tolyldiphenylmethane.

The free pararosaniline, $C_{19}H_{19}N_3O$, and rosaniline, $C_{20}H_{21}N_3O$, may be obtained by precipitating solutions of their salts with a caustic alkali, colourless precipitates being obtained, which crystallize from hot water in the form of needles or plates. The position of the amino groups in pararosaniline was determined by the work of H. Caro and C. Graebe (*Ber.*, 1878, II, p. 1348) and of E. and O. Fischer (*Ber.*, 1880, 13, p. 2204) as follows: Nitrous acid converts pararosaniline into aurin, which when superheated with water yields para-dioxybenzophenone. As the hydroxyl groups in aurin correspond to the amino groups in pararosaniline, two of these in the latter compound must be in the para position. The third is also in the para position; for if benzaldehyde be condensed with aniline, condensation occurs in the para position, for the compound formed may be converted into para-dioxybenzophenone,

$$C_6H_5CHO \rightarrow C_6H_5CH(C_6H_4NH_2)_2 \rightarrow C_6H_5CH(C_6H_4OH)_2 \rightarrow CO(C_6H_4OH)_2;$$

but if para-nitrobenzaldehyde be used in the above reaction and the resulting nitro compound $NO_2.C_6H_4.CH(C_6H_4NH_2)_2$ be reduced, then pararosaniline is the final product, and consequently the third amino group occupies the para position. Many derivatives of pararosaniline and rosaniline are known, in which the hydrogen atoms of the amino groups are replaced by alkyl groups; this has the effect of producing a blue or violet shade, which becomes deeper as the number of groups increases (see DYEING).

FUCINO, LAGO DI [Lat. *Lacus Fucinus*], a lake bed of the Abruzzi, Italy, in the province of Aquila, 2 m. E. of the town of Avezzano. The lake was 37 m. in circumference and 65 ft. deep. From the lack of an outlet, the level of the lake was subject to great variations, often fraught with disastrous consequences. As early as A.D. 52 the emperor Claudius, realizing a project of Julius Caesar, constructed a tunnel 3½ m. long, with 40 shafts at intervals, by which the surplus waters found an outlet to the Liris (or Garigliano). No less than 30,000 workmen were employed for eleven years in driving this tunnel. In the following reign the tunnel was allowed to fall into disrepair, but was repaired by Trajan. When, however, it finally went out of use is uncertain. The various attempts made to reopen it from 1240 onwards were unsuccessful. By 1852 the lake had gradually risen until it was 30 ft. above its original level, and had become a source of danger to the surrounding countryside. A company undertook to drain it on condition of becoming proprietors of the site when dry; in 1854, however, the rights and privileges were purchased by Prince Giulio Torlonia (d. 1886), the great Roman banker, who carried on the work at his own expense until, in 1876, the lake was finally drained at the cost of some £1,700,000. The reclaimed area is 12½ m. long, 7 m. broad, and is cultivated by families from the Torlonia estates. The outlet by which it was drained is 4 m. long and 24 sq. yds. in section. 274

See A. Brisse and L. de Rotron, *Le Desséchement du lac Fucin, exécuté par S.E. le Prince A. Torlonia* (Rome, 1876).

(T. As.)

FUEL (O. Fr. *feuaile*, popular Lat. *focalia*, from *focus*, hearth, fire), a term applicable to all substances that can be usefully employed for the production of heat by combustion. Any element or combination of elements susceptible of oxidation may under appropriate conditions be made to burn; but only those that ignite at a moderate initial temperature and burn with comparative rapidity, and, what is practically of more importance, are obtainable in quantity at moderate prices, can fairly be regarded as fuels. The elementary substances that can be so classed are primarily hydrogen, carbon and sulphur, while others finding more special applications are silicon, phosphorus, and the more readily oxidizable metals, such as iron, manganese, aluminium and magnesium. More important, however, than the elements are the carbohydrates or compounds of carbon, oxygen and hydrogen, which form the bulk of the natural fuels, wood, peat and coal, as well as of their liquid and gaseous derivatives—coal-gas, coal-tar, pitch, oil, &c., which have high values as fuel. Carbon in the elementary form has its nearest representative in the carbonized fuels, charcoal from wood and coke from coal.

Solid Fuels.

Wood.

Wood may be considered as having the following average composition when in the air-dried state: Carbon, 39.6; hydrogen, 4.8; oxygen, 34.8; ash, 1.0; water, 20%. When it is freshly felled, the water may be from 18 to 50%. Air-dried or even green wood ignites readily when a considerable surface is exposed to the kindling flame, but in large masses with regular or smooth surfaces it is often difficult to get it to burn. When previously torrefied or scorched by heating to a temperature of about 200°, at which incipient charring is set up, it is exceedingly inflammable. The ends of imperfectly charred boughs from the charcoal heaps in this condition are used in Paris and other large towns in France for kindling purposes, under the name of *fumerons*. The inflammability, however, varies with the density,—the so-called hard woods, oak, beech and maple, taking fire less readily than the softer, and, more especially, the coniferous varieties rich in resin. The calorific power of absolutely dry woods may as an average be taken at about 4000 units, and when air-dried, *i.e.* containing 25% of water, at 2800 to 3000 units. Their evaporative values, *i.e.* the quantities of water evaporated by unit weight, are 3.68 and 4.44.

Wood being essentially a flaming fuel is admirably adapted for use with heat-receiving surfaces of large extent, such as locomotive and marine boilers, and is also very clean in use. The absence of all cohesion in the cinders or unburnt carbonized residue causes a large amount of ignited particles to be projected from the chimney, when a rapid draught is used, unless special spark-catchers of wire gauze or some analogous contrivance are used. When burnt in open fireplaces the volatile products given off in the apartment on the first heating have an acrid penetrating odour, which is, however, very generally considered to be agreeable. Owing to the large amount of water present, no very high temperatures can be obtained by the direct combustion of wood, and to produce these for metallurgical purposes it is necessary to convert it previously either into charcoal or into inflammable gas.

Peat.

Peat includes a great number of substances of very unequal fuel value, the most recently formed spongy light brown kind approximating in composition to wood, while the dense pitchy brown compact substance, obtained from the bottom of bogs of ancient formation, may be compared with lignite or even in some instances with coal. Unlike wood, however, it contains incombustible matter in variable but large quantity, from 5 to 15% or even more. Much of this, when the amount is large, is often due to sand mechanically intermixed; when air-dried the proportion of water is from 8 to 20%. When these constituents are deducted the average composition may be stated to be—carbon, 52 to 66; hydrogen, 4.7 to 7.4; oxygen, 28 to 39; and nitrogen, 1.5 to 3%. Average air-dried peat may be taken as having a calorific value of 3000 to 3500 units, and when dried at 100° C., and with a minimum of ash (4 to 5%), at about 5200 units, or from a quarter to one-third more than that of an equal weight of wood. The lighter and more spongy varieties of peat when air-dried are exceedingly inflammable, firing at a temperature of 200° C.; the denser pulpy kinds ignite less readily when in the natural state, and often require a still higher temperature when prepared by pulping and compression or partial carbonization. Most kinds burn with a red smoky flame, developing a very strong odour, which, however, has its admirers in the same way that wood smoke has. This arises from the destructive distillation of imperfectly carbonized organic matter. The ash, like that of wood, is light and powdery, except when much sand is present, when it is of a denser character.

Peat is principally found in high latitudes, on exposed high tablelands and treeless areas in more temperate climates, and in the valleys of slow-flowing rivers,—as in Ireland, the west of Scotland, the tableland of Bavaria, the North German plain, and parts of the valleys of the Somme, Oise and a few other rivers in northern France. A principal objection to its use is its extreme bulk, which for equal evaporative effect is from 8 to 18 times that of coal. Various methods have been proposed, and adopted more or less successfully, for the purpose of increasing the density of raw peat by compression, either with or without pulping; the latter process gives the heaviest products, but the improvement is scarcely sufficient to compensate for the cost.

Lignite.

Lignite or brown coal is of intermediate character between peat and coal proper. The best kinds are undistinguishable in quality from free-burning coals, and the lowest earthy kinds are not equal to average peat. When freshly raised, the proportion of water may be from 45 to 50% and even more, which is reduced from 28 to 20% by exposure to dry air. Most varieties, however, when fully dried, break up into powder, which considerably diminishes their utility as fuel, as they cannot be consolidated by coking. Lignite dust may, however, be compacted into serviceable blocks for burning, by pressure in machines similar to those used for brickmaking, either in the wet state as raised from the mines or when kiln-dried at 200° C. This method was adopted to a very large extent in Prussian Saxony. The calorific value varies between 3500 and 5000 units, and the evaporative factor from 2.16 when freshly raised to 5.84 for the best kinds of lignite when perfectly dried.

Other natural fuels.

Of the other natural fuels, apart from coal (*q.v.*), the most important is so-called vegetable refuse, such as cotton stalks, brushwood, straw, and the woody residue of sugar-cane after the extraction of the saccharine juice known as megasse or cane trash. These are extensively used in countries where wood and coal are scarce, usually for providing steam in the manufactures where they arise, *e.g.* straw for thrashing, cotton stalks for ploughing, irrigating, or working presses, and cane trash for boiling down sugar or driving the cane mill. According to J. Head (*Proc. Inst. of Civil Engineers*, vol. xlviii. p. 75), the evaporative values of 1 ℔ of these different articles when burnt in a tubular boiler are—coal, 8 ℔; dry peat, 4 ℔; dry wood, 3.58-3.52 ℔; cotton stalks or megasse, 3.2-2.7 ℔; straw, 2.46-2.30 ℔. Owing to the siliceous nature of the ash of straw, it is desirable to have a means of clearing the grate bars from slags and clinkers at short intervals, and to use a steam jet to clear the tubes from similar deposits.

The common fuel of India and Egypt is derived from the dung of camels and oxen, moulded into thin cakes, and dried in the sun. It has a very low heating power, and in burning gives off acrid ammoniacal smoke and vapour.

Somewhat similar are the tan cakes made from spent tanners’ bark, which are used to some extent in eastern France and in Germany. They are made by moulding the spent bark into cakes, which are then slowly dried by exposure to the air. Their effect is about equivalent to 80 and 30% of equal weights of wood and coal respectively.

Sulphur, phosphorus and silicon, the other principal combustible elements, are only of limited application as fuels. The first is used in the liquidation of sulphur-bearing rocks. The ore is piled into large heaps, which are ignited at the bottom, a certain proportion, from one-fourth to one-third, of the sulphur content being sacrificed, in order to raise the mass to a sufficient temperature to allow the remainder to melt and run down to the collecting basin. Another application is in the so-called “pyritic smelting,” where ores of copper (*q.v.*) containing iron pyrites, FeS_2, are smelted with appropriate fluxes in a hot blast, without preliminary roasting, the sulphur and iron of the pyrites giving sufficient heat by oxidation to liquefy both slag and metal. Phosphorus, which is of value from its low igniting point, receives its only application in the manufacture of lucifer matches. The high temperature produced by burning phosphorus is in part due to the product of combustion (phosphoric acid) being solid, and therefore there is less heat absorbed than would be the case with a gaseous product. The same effect is observed in a still more striking manner with silicon, which in the only special case of its application to the production of heat, namely, in the Bessemer process of steel-making, gives rise to an enormous increase of temperature in the metal, sufficient indeed to keep the iron melted. The absolute calorific value of silicon is lower than that of carbon, but the product of combustion (silica) being non-volatile at all furnace temperatures, the whole of the heat developed is available for heating the molten iron, instead of a considerable part being consumed in the work of volatilization, as is the case with carbonic oxide, which burns to waste in the air.

Assay and Valuation of Carbonaceous Fuels.—The utility or value of a fuel depends upon two principal factors, namely, its calorific power and its calorific intensity or pyrometric effect, that is, the sensible temperature of the products of combustion. The first of these is constant for any particular product of combustion independently of the method by which the burning is effected, whether by oxygen, air or a reducible metallic oxide. It is most conveniently determined in the laboratory by measuring the heat evolved during the combustion of a given weight of the fuel. The method of Lewis Thompson is one of the most useful. The calorimeter consists of a copper cylinder in which a weighed quantity of coal intimately mixed with 10-12 parts of a mixture of 3 parts of potassium chlorate and 1 of potassium nitrate is deflagrated under a copper case like a diving-bell, placed at the bottom of a deep glass jar filled with a known weight of water. The mixture is fired by a fuse of lamp-cotton previously soaked in a nitre solution and dried. The gases produced by the combustion rising through the water are cooled, with a corresponding increase of temperature in the latter, so that the difference between the temperature observed before and after the experiment measures the heat evolved. The instrument is so constructed that 30 grains (2 grammes) of coal are burnt in 29,010 grains of water, or in the proportion of 1 to 937, these numbers being selected that the observed rise of temperature in Fahrenheit degrees corresponds to the required evaporative value in pounds, subject only to a correction for the amount of heat absorbed by the mass of the instrument, for which a special coefficient is required and must be experimentally determined. The ordinary bomb calorimeter is also used. An approximate method is based upon the reduction of lead oxide by the carbon and hydrogen of the coal, the amount of lead reduced affording a measure of the oxygen expended, whence the heating power may be calculated, 1 part of pure carbon being capable of producing 34½ times its weight of lead. The operation is performed by mixing the weighed sample with a large excess of litharge in a crucible, and exposing it to a bright red heat for a short time. After cooling, the crucible is broken and the reduced button of lead is cleaned and weighed. The results obtained by this method are less accurate with coals containing much disposable hydrogen and iron pyrites than with those approximating to anthracite, as the heat equivalent of the hydrogen in excess of that required to form water with the oxygen of the coal is calculated as carbon, while it is really about four times as great. Sulphur in iron pyrites also acts as a reducing agent upon litharge, and increases the apparent effect in a similar manner.

Calorific power.

The evaporative power of a coal found by the above methods, and also by calculating the separate calorific factors of the components as determined by the chemical analysis, is always considerably above that obtained by actual combustion under a steam boiler, as in the latter case numerous sources of loss, such as imperfect combustion of gases, loss of unburnt coal in cinders, &c., come into play, which cannot be allowed for in laboratory experiments. It is usual, therefore, to determine the value of a coal by the combustion of a weighed quantity in the furnace of a boiler, and measuring the amount of water evaporated by the heat developed.

In a research upon the heating power and other properties of coal for naval use, carried out by the German admiralty, the results tabulated below were obtained with coals from different localities.

	Slag left in Grate.	Ashes in Ashpit.	Soot in Flues.	Water evaporated by 1 ℔ of Coal
Westphalian gas coals	0.33-6.42	2.83-6.53	0.32-0.46	6.60-7.45 ℔
Do. bituminous coals	0.98-9.10	1.97-9.63	0.24-0.88	7.30-8.66
Do. dry coals	1.93-5.70	4.37-10.63	0.24-0.48	7.03-8.51
Silesian coals	0.92-1.30	3.15-3.50	0.24-0.30	6.73-7.10
Welsh steam coals	1.20-4.07	4.07	0.32	8.41
Newcastle coals	1.92	2.57	0.35	7.28

The heats of combustion of elements and compounds will be found in most of the larger works on physical and chemical constants; a convenient series is given in the *Annuaire du Bureau des Longitudes*, appearing in alternate years. The following figures for the principal fuel elements are taken from the issue for 1908; they are expressed in gramme “calories” or heat units, signifying the weight of water in grammes that can be raised 1° C. in temperature by the combustion of 1 gramme of the substance, when it is oxidized to the condition shown in the second column:

Element.	Product of Combustion.	Calories.
Hydrogen	Water, H_2O, condensed to liquid	34,500
	” as vapour	29,650
Carbon—		
Diamond	Carbon Dioxide, CO_2	7,868
Graphite	” ”	7,900
Amorphous	” ”	8,133
Silicon—		
Amorphous	Silicon Dioxide, SiO_2	6,414
Crystallized	” ”	6,570
Phosphorus	Phosphoric pentoxide, P_2O_5	5,958
Sulphur	Sulphur dioxide, SO_2, gaseous	2,165

The results may also be expressed in terms of the atomic equivalent of the combustible by multiplying the above values by the atomic weight of the substance, 12 for carbon, 28 for silicon, &c.

In all fuels containing hydrogen the calorific value as found by the calorimeter is higher than that obtainable under working conditions by an amount equal to the latent heat of volatilization of water which reappears as heat when the vapour is condensed, though under ordinary conditions of use the vapour passes away uncondensed. This gives rise to the distinction of higher and lower calorific values for such substances, the latter being those generally used in practice. The differences for the more important compound gaseous fuels are as follows:—

	Calorific Value.	
	Higher.	Lower.
Acetylene, C_2H_2	11,920	11,500
Ethylene, C_2H_4	11,880	11,120
Methane, CH_4	13,240	11,910
Carbon monoxide, CO	2,440	2,440

Caloric intensity.

The calorific intensity or pyrometric effect of any particular fuel depends upon so many variable elements that it cannot be determined except by actual experiment. The older method was to multiply the weight of the products of combustion by their specific heats, but this gave untrustworthy results as a rule, on account of two circumstances—the great increase in specific heat at high temperatures in compound gases such as water and carbon dioxide, and their instability when heated to 1800° or 2000°. At such temperatures dissociation to a notable extent takes place, especially with the latter substance, which is also readily reduced to carbon monoxide when brought in contact with carbon at a red heat—a change which is attended with a large heat absorption. This effect is higher with soft kinds of carbon, such as charcoal or soft coke, than with dense coke, gas retort carbon or graphite. These latter substances, therefore, are used when an intense local heat is required, as for example, in the Deville furnace, to which air is supplied under pressure. Such a method is, however, only of very special application, the ordinary method being to supply air to the fire in excess of that required to burn the fuel to prevent the reduction of the carbon dioxide. The volume of flame, however, is increased by inert gas, and there is a proportionate diminution of the heating effect. Under the most favourable conditions, when the air employed has been previously raised to a high temperature and pressure, the highest attainable flame temperature from carbonaceous fuel seems to be about 2100°-2300° C.; this is realized in the bright spots or “eyes” of the tuyeres of blast furnaces.

Very much higher temperatures may be reached when the products of combustion are not volatile, and the operation can be effected by using the fuel and oxidizing agent in the proportions exactly required for perfect combustion and intimately mixed. These conditions are met in the “Thermit” process of Goldschmidt, where finely divided aluminium is oxidized by the oxide of some similar metal, such as iron, manganese or chromium, the reaction being started by a primer of magnesium and barium peroxide. The reaction is so rapidly effected that there is an enormous rise in temperature, estimated to be 5400° F. (3000° C.), which is sufficient to melt the most refractory metals, such as chromium. The slag consists of alumina which crystallizes in the forms of corundum and ruby, and is utilized as an abrasive under the name of corubin. 276

The chemical examination includes the determination of (1) moisture, (2) ash, (3) coke, (4) volatile matter, (5) fixed carbon in coke, (6) sulphur, (7) chlorine, (8) phosphorus. Moisture is determined by noting the loss in weight when a sample is heated at 100° for about one hour. The ash is determined by heating a sample in a muffle furnace until all the combustible matter has been burnt off. The ash, which generally contains silica, oxides of the alkaline earths, ferric oxide (which gives the ash a red colour), sulphur, &c., is analysed by the ordinary gravimetric methods. The determination of coke is very important on account of the conclusions concerning the nature of the coal which it permits to be drawn. A sample is finely powdered and placed in a covered porcelain crucible, which is surrounded by an outer one, the space between them being packed with small coke. The crucibles are heated in a wind furnace for 1 to 1½ hours, then allowed to cool, the inner crucible removed, and the coke weighed. The coke may be (1) pulverulent, (2) slightly fritted, (3) spongy and swelled, (4) compact. Pulverulent cokes indicate a non-caking bituminous coal, rich in oxygen if the amount be below 60%, but if the amount be very much less it generally indicates a lignite; if the amount be above 80% it indicates an anthracite containing little oxygen or hydrogen. A fritted coke indicates a slightly coking coal, while the spongy appearance points to a highly coking coal which has been partly fused in the furnace. A compact coke is yielded by good coking coals, and is usually large in amount. The volatile matters are determined as the loss of weight on coking less the amount of moisture. The “fixed carbon” is the carbon retained in the coke, which contains in addition the ash already determined. The fixed carbon is therefore the difference between the coke and the ash, and may be determined from these figures; or it may be determined directly by burning off the coke in a muffle and noting the loss in weight. Sulphur may be present as (1) organic sulphur, (2) as iron pyrites or other sulphides, (3) as the sulphates of calcium, aluminium and other metals; but the amount is generally so small that only the total sulphur is determined. This is effected by heating a mixture of the fuel with lime and sodium carbonate in a porcelain dish to redness in a muffle until all the carbonaceous matter has been burnt off. The residue, which contains the sulphur as calcium sulphate, is transferred to a beaker containing water to which a little bromine has been added. Hydrochloric acid is carefully added, the liquid filtered and the residue washed. To the filtrate ammonia is added, and then barium chloride, which precipitates the sulphur as barium sulphate. Sulphur existing in the form of sulphates may be removed by washing a sample with boiling water and determining the sulphuric acid in the solution. The washed sample is then fused in the usual way to determine the proportion of sulphur existing as iron pyrites. The distinction between sulphur present as sulphate and sulphide is of importance in the examination of coals intended for iron smelting, as the sulphates of the earthy metals are reduced by the gases of the furnace to sulphides, which pass into the slag without affecting the quality of the iron produced, while the sulphur of the metallic sulphides in the ash acts prejudicially upon the metal. Coals for gas-making should contain little sulphur, as the gases produced in the combustion are noxious and have very corrosive properties. Chlorine is rarely determined, but when present in quantity it corrodes copper and brass boiler tubes, with which consequently chlorine-bearing coals cannot be used. The element is determined by fusing with soda lime in a muffle, dissolving the residue in water and precipitating with silver nitrate. Phosphorus is determined in the ash by fusing it with a mixture of sodium and potassium carbonates, extracting the residue with hydrochloric acid, and twice evaporating to dryness with the same acid. The residue is dissolved in hydrochloric acid, a few drops of ferric chloride added, and then ammonia in excess. The precipitate of ferric phosphate is then treated as in the ordinary estimation of phosphates. If it be necessary to determine the absolute amount of carbon and hydrogen in a fuel, the dried sample is treated with copper oxide as in the ordinary estimation of these elements in organic compounds.

(H. B.)

Vegetable oil is not used for fuel except for laboratory purposes, partly because its constituent parts are less adaptable for combustion under the conditions necessary for steam-raising, but chiefly because of the commercial difficulty of producing it with sufficient economy to compete with mineral fuel either solid or liquid.

The use of petroleum as fuel had long been recognized as a scientific possibility, and some attempts had been made to adopt it in practice upon a commercial scale, but the insufficiency, and still more the irregularity, of the supplies prevented it from coming into practical use to any important extent until about 1898, when discoveries of oil specially adapted by chemical composition for fuel purposes changed the aspect of the situation. These discoveries of special oil were made first in Borneo and later in Texas, and experience in treating the oils from both localities has shown that while not less adapted to produce kerosene or illuminating oil, they are better adapted to produce fuel oil than either the Russian or the Pennsylvanian products. Texas oil did not hold its place in the market for long, because the influx of water into the wells lowered their yield, but discoveries of fuel oil in Mexico have come later and will help to maintain the balance of the world's supply, although this is still a mere fraction of the assured supply of coal.

With regard to the chemical properties of petroleum, it is not necessary to say more in the present place than that the lighter and more volatile constituents, known commercially as naphtha and benzene, must be removed by distillation in order to leave a residue composed principally of hydrocarbons which, while containing the necessary carbon for combustion, shall be sufficiently free from volatile qualities to avoid premature ignition and consequent danger of explosion. Attempts have been made to use crude oil for fuel purposes, and these have had some success in the neighbourhood of the oil wells and under boilers of unusually good ventilation both as regards their chimneys and the surroundings of their stokeholds; but for reasons both of commerce and of safety it is not desirable to use crude oil where some distillation is possible. The more complete the process of distillation, and the consequent removal of the volatile constituents, the higher the flash-point, and the more turgid and viscous is the fuel resulting; and if the process is carried to an extreme, the residue or fuel becomes difficult to ignite by the ordinary process of spraying or atomizing mechanically at the moment immediately preceding combustion. The proportions which have been found to work efficiently in practice are as follows:—

Carbon	88.00 %
Hydrogen	10.75 %
Oxygen	1.25 %
Total	100

The standards of safety for liquid fuel as determined by flash-point are not yet finally settled, and are changing from time to time. The British admiralty require a flash-point of 270° F., and to this high standard, and the consequent viscosity of the fuel used by vessels in the British fleet, may partly be attributed the low rate of combustion that was at first found possible in them. The German admiralty have fixed a flash-point of 187° F., and have used oil of this standard with perfect safety, and at the same time with much higher measure of evaporative duty than has been attained in British war-vessels. In the British mercantile marine Lloyd's Register has permitted fuel with a flash-point as low as 150° F. as a minimum, and no harm has resulted. The British Board of Trade, the department of the government which controls the safety of passenger vessels, has fixed a higher standard upon the basis of a minimum of 185°. In the case of locomotives the flash-point as a standard of safety is of less importance than in the case of stationary or marine boilers, because the storage is more open, and the ventilation, both of the storage tanks and the boilers during combustion, much more perfect than in any other class of steam-boilers.

The process of refining by distillation is also necessary to reduce two impurities which greatly retard storage and combustion, *i.e.* water and sulphur. Water is found in all crude petroleum as it issues from the wells, and sulphur exists in important quantities in oil from the Texas wells. Its removal was at first found very expensive, but there no longer exists difficulty in this respect, and large quantities of petroleum fuel practically free from sulphur are now regularly exported from Texas to New York and to Europe.

Water mixed with fuel is in intimate mechanical relation, and frequently so remains in considerable 277
quantities even after the process of distillation. It is in fact so thoroughly mixed as to form an emulsion. The effect of feeding such a mixture into a furnace is extremely injurious, because the water must be decomposed chemically into its constituents, hydrogen and oxygen, thus absorbing a large quantity of heat which would otherwise be utilized for evaporation. Water also directly delays combustion by producing from the jet a long, dull, red flame instead of a short bright, white flame, and the process of combustion, which should take place by vaporization of the oil near the furnace mouth, is postponed and transferred to the upper part of the combustion-box, the tubes, and even the base of the chimney, producing loss of heat and injury to the boiler structure. The most effective means of ridding the fuel of this dangerous impurity is by heat and settlement. The coefficients of expansion of water and oil by heat are substantially different, and a moderate rise of temperature therefore separates the particles and precipitates the water, which is easily drawn off—leaving the oil available for use. The heating and precipitation are usually performed upon a patented system of settling tanks and heating apparatus known as the Flannery-Boyd system, which has proved itself indispensable for the successful use at sea of petroleum fuel containing any large proportion of water.

Progress of liquid fuel.

The laboratory and mechanical use of petroleum for fuel has already been referred to, but it was not until the year 1870 that petroleum was applied upon a wider and commercial scale. In the course of distillation of Russian crude petroleum for the production of kerosene or lamp oil, large quantities of refuse were produced—known by the Russian name of *astatki*—and these were found an incumbrance and useless for any commercial purpose. To a Russian oil-refiner gifted with mechanical instinct and the genius for invention occurred the idea of utilizing the waste product as fuel by spraying or atomizing it with steam, so that, the thick and sluggish fluid being broken up into particles, the air necessary for combustion could have free access to it. The earliest apparatus for this purpose was a simple piece of gas-tube, into which the thick oil was fed; by another connexion steam at high pressure was admitted to an inner and smaller tube, and, the end of the tube nearest to the furnace being open, the pressure of the steam blew the oil into the furnace, and by its velocity broke it up into spray. The apparatus worked with success from the first. Experience pointed out the proper proportionate sizes for the inlets of steam and oil, the proper pressure for the steam, and the proportionate sizes for the orifices of admission to the furnaces, as well as the sizes of air-openings and best arrangements of fire-bricks in the furnaces themselves; and what had been a waste product now became a by-product of great value. Practically all the steam power in South Russia, both for factories and navigation of the inland seas and rivers, is now raised from *astatki* fuel.

In the Far East, including Burma and parts of China and Japan, the use of liquid fuel spread rapidly during the years 1899, 1900 and 1901, owing entirely to the development of the Borneo oil-fields by the enterprise of Sir Marcus Samuel and the large British corporation known as the Shell Transport and Trading Company, of which he is the head. This corporation has since amalgamated with the Royal Dutch Petroleum Company controlling the extensive wells in Dutch Borneo, and together they supply large quantities of liquid fuel for use in the Far East. In the United States of America liquid fuel is not only used for practically the whole of the manufacturing and locomotive purposes of the state of Texas, but factories in New York, and a still larger number in California, are now discarding the use of coal and adopting petroleum, because it is more economical in its consumption and also more easily handled in transit, and saves nearly all the labour of stoking. So far the supplies for China and Japan have been exported from Borneo, but the discoveries of new oil-fields in California, of a character specially adapted for fuel, have encouraged the belief that it may be possible to supply Chile and Peru and other South American countries, where coal is extremely expensive, with Californian fuel; and it has also found its way across the Pacific to Japan. There are believed to be large deposits in West Africa, but in the meantime the only sources of supply to those parts of Africa where manufacture is progressing, *i.e.* South Africa and Egypt, are the oil-fields of Borneo and Texas, from which the import has well begun, from Texas to Alexandria via the Mediterranean, and from Borneo to Cape Town via Singapore.

In England, notwithstanding the fact that there exist the finest coal-fields in the world, there has been a surprising development of the use of petroleum as fuel. The Great Eastern railway adapted 120 locomotive engines to its use, and these ran with regularity and success both on express passenger and goods trains until the increase in price due to short supply compelled a return to coal fuel. The London, Brighton & South Coast railway also began the adaptation of some of their locomotive engines, but discontinued the use of liquid fuel from the same cause. Several large firms of contractors and cement manufacturers, chiefly on the banks of the Thames, made the same adaptations which proved mechanically successful, but were not continued when the price of liquid fuel increased with the increased demand.

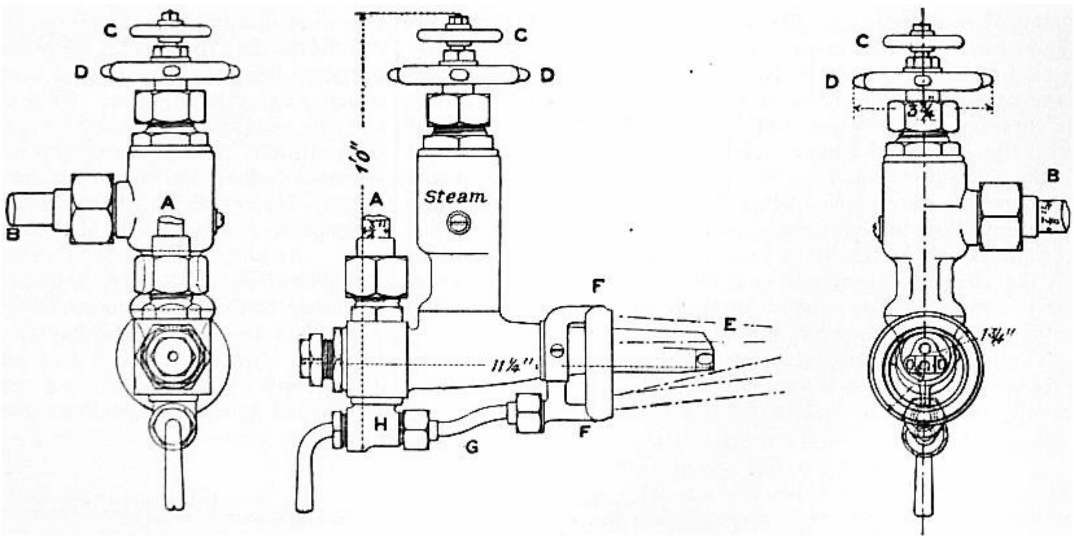


Fig. 1.—Holden Burner.

Economy of liquid fuel.

The chief factors of economy are the greater calorific value of oil than coal (about 16 ℔ of water per ℔ of oil fuel evaporated from a temperature of 212° F.), not only in laboratory practice, but in actual use on a large scale, and the saving of labour both in transit from the source of supply to the place of use and in the act of stoking the furnaces. The use of cranes, hand labour with shovels, wagons and locomotives, horses and carts, is unavoidable for the transit of coal; and labour to trim the coal, to stoke it when under combustion, and to handle the residual ashes, are all indispensable to steam-raising by coal. On the other hand, a system of pipes and pumps, and a limited quantity of skilled labour to manage them, is all that is necessary for the transit and combustion of petroleum 278 fuel; and it is certain that even in England will be found places which, from topographical and other circumstances, will use petroleum more economically than coal as fuel for manufacturing purposes under reasonable conditions of price for the fuel.

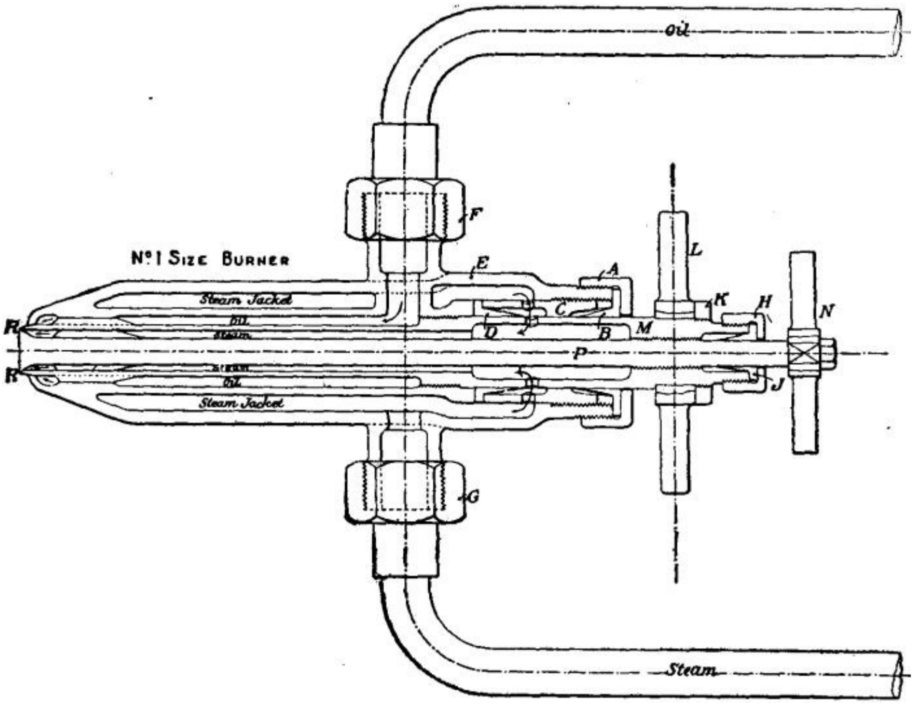


Fig. 2.—Rusden and Eeles Burner.

The theoretical calorific value of oil fuel is more nearly realized in practice than the theoretical calorific value of coal, because the facilities for complete combustion, due to the artificial admixture of the air by the atomizing process, are greater in the case of oil than coal, and for this reason, among others, the practical evaporative results are proportionately higher with liquid fuel. In some cases the work done in a steam-engine by 2 tons of coal has been performed by 1 ton of oil fuel, but in others the proportions have been as 3 to 2, and these latter can be safely relied on in practice as a minimum. This saving, combined with the savings of labour and transit already explained, will in the near future make the use of liquid fuel compulsory, except in places so near to coal-fields that the cost of coal becomes sufficiently low to counterbalance the savings in weight of fuel consumed and in labour in handling it. In some locomotives on the Great Eastern railway the consumption of oil and coal for the same development of horse-power was as 17 ℔ oil is to 35 ℔ coal; all, however, did not realize so high a result.

Liquid fuel in locomotives.

The mechanical apparatus for applying petroleum to steam-raising in locomotives is very simple. The space in the tender usually occupied by coal is closed up by steel-plating closely riveted and tested, so as to form a storage tank. From this tank a feed-pipe is led to a burner of the combined steam-and-oil type already indicated, and this burner is so arranged as to enter a short distance inside the furnace mouth. The ordinary fire-bars are covered with a thin layer of coal, which starts the ignition in the first place, and the whole apparatus is ready for work. The burner best adapted for locomotive practice is the Holden Burner (fig. 1), which was used on the Great Eastern railway. The steam-pipe is connected at A, the oil-pipe at B, and the hand-wheels C and D are for the adjustment of the internal orifices according to the rate of combustion required. The nozzle E is directed towards the furnace, and the external ring FF, supplied by the small pipe G and the by-pass valve H, projects a series of steam jets into the furnace, independent of the injections of atomized fuel, and so induces an artificial inrush of air for the promotion of combustion. This type of burner has also been tried on stationary boilers and on board ship. It works well, although the great consumption of steam by the supplementary ring is a difficulty at sea, where the water lost by the consumption of steam cannot easily be made up.

Liquid fuel at sea.

Although the application of the new fuel for land and locomotive boilers has already been large, the practice at sea has been far more extensive. The reason is chiefly to be found in the fact that although the sources of supply are at a distance from Great Britain, yet they are in countries to whose neighbourhood British steamships regularly trade, and in which British naval squadrons are regularly stationed, so that the advantages of adopting liquid fuel have been more immediate and the economy more direct. The certainty of continuous supply of the fuel and the wide distribution of storage stations have so altered the conditions that the general adoption of the new fuel for marine purposes becomes a matter of urgency for the statesman, the merchant and the engineer. None of these can afford to neglect the new conditions, lest they be noted and acted upon by their competitors. Storage for supply now exists at a number of sea ports: London, Barrow, Southampton, Amsterdam, Copenhagen, New Orleans, Savannah, New York, Philadelphia, Singapore, Hong Kong, Madras, Colombo, Suez, Hamburg, Port Arthur, Rangoon, Calcutta, Bombay, Alexandria, Bangkok, Saigon, Penang, Batavia, Surabaya, Amoy, Swatow, Fuchow, Shanghai, Hankow, Sydney, Melbourne, Adelaide, Zanzibar, Mombasa, Yokohama, Kobe and Nagasaki; also in South African and South American ports.

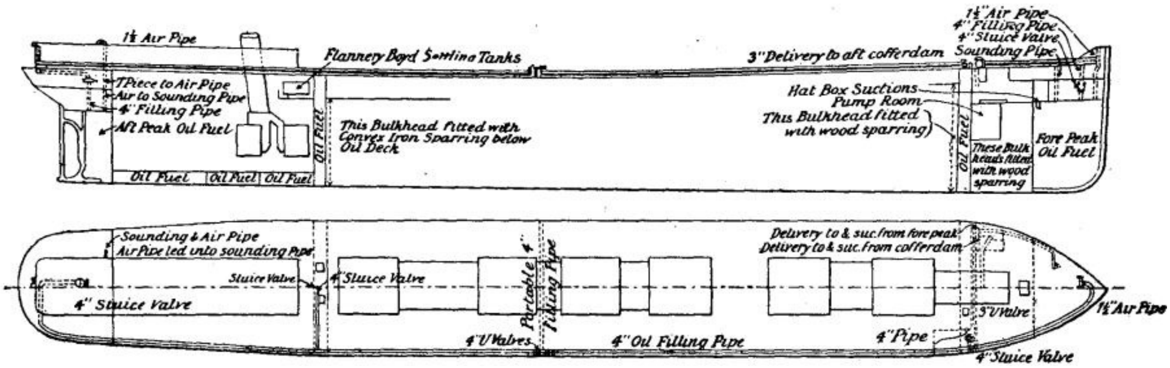


Fig. 3.—Storage of Liquid Fuel on Oil-carrying Steamers (Flannery-Boyd System).

The British admiralty have undertaken experiments with liquid fuel at sea, and at the same time investigations of the possibility of supply from sources within the regions of the British empire. There is an 279 enormous supply of shale under the north-eastern counties of England, but no oil that can be pumped—still less oil with a pressure above it so as to “gush” like the wells in America—and the only sources of liquid supply under the British flag appear to be in Burma and Trinidad. The Borneo fields are not under British control, although developed entirely by British capital. The Italian admiralty have fitted several large warships with boiler apparatus to burn petroleum. The German admiralty are regularly using liquid fuel on the China station. The Dutch navy have fitted coal fuel and liquid fuel furnaces in combination, so that the smaller powers required may be developed by coal alone, and the larger powers by supplementing coal fuel with oil fuel. The speeds of some vessels of the destroyer type have by this means been accelerated nearly two knots.

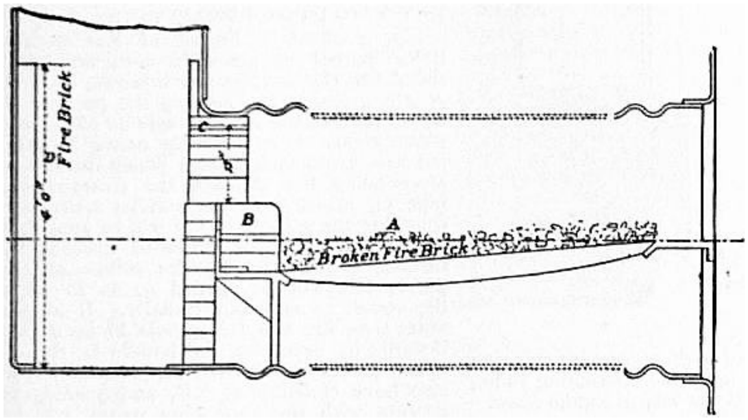


Fig. 4.—Installation on ss. “Trochas.”

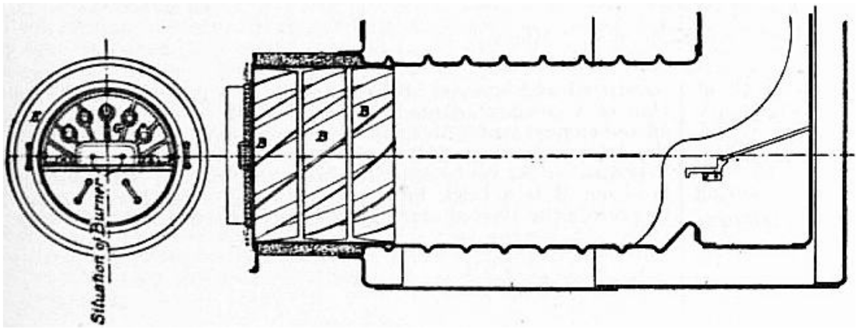


Fig. 5.—Details of Furnace, Meyer System.

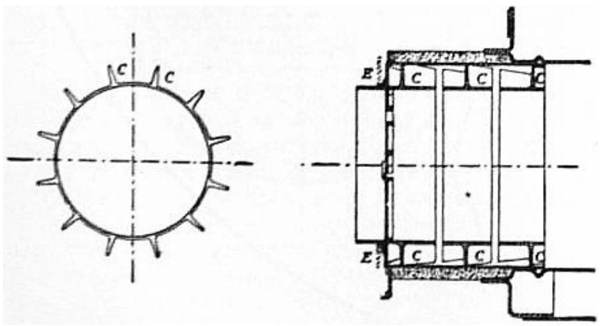


Fig. 6.—Details of Exterior Elongation of Furnace, Meyer System.

Advantages

The questions which govern the use of fuel in warships are more largely those of strategy and fighting efficiency than economy of evaporation. Indeed, the cost of constructing and maintaining in fighting efficiency a modern warship is so great that the utmost use strategically must be obtained from the vessel, and in this comparison the cost of fuel is relatively so small an item that its increase *in warships.* or decrease may be considered almost a negligible quantity. The desideratum in a warship is to obtain the greatest fighting efficiency based on the thickest armour, the heaviest and most numerous guns, the highest maximum speed, and, last and not least, the greatest range of effective action based upon the maximum supplies of fuel, provisions and other consumable stores that the ship can carry. Now, if by changing the type of fuel it be possible to reduce its weight by 30%, and to abolish the stokers, who are usually more than half the ship’s company, the weight saved will be represented not merely by the fuel, but by the consumable stores otherwise necessary for the stokers. Conversely, the radius of effective action of the ship will be doubled as regards consumable stores if the crew be halved, and will be increased by 50% if the same weight of fuel be carried in the form of liquid instead of coal. In space the gain by using oil fuel is still greater, and 36 cubic feet of oil as stored are equal in practical calorific value to 67 cubic feet of coal according to the allowance usual for ship’s bunkering. On the other hand, coal has been relied upon, when placed in the side bunkers of unarmoured ships, as a protection against shot and shell, and this advantage, if it really exists, could not be claimed in regard to liquid fuel.

Recent experiments in coaling warships at sea have not been very successful, as the least bad weather has prevented the safe transmission of coal bags from the collier to the ship. The same difficulty does not exist for oil fuel, which has been pumped through flexible tubing from one ship to the other even in comparatively rough weather. Smokelessness, so important a feature of sea strategy, has not always been attained by liquid fuel, but where the combustion is complete, by reason of suitable furnace arrangements and careful management, there is no smoke. The great drawback, however, to the use of liquid fuel in fast small vessels is the confined space allotted to the boilers, such confinement being unavoidable in view of the high power concentrated in a small hull. The British admiralty’s experiments, however, have gone far to solve the problem, and the quantity of oil which can be consumed by forced draught in confined boilers now more nearly equals the quantity of coal consumed under similar conditions. All recent vessels built for the British navy are so constructed that the spaces between their double bottoms are oil-tight and capable of storing liquid fuel in the tanks so formed. Most recent battleships and cruisers have also liquid fuel furnace fittings, and in 1910 it already appeared probable that the use of oil fuel in warships would rapidly develop.

In view of recent accusations of insufficiency of coal storage in foreign naval depots, by reason of the allegation that coal so stored quickly perishes, it is interesting to note that liquid fuel may be stored in tanks for an indefinite time without any deterioration whatever.

Advantages in merchant ships. In the case of merchant steamers large progress has also been made. The Shell Transport and Trading Company have twenty-one vessels successfully navigating in all parts of the world and using liquid fuel. The Hamburg-American Steamship Company have four large vessels similarly fitted for oil fuel, which, however, differ in furnace arrangements, as will be hereafter described, although using coal when the fluctuation of the market renders that the more economical fuel. One of the large American transatlantic lines is adopting liquid fuel, and French, German, Danish and American mercantile vessels are also beginning to use it in considerable amounts.

In the case of very large passenger steamers, such as those of 20 knots and upwards in the Atlantic trade, the saving in cost of fuel is trifling compared with the advantage arising from the greater weight and space available for freight. Adopting a basis of 3 to 2 as between coal consumption and oil consumption, there is an increase of 1000 tons of dead weight cargo in even a medium-sized Atlantic steamer, and a collateral gain of 280
about 100,000 cub. ft. of measurement cargo, by reason of the ordinary bunkers being left quite free, and the oil being stored in the double bottom spaces hitherto unutilized except for the purpose of water ballast. The cleanliness and saving of time from bunkering by the use of oil fuel is also an important factor in passenger ships, whilst considerable additional speed is obtainable. The cost of the installation, however, is very considerable, as it includes not only burners and pipes for the furnaces, but also the construction of oil-tight tanks, with pumps and numerous valves and pipe connexions.

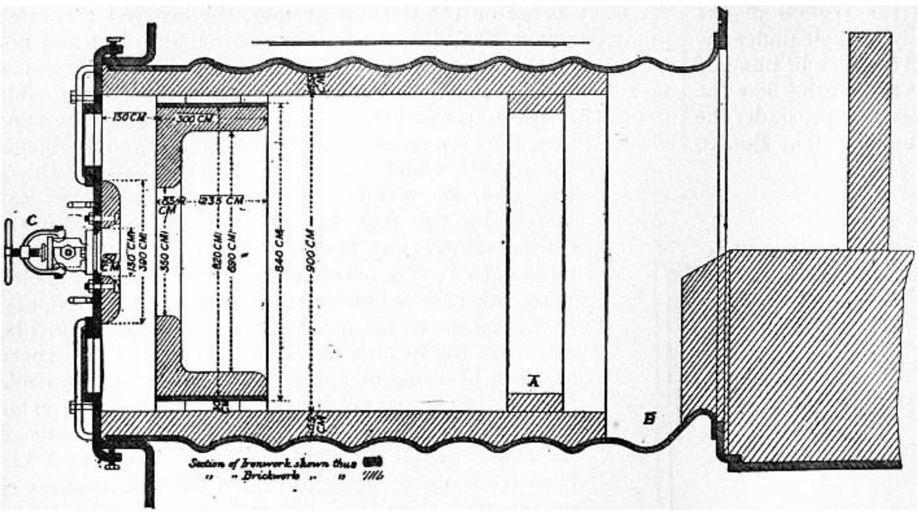


Fig. 7.—Furnace on ss. “Ferdinand Laeisz.” A, it is proposed to do away with this ring of brickwork as being useless; B, it is proposed to fill this space up, thus continuing lining of furnace to combustion chamber, and also to fit protection bricks in way of saddle plate.

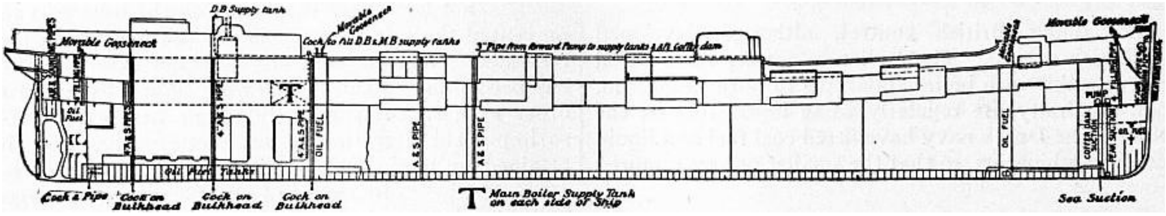


Fig. 8.—Fuel Tanks, &c., of ss. “Murex.”

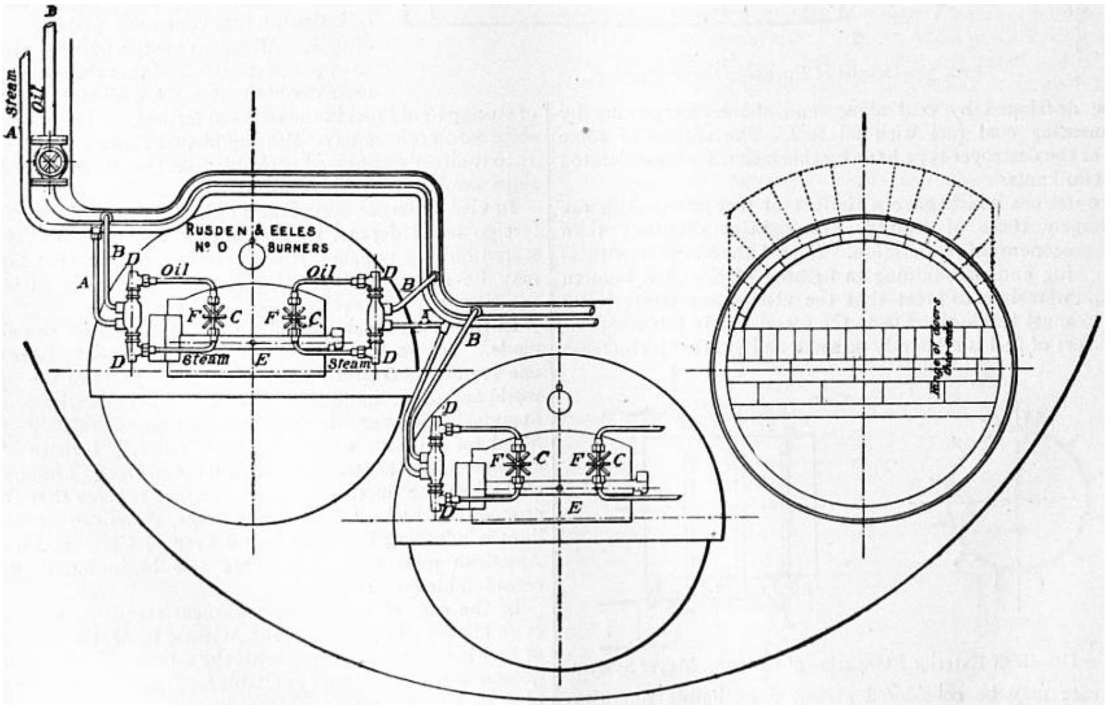


Fig. 9.—Furnace Gear of ss. “Murex.”

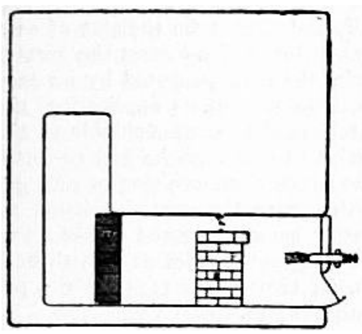


Fig. 10.—Section through Furnace of ss. “Murex.”

Fig. 2 shows a burner of Rusden and Eeles’ patent as generally used on board ships for the purpose of injecting the oil. A is a movable cap holding the packing B, which renders the annular spindle M oil and steam tight. E is the outer casing containing the steam jacket from which the steam, after being fed through the steam-supply pipe G, passes into the annular space surrounding the spindle P. It will be seen that if the spindle P be travelled inwards by turning the handle N, the orifice at the nozzle RR will be opened so as to allow the steam to flow out radially. If at the same time the annular spindle M be drawn inwards by revolving the handle L, the oil which passes through the supply pipe F will also have emission at RR, and, coming in contact with the outflowing steam, will be pulverized and sprayed into the furnace. Fig. 3 is a profile and plan of a steamer adapted for carrying oil in bulk, and showing all the storage arrangements for handling liquid fuel. Fig. 4 shows the interior arrangement of the boiler furnace of the steamship “Trocas.” A is broken fire-brick resting on the ordinary fire-bars, B is a brick bridge, C a casing of fire-brick intended to protect the riveted seam immediately above it from the direct impact of the flame, and D is a lining of fire-brick at the back of the 281
combustion-box, also intended to protect the plating from the direct impact of the petroleum flame. The arrangement of the furnace on the Meyer system is shown in fig. 5, where E is an annular projection built at the mouth of the furnace, and BB are spiral passages for heating the air before it passes into the furnace. Fig. 6 shows the rings CC and details of the casting which forms the projection or exterior elongation of the furnace. The brickwork arrangement adopted for the double-ended boilers on the Hamburg-American Steamship Company’s “Ferdinand Laeisz” is represented in fig. 7. The whole furnace is lined with fire-brick, and the burner is mounted upon a circular disk plate which covers the mouth of the furnace. The oil is injected not by steam pulverization, but by pressure due to a steam-pump. The oil is heated to about 60°C. before entering the pump, and further heated to 90°C. after leaving the pump. It is then filtered, and passes to the furnace injector C at about 30-℔ pressure; and its passage through this injector and the spiral passages of which it consists pulverizes the oil into spray, in which form it readily ignites on reaching the interior of the furnace. The injector is on the Körting principle, that is, it atomizes by fracture of the liquid oil arising from its own momentum under pressure. The advantage of this system as compared with the steam-jet system is the saving of fresh water, the abstraction of which is so injurious to the boiler by the formation of scale.

The general arrangement of the fuel tanks and filling pipes on the ss. “Murex” is shown in fig. 8; and fig. 9 represents the furnace gear of the same vessel, A being the steam-pipe, B the oil-pipe, C the injector, D the swivel upon which the injector is hung so that it may be swung clear of the furnace, E the fire-door, and F the handle for adjusting the injector. In fig. 10, which represents a section of the furnace, H is a fire-brick pier and K a fire-brick baffling bridge.

It is found in practice that to leave out the fire-bars ordinarily used for coal produces a better result with liquid fuel than the alternative system of keeping them in place and protecting them by a layer of broken fire-brick.

Boilers fitted upon all the above systems have been run for thousands of miles without trouble. In new construction it is desirable to give larger combustion chambers and longer and narrower boiler tubes than in the case of boilers intended for the combustion of coal alone.

(F. F.*)

Gaseous Fuel.

Strictly speaking, much, and sometimes even most, of the heating effected by solid or liquid fuel is actually performed by the gases given off during the combustion. We speak, however, of gaseous fuel only in those cases where we supply a combustible gas from the outset, or where we produce from ordinary solid (or liquid) fuel in one place a stream of combustible gas which is burned in another place, more or less distant from that where it has been generated.

The various descriptions of gaseous fuel employed in practice may be classified under the following heads:

I. Natural Gas.

II. Combustible Gases obtained as by-products in various technical operations.

III. Coal Gas (Illuminating Gas).

IV. Combustible Gases obtained by the partial combustion of coal, &c.

I. *Natural Gas.*—From time immemorial it has been known that in some parts of the Caucasus and of China large quantities of gases issue from the soil, sometimes under water, which can be lighted and burn with a luminous flame. The "eternal fires" of Baku belong to this class. In coal-mines frequently similar streams of gas issue from the coal; these are called "blowers," and when they are of somewhat regular occurrence are sometimes conducted away in pipes and used for underground lighting. As a regular source of heating power, however, natural gas is employed only in some parts of the United States, especially in Pennsylvania, Kansas, Ohio and West Virginia, where it always occurs in the neighbourhood of coal and petroleum fields. The first public mention of it was made in 1775, but it was not till 1821 that it was turned to use at Fredonia, N.Y. In Pennsylvania natural gas was discovered in 1859, but at first very little use was made of it. Its industrial employment dates only from 1874, and became of great importance about ten years later. Nobody ever doubted that the gas found in these localities was an accumulation of many ages and that, being tapped by thousands of bore-holes, it must rapidly come to an end. This assumption was strengthened by the fact that the "gas-wells," which at first gave out the gas at a pressure of 700 or 800, sometimes even of 1400 ℔ per sq. in., gradually showed a more and more diminishing pressure and many of them ceased to work altogether. About the year 1890 the belief was fairly general that the stock of natural gas would soon be entirely exhausted. Indeed, the value of the annual production of natural gas in the United States, computed as its equivalent of coal, was then estimated at twenty-one million dollars, in 1895 at twelve millions, in 1899 at eleven and a half millions. But the output rose again to a value of twenty-seven millions in 1901, and to fifty million dollars in 1907. Mostly the gas, derived from upwards of 10,000 gas-wells, is now artificially compressed to a pressure of 300 or 400 ℔ per sq. in. by means of steam-power or gas motors, fed by the gas itself, and is conveyed over great distances in iron pipes, from 9 or 10 to 36 in. in diameter. In 1904 nearly 30,000 m. of pipe lines were in operation. In 1907 the quantity of natural gas consumed in the United States (nearly half of which was in Pennsylvania) was 400,000 million cub. ft., or nearly 3 cub. m. Canada (Ontario) also produces some natural gas, reaching a maximum of about $746,000 in 1907.

The principal constituent of natural gas is always methane, CH_4, of which it contains from 68.4 to 94.0% by volume. Those gases which contain less methane contain all the more hydrogen, viz. 2.9 to 29.8%. There is also some ethylene, ethane and carbon monoxide, rarely exceeding 2 or 3%. The quantity of incombustible gases—oxygen, carbon dioxide, nitrogen—ranges from mere traces to about 5%. The density is from 0.45 to 0.55. The heating power of 1000 cub. ft. of natural gas is equal to from 80 to 120 ℔, on the average 100 ℔, of good coal, but it is really worth much more than this proportion would indicate, as it burns completely, without smoke or ashes, and without requiring any manual labour. It is employed for all domestic and for most industrial purposes.

The origin of natural gas is not properly understood, even now. The most natural assumption is, of course, that its formation is connected with that of the petroleum always found in the same neighbourhood, the latter principally consisting of the higher-boiling aliphatic hydrocarbons of the methane series. But whence do they both come? Some bring them into connexion with the formation of coal, others with the decomposition of animal remains, others with that of *diatomaceae*, &c., and even an inorganic origin of both petroleum and natural gas has been assumed by chemists of the rank of D.I. Mendeléeff and H. Moissan.

II. *Gases obtained as By-products.*—There are two important cases in which gaseous by-products are utilized as fuel; both are intimately connected with the manufacture of iron, but in a very different way, and the gases are of very different composition.

(*a*) *Blast-furnace Gases.*—The gases issuing from the mouths of blast-furnaces (see Iron and Steel) were first utilized in 1837 by Faber du Faur, at Wasseralfingen. Their use became more extensive after 1860, and practically universal after 1870. The volume of gas given off per ton of iron made is about 158,000 cub. ft. Its percentage composition by volume is:

Carbon monoxide	21.6	to	29.0,	mostly	about	26	%
Hydrogen	1.8	″	6.3,	″	″	3	%
Methane	0.1	″	0.8,	″	″	0.5	%
Carbon dioxide	6	″	12,	″	″	9.5	%
Nitrogen	51	″	60,	″	″	56	%
Steam	5	″	12,	″	″	5	%
						———	
						100	%

There is always a large amount of mechanically suspended flue-dust in this gas. It is practically equal to a poor producer-gas (see below), and is everywhere used, first for heating the blast in Cowper stoves or similar apparatus, and secondly for raising all the steam required for the operation of the blast-furnace, that is, for driving the blowing-engines, hoisting the materials, &c. Where the iron ore is roasted previously to being fed into the furnace, this can also be done by this gas, but in some cases the waste in using it is so great that there is not enough left for the last purpose. The calorific power of this gas per cubic foot is from 80 to 120 B.Th.U.

Since about 1900 a great advance has been made in this field. Instead of burning the blast-furnace gas under steam boilers and employing the steam for producing mechanical energy, the gas is directly burned in gas-motors on the explosion principle. Thus upwards of three times the mechanical energy is obtained in comparison with the indirect way through the steam boiler. After all the power required for the operations of the blast-furnace has been supplied, there is a surplus of from 10 to 20 h.p. for each ton of pig-iron made, which may be applied to any other purpose.

(*b*) *Coke-oven Gases.*—Where the coking of coal is performed in the old beehive ovens or similar apparatus the gas issuing at the mouth of the ovens is lost. The attempts at utilizing the gases in such cases have not been very successful. It is quite different where coke is manufactured in the same way as illuminating gas, viz. by the destructive distillation of coal in closed apparatus (retorts), heated from the outside. This industry, which is described in detail in G. Lunge’s *Coal-Tar and Ammonia* (4th ed., 1909), originated in France, but has spread far more in Germany, where more than half of the coke produced is made by it; in the United Kingdom and the United States its progress has been much slower, but there also it has long been recognized as the only proper method. The output of coke is increased by about 15% in comparison with the beehive ovens, as the heat required for the process of distillation is not produced by burning part of the coal itself (as in the beehive ovens), but by burning part of the gas. The quality of the coke for iron-making is quite as good as that of beehive coke, although it differs from it in appearance. Moreover, the gases can be made to yield their ammonia, their tar, and even their benzene vapours, the value of which products sometimes exceeds that of the coke itself. And after all this there is still an excess of gas available for any other purpose.

As the principle of distilling the coal is just the same, whether the object is the manufacture of coal gas proper or of coke as the main product, although there is much difference in the details of the manufacture, it follows that the quality of the gas is very similar in both cases, so far as its heating value is concerned. Of course this heating value is less where the benzene has been extracted from coke-oven gas, since this compound is the richest heat-producer in the gas. This is, however, of minor importance in the present case, as there is only about 1% benzene in these gases.

The composition of coke-oven gases, after the extraction of the ammonia and tar, is about 53% hydrogen, 36% methane, 6% carbon monoxide, 2% ethylene and benzene, 0.5% sulphuretted hydrogen, 1.5% carbon dioxide, 1% nitrogen.

III. *Coal Gas* (*Illuminating Gas*).—Although ordinary coal gas is primarily manufactured for illuminating purposes, it is also extensively used for cooking, frequently also for heating domestic rooms, baths, &c., and to some extent also for industrial operations on a small scale, where cleanliness and exact regulation of the work are of particular importance. In chemical laboratories it is preferred to every other kind of fuel wherever it is available. The manufacture of coal gas being described elsewhere in this work (see Gas, § *Manufacture*), we need here only point out that it is obtained by heating bituminous coal in fireclay retorts and purifying the products of this destructive distillation by cooling, washing and other operations. The residual gas, the ordinary composition of which is given in the table below, amounts to about 10,000 cub. ft. for a ton of coal, and represents about 21% of its original heating value, 56.5% being left in the coke, 5.5% in the tar and 17% being lost. As we must deduct from the coke that quantity which is required for the heating of the retorts, and which, even when good gas producers are employed, amounts to 12% of the weight of the coal, or 10% of its heat value, the total loss of heat rises to 27%. Taking, further, into account the cost of labour, the wear and tear, and the capital interest on the plant, coal gas must always be an expensive fuel in comparison with coal itself, and cannot be thought of as a general substitute for the latter. But in many cases the greater expense of the coal gas is more than compensated by its easy distribution, the facility and cleanliness of its application, the general freedom from the mechanical loss, unavoidable in the case of coal fires, the prevention of black smoke and so forth. The following table shows the average composition of coal gas by volume and weight, together with the heat developed by its single constituents, the latter being expressed in kilogram-calories per cub. metre (0.252 kilogram-calories = 1 British heat unit; 1 cub. metre = 35.3 cub. ft.; therefore 0.1123 calories per cub. metre = 1 British heat unit per cub. foot).

Constituents.	Volume per cent.	Weight per cent.	Heat-value per Cubic Metre Calories.	Heat-value per Quantity contained in 1 Cub. Met.	Heat-value per cent. of Total.
Hydrogen, H_2	47	7.4	2,582	1213	22.8
Methane, CH_4	34	42.8	8,524	2898	54.5
Carbon monoxide, CO	9	19.9	3,043	273	5.1
Benzene vapour, C_6H_6	1.2	7.4	33,815	405	7.7
Ethylene, C_2H_4	3.8	8.4	13,960	530	9.9
Carbon dioxide, CO_2	2.5	8.6	..	..	..
Nitrogen, N_2	2.5	5.5	..	..	..
Total	100.0	100.0	..	5319	100.0

One cubic metre of such gas weighs 568 grammes. *Rich gas*, or gas made by the destructive distillation of certain bituminous schists, of oil, &c., contains much more of the heavy hydrocarbons, and its heat-value is therefore much higher than the above. The carburetted water gas, very generally made in America, and sometimes employed in England for mixing with coal gas, is of varying composition; its heat-value is generally rather less than that of coal gas (see below).

IV. *Combustible Gases produced by the Partial Combustion of Coal, &c.*—These form by far the most important kind of gaseous fuel. When coal is submitted to destructive distillation to produce the illuminating gas described in the preceding paragraph, only a comparatively small proportion of the heating value of the coal (say, a sixth or at most a fifth part) is obtained in the shape of gaseous fuel, by far the greater proportion remaining behind in the shape of coke.

An entirely different class of gaseous fuels comprises those produced by the incomplete combustion of the total carbon contained in the raw material, where the result is a mixture of gases which, being capable of combining with more oxygen, can be burnt and employed for heating purposes. Apart from some descriptions of waste gases belonging to this class (of which the most notable are those from blast-furnaces), we must distinguish two ways of producing such gaseous fuels entirely different in principle, though sometimes combined in one operation. The incomplete combustion of carbon may be brought about by means of atmospheric oxygen, by means of water, or by a simultaneous combination of these two actions. In the first case the chemical reaction is

$$C + O = CO \qquad (a);$$

the nitrogen accompanying the oxygen in the atmospheric air necessarily remains mixed with carbon
monoxide, and the resulting gases, which always contain some carbon dioxide, some products of the 283
destructive distillation of the coal, &c., are known as *producer gas* or *Siemens gas*. In the second case the chemical reaction is mainly

$$C + H_2O = CO + H_2 \qquad (b);$$

that is to say, the carbon is converted into monoxide and the hydrogen is set free. As both of these substances can combine with oxygen, and as there is no atmospheric nitrogen to deal with, the resulting gas (*water gas*) is, apart from a few impurities, entirely combustible. Another kind of water gas is formed by the reaction

$$C + 2H_2O = CO_2 + 2H_2 \qquad (c),$$

but this reaction, which converts all the carbon into the incombustible form of CO_2, is considered as an unwelcome, although never entirely avoidable, concomitant of (*b*).

The reaction by which water gas is produced being endothermic (as we shall see), this gas cannot be obtained except by introducing the balance of energy in another manner. This might be done by heating the apparatus from without, but as this method would be uneconomical, the process is carried out by alternating the endothermic production of water gas with the exothermic combustion of carbon by atmospheric air. Pure water gas is not, therefore, made by a continuous process, but alternates with the production of other gases, combustible or not. But instead of constantly interrupting the process in this way, a continuous operation may be secured by simultaneously carrying on both the reactions (*a*) and (*b*) in such proportions that the heat generated by (*a*) at least equals the heat absorbed by (*b*). For this purpose the apparatus is fed at the same time with atmospheric air and with a certain quantity of steam, preferably in a superheated state. Gaseous mixtures of this kind have been made, more or less intentionally, for a long time past. One of the best known of them, intended less for the purpose of serving as ordinary fuel than for that of driving machinery, is the Dowson gas.

An advantage common to all kinds of gaseous fuel, which indeed forms the principal reason why it is intentionally produced from solid fuel, in spite of inevitable losses in the course of the operation, is the following. The combustion of solid fuel (coal, &c.) cannot be carried on with the theoretically necessary quantity of atmospheric air, but requires a considerable excess of the latter, at least 50%, sometimes 100% and more. This is best seen from the analyses of smoke gases. If all the oxygen of the air were converted into CO_2 and H_2O, the amount of CO_2 in the smoke gases should be in the case of pure carbon nearly 21 volumes %, as carbon dioxide occupies the same volume as oxygen; while ordinary coal, where the hydrogen takes up a certain quantity of oxygen as well, should show about 18.5% CO_2. But the best smoke gases of steam boilers show only 12 or 13%, much more frequently only 10% CO_2, and gases from reverberatory furnaces often show less than 5%. This means that the volume of the smoke gases escaping into the air is from 1½ to 2 times (in the case of high-temperature operations often 4 times) greater than the theoretical minimum; and as these gases always carry off a considerable quantity of heat, the loss of heat is all the greater the less complete is the utilization of the oxygen and the higher the temperature of the operation. This explains why, in the case of the best-constructed steam-boiler fires provided with heat economizers, where the smoke gases are deprived of most of their heat, the proportion of the heat value of the fuel actually utilized may rise to 70 or even 75%, while in some metallurgical operations, in glass-making and similar cases, it may be below 5%.

One way of overcoming this difficulty to a certain extent is to reduce the solid fuel to a very fine powder, which can be intimately mixed with the air so that the consumption of the latter is only very slightly in excess of the theoretical quantity; but this process, which has been only recently introduced on a somewhat extended scale, involves much additional expense and trouble, and cannot as yet be considered a real success. Generally, too, it is far less easily applied than gaseous fuel. The latter can be readily and intimately mixed with the exact quantity of air that is required and distributed in any suitable way, and much of the waste heat can be utilized for a preliminary heating of the air and the gas to be burned by means of "recuperators."

We shall now describe the principal classes of gaseous fuel, produced by the partial combustion of coal.

A. *Producer Gas, Siemens Gas.*—As we have seen above, this gas is made by the incomplete combustion of fuel. The materials generally employed for its production are anthracite, coke or other fuels which are not liable to cake during the operation, and thus stop the draught or otherwise disturb the process, but by special measures also bituminous coal, lignite, peat and other fuel may be utilized for gas producers. The fuel is arranged in a deep layer, generally from 4 ft. up to 10 ft., and the air is introduced from below, either by natural draught or by means of a blast, and either by a grate or only by a slit in the wall of the "gas producer.” Even if the primary action taking place at the entrance of the air consisted in the complete combustion of the carbon to dioxide, CO_2, the latter, in rising through the high column of incandescent fuel, must be reduced to monoxide: $CO_2 + C = 2CO$. But as the temperature in the producer rises rather high, and as in ordinary circumstances the action of oxygen on carbon above 1000° C. consists almost entirely in the direct formation of CO, we may regard this compound as primarily formed in the hotter parts of the gas-producer. It is true that ordinary producer gas always contains more or less CO_2, but this may be formed higher up by air entering through leakages in the apparatus. If we ignore the hydrogen contained in the fuel, the theoretical composition of producer gas would be 33.3% CO and 66.7% N, both by volume and weight. Its weight per cubic metre is 1.251 grammes, and its heat value 1013 calories per cubic metre, or less than one-fifth of the heat-value of coal gas. Practically, however, producer gas contains a small percentage of gases, increasing its heat-value, like hydrogen, methane, &c., but on the other hand it is never free from carbon dioxide to the extent of from 2 to 8%. Its heat-value may therefore range between 800 and 1100 calories per cubic metre. Even when taking as the basis of our calculation a theoretical gas of 33.3% CO, we find that there is a great loss of heat-value in the manufacture of this gas. Thermochemistry teaches us that the reaction C + O develops 29.5% of the heat produced by the complete oxidation of C to CO_2, thus leaving only 70.5% for the stage $CO + O = CO_2$. If, therefore, the gas given off in the producer is allowed to cool down to ordinary temperature, nearly 30% of the heat-value of the coal is lost by radiation. If, however, the gas producer is built in close proximity to the place where the combustion takes place, so that the gas does not lose very much of its heat, the loss is correspondingly less. Even then there is no reason why this mode of burning the fuel, *i.e.* first with “primary air” in the producer (C + O = CO), then with “secondary air” in the furnace ($CO + O = CO_2$), should be preferred to the direct complete burning of the fuel on a grate, unless the above-mentioned advantage is secured, viz. reduction of the smoke gases to a minimum by confining the supply of air as nearly as possible to that required for the formation of CO_2, which is only possible by producing an intimate mixture of the producer gas with the secondary air. The advantage in question is not very great where the heat of the smoke gases can be very fully utilized, *e.g.* in well-constructed steam boilers, salt-pans and the like, and as a matter of fact gas producers have not found much use in such cases. But a very great advantage is attained in high-temperature operations, where the smoke gases escape very hot, and where it is on that account all-important to confine their quantity to a minimum.

It is precisely in these cases that another requirement frequently comes in, viz. the production at a given point of a higher temperature than is easily attained by ordinary fires. Gas-firing lends itself very well to this end, as it is easily combined with a preliminary heating up of the air, and even of the gas itself, by means of “recuperators.” The original and best-known form of these, due to Siemens Brothers, consists of two brick chambers filled with loosely stacked fire-bricks in such manner that any gases passed through the chambers must seek their way through the interstices left between the bricks, by which means a thorough interchange 284
of temperature takes place. The smoke gases, instead of escaping directly into the atmosphere, are made to pass through one of these chambers, giving up part of their heat to the brickwork. After a certain time the draught is changed by means of valves, the smoke gases are passed through another chamber, and the cold air intended to feed the combustion is made to pass through the first chamber, where it takes up heat from the white-hot bricks, and is thus heated up to a bright red heat until the chamber is cooled down too far, when the draughts are again reversed. Sometimes the producer gas itself is heated up in this manner (especially when it has been cooled down by travelling a long distance); in that case four recuperator chambers must be provided instead of two. Another class of recuperators is not founded on the alternating system, but acts continuously; the smoke gases travel always in the same direction in flues contiguous to other flues or pipes in which the air flows in the opposite direction, an interchange of heat taking place through the walls of the flues or pipes. Here the surface of contact must be made very large if a good effect is to be produced. In both cases not merely is a saving effected of all the calories which are abstracted by the cold air from the recuperator, but as less fuel has to be burned to get a given effect, the quantity of smoke gas is reduced. For details and other producer gases, see Gas, II. *For Fuel and Power.*

Gas-firing in the manner just described can be brought about by very simple means, viz. by lowering the fire-grate of an ordinary fire-place to at least 4 ft. below the fire-bridge, and by introducing the air partly below the grate and partly behind the fire-place, at or near the point where the greatest heat is required. Usually, however, more elaborate apparatus is employed, some of which we shall describe below. Gas-firing has now become universal in some of the most important industries and nearly so in others. The present extension of steel-making and other branches of metallurgy is intimately connected with this system, as is the modern method of glass-making, of heating coal gas retorts and so forth.

The composition of producer gas differs considerably, principally according to the material from which it is made. Analyses of ordinary producer gas (not such as falls under the heading of “semi-water gas,” see *sub* C) by volume show 22 to 33% CO, 1 to 7% CO_2, 0.5 to 2% H_2, 0.5 to 3% hydrocarbons, and 64 to 68% N_2.

B. *Water Gas.*—The reaction of steam on highly heated carbonaceous matter was first observed by Felice Fontana in 1780. This was four years before Henry Cavendish isolated hydrogen from water, and thirteen years before William Murdoch made illuminating gas by the distillation of coal, so that it was no wonder that Fontana’s laboratory work was soon forgotten. Nor had the use of carburetted water gas, as introduced by Donovan in 1830 for illuminating purposes, more than a very short life. More important is the fact that during nine years the illumination of the town of Narbonne was carried on by incandescent platinum wire, heated by water gas, where also internally heated generators were for the first time regularly employed. The Narbonne process was abandoned in 1865, and for some time no real progress was made in this field in Europe. But in America, T.S.C. Lowe, Strong, Tessié du Motay and others took up the matter, the first permanent success being obtained by the introduction (1873) of Lowe’s system at Phoenixville, Pa. In the United States the abundance of anthracite, as well as of petroleum naphtha, adapted for carburetting the gas, secures a great commercial advantage to this kind of illuminant over coal gas, so that now three-fourths of all American gas-works employ carburetted water gas. In Europe the progress of this industry was naturally much less rapid, but here also since 1882, when the apparatus of Lowe and Dwight was introduced in the town of Essen, great improvements have been worked out, principally by E. Blass, and by these improvements water gas obtained a firm footing also for certain heating purposes. The American process for making carburetted water gas, as an auxiliary to ordinary coal gas, was first introduced by the London Gas Light and Coke Company on a large scale in 1890.

Water gas in its original state is called “blue gas,” because it burns with a blue, non-luminous flame, which produces a very high temperature. According to the equation $C + H_2O = CO + H_2$, this gas consists theoretically of equal volumes of carbon monoxide and hydrogen. We shall presently see why it is impossible to avoid the presence of a little carbon dioxide and other gases, but we shall for the moment treat of water gas as if it were composed according to the above equation. The reaction $C + H_2O = CO + H_2$ is endothermic, that is, its thermal value is negative. One gram-molecule of carbon produces 97 great calories (1 great calorie or kilogram-calorie = 1000 gram-calories) when burning to CO_2, and this is of course the maximum effect obtainable from this source. If the same gram-molecule of carbon is used for making water gas, that is, $CO + H_2$, the heat produced by the combustion of the product is 68.4 + 57.6 = 126 great calories, an apparent surplus of 29 calories, which cannot be got out of nothing. This is made evident by another consideration. In the above reaction C is not burned to CO_2, but to CO, a reaction which produces 28.6 calories per gram-molecule. But as the oxygen is furnished from water, which must first be decomposed by the expenditure of energy, we must introduce this amount, 68.5 calories in the case of liquid water, or 57.6 calories in the case of steam, as a negative quantity, and the difference, viz. + 28.6 − 57.6 = 29 great calories, represents the amount of heat to be expended from another source in order to bring about the reaction of one gram-molecule of carbon on one gram-molecule of H_2O in the shape of steam. This explains why steam directed upon incandescent coal will produce water gas only for a very short time: even a large mass of coal will quickly be cooled down so much that at first a gas of different composition is formed and soon the process will cease altogether. We can avoid this result by carrying on the process in a retort heated from without by an ordinary coal fire, and all the early water gas apparatus was constructed in this way; but such a method is very uneconomical, and was long ago replaced by a process first patented by J. and T.N. Kirkham in 1854, and very much improved by successive inventors. This process consists in conducting the operation in an upright brick shaft, charged with anthracite, coke or other suitable fuel. This shaft resembles an ordinary gas producer, but it differs in being worked, not in a continuous manner, which, as shown above, would be impossible, but by alternately blowing air and steam through the coal for periods of a few minutes each. During the first phase, when carbon is burned by atmospheric oxygen, and thereby heat is produced, this heat, or rather that part of it which is not carried away by radiation and by the products of combustion on leaving the apparatus, is employed in raising the temperature of the remaining mass of fuel, and is thus available for the second phase, in which the reaction (*b*) $C + H_2O = CO + H_2$ goes on with the abstraction of a corresponding amount of heat from the incandescent fuel, so that the latter rapidly cools down, and the process must be reversed by blowing in air and so forth. The formation of exactly equal volumes of carbon monoxide and hydrogen goes on only at temperatures over 1200° C., that is, for a very few minutes. Even at 1100° C. a little CO_2 can be proved to exist in the gas, and at 900° its proportion becomes too high to allow the process to go on. About 650° C. the CO has fallen to a minimum, and the reaction is now essentially (*c*) $C + 2H_2O = CO_2 + 2H_2$; soon after the temperature of the mass will have fallen to such a low point that the steam passes through it without any perceptible action. The gas produced by reaction (*c*) contains only two-thirds of combustible matter, and is on that account less valuable than proper water gas formed by reaction (*b*); moreover, it requires the generation of twice the amount of steam, and its presence is all the less desirable since it must soon lead to a total cessation of the process. In ordinary circumstances it is evident that the more steam is blown in during a unit of time, the sooner reaction (*c*) will set in; on the other hand, the more heat has been accumulated in the producer the longer can the blowing-in of steam be continued.

The process of making water gas consequently comprises two alternating operations, viz. first “blowing-up” 285
by means of a current of air, by which the heat of the mass of fuel is raised to about 1200° C.; and, secondly “steaming,” by injecting a current of (preferably superheated) steam until the temperature of the fuel had fallen to about 900° C., and too much carbon dioxide appears in the product. During the steaming the gas is carried off by a special conduit into a scrubber, where the dust mechanically carried away in the current is washed out, and the gas is at the same time cooled down nearly to the ordinary temperature. It is generally stored in a gas-holder, from which it is conducted away as required. It is never quite free from nitrogen, as the producer at the beginning of steaming contains much of this gas, together with CO or CO_2. The proportion of hydrogen may exceed 50%, in consequence of reaction (*c*) setting in at the close of the steaming. Ordinary “blue” water gas, if, as usual, made from coke or anthracite, contains 48-52% H_2, 40-41% CO, 1-5% CO_2, 4-5% N_2, and traces of hydrocarbons, especially methane. If made from bituminous coal, it contains more of the latter. If “carburetted” (a process which increases its volume 50% and more) by the vapours from superheated petroleum naphtha, the proportion of CO ranges about 25%, with about as much methane, and from 10 to 15% of “illuminants” (heavy hydrocarbons). The latter, of course, greatly enhance the fuel-value of the gas. Pure water gas would possess the following fuel-value per cubic metre:

0.5	cub. met.	H_2	= 1291	calories
0.5	” ”	CO	= 1522	”
			2813	”

Ordinary “blue” water gas has a fuel-value of at least 2500 calories. Carburetted water gas, which varies very much in its percentage of hydrocarbons, sometimes reaches nearly the heat-value of coal gas, but such gas is only in exceptional cases used for heating purposes.

We must now turn to the “blowing-up” stage of the process. Until recently it was assumed that during this stage the combustion of carbon cannot be carried on beyond the formation of carbon monoxide, for as the gas-producer must necessarily contain a deep layer of fuel (generally about 6 to 10 ft.), any CO_2 formed at first would be reduced to CO; and it was further assumed that hardly any CO_2 would be formed from the outset, as the temperature of the apparatus is too high for this reaction to take place. But as the combustion of C to CO produces only about 30% of the heat produced when C is burned into CO_2, the quantity of fuel consumed for “blowing-up” is very large, and in fact considerably exceeds that consumed in “steaming.” There is, of course, a further loss by radiation and minor sources, and the result is that 1 kilogram of carbon yields only about 1.2 cub. met. of water gas. Each period of blowing-up generally occupies from 8 to 12 minutes, that of steaming only 4 or 5 minutes. This low yield of water gas until quite recently appeared to be unavoidable, and the only question seemed to be whether and to what extent the gas formed during blowing-up, which is in fact identical with ordinary producer gas (Siemens gas), could be utilized. In America, where the water gas is mostly employed for illuminating purposes, at least part of the blowing-up gas is utilized for heating the apparatus in which the naphtha is volatilized and the vapours are "fixed" by superheating. This process, however, never utilizes anything like the whole of the blowing-up gas, nor can this be effected by raising and superheating the steam necessary for the second operation; indeed, the employment of this gas for raising steam is not very easy, owing to the irregularities of and constant interruptions in the supply. In some systems the gas made during the blowing-up stage is passed through chambers, loosely filled with bricks, like Siemens recuperators, where it is burned by "secondary" air: the heat thus imparted to the brickwork is utilized by passing through the recuperator, and thus superheating, the steam required for the next steaming operation. In many cases, principally where no carburetting is practised, the blowing-up gas is simply burned at the mouth of the producer, and is thus altogether lost; and in no case can it be utilized without great waste. A very important improvement in this respect was effected by C. Dellwik and E. Fleischer. They found that the view that it is unavoidable to burn the carbon to monoxide during the blowing-up holds good only for the pressure of blast formerly applied. This did not much exceed that which is required for overcoming the frictional resistance within the producer. If, however, the pressure is considerably increased, and the height of the column of fuel reduced, both of these conditions being strictly regulated in accordance with the result desired, it is easy to attain a combustion of the carbon to dioxide, with only traces of monoxide, in spite of the high temperature. Evidently the excess of oxygen coming into contact with each particle of carbon in a given unit of time produces other conditions of chemical equilibrium than those existing at lower pressures. At any rate, experience has shown that by this process, in which the full heat-value of carbon is utilized during the blowing-up stage, the time of heating-up can be reduced from 10 to 1½ or 2 minutes, and the steaming can be prolonged from 4 or 5 to 8 or 10 minutes, with the result that twice the quantity of water gas is obtained, viz. upwards of 2 cub. metres from 1 kilogram of carbon.

The application of water gas as a fuel mainly depends upon the high temperatures which it is possible to attain by its aid, and these are principally due to the circumstance that it forms a much smaller flame than coal gas, not to speak of Siemens gas, which contains at most 33% of combustible matter against 90% or more in water gas. The latter circumstance also allows the gas to be conducted and distributed in pipes of moderate dimensions. Its application, apart from its use as an illuminant (with which we are not concerned here), was formerly retarded by its high cost in comparison with Siemens gas and other sources of heat, but as this state of affairs has been changed by the modern improvements, its use is rapidly extending, especially for metallurgical purposes.

C. *Mixed Gas* (*Semi-Water Gas*).—This class is sometimes called Dowson gas, irrespective of its method of production, although it was made and extensively used a long time before J.E. Dowson constructed his apparatus for generating such a gas principally for driving gas-engines. By a combination of the processes for generating Siemens gas and water gas, it is produced by injecting into a gas-producer at the same time a certain quantity of air and a corresponding quantity of steam, the latter never exceeding the amount which can be decomposed by the heat-absorbing reaction, $C + H_2O = CO + H_2$, at the expense of the heat generated by the action of the air in the reaction $C + O = CO$. Such gas used to be frequently obtained in an accidental way by introducing liquid water or steam into an ordinary gas-producer for the purpose of facilitating its working by avoiding an excessive temperature, such as might cause the rapid destruction of the brickwork and the fusion of the ashes of the fuel into troublesome cakes. It was soon found that by proceeding in this way a certain advantage could be gained in regard to the consumption of fuel, as the heat abstracted by the steam from the brickwork and the fuel itself was usefully employed for decomposing water, its energy thus reappearing in the shape of a combustible gas. It is hardly necessary to mention explicitly that the total heat obtained by any such process from a given quantity of carbon (or hydrogen) can in no case exceed that which is generated by direct combustion; some inventors, however, whether inadvertently or intentionally, have actually represented this to be possible, in manifest violation of the law of the conservation of energy.

Roughly speaking, this gas may be said to be produced by the combination of the reactions, described *sub* A and B, to the joint reaction: $2C + O + H_2O = 2CO + H_2$. The decomposition of H_2O (applied in the shape of steam) absorbs 57.6 gram calories, the formation of 2CO produces 59 gram calories; hence there is a small positive excess of 1.4 calories at disposal. This in reality would not be sufficient to cover the loss by radiation, &c.; hence rather more free oxygen (*i.e.* atmospheric air) must be employed than is represented by the above equation. All this free oxygen is, of course, accompanied by nearly four times its volume of nitrogen.

The mixed gas thus obtained differs very much in composition, but is always much richer in hydrogen (of which it contains sometimes as much as 20%) and poorer in carbon monoxide (sometimes down to 20%) than Siemens gas; generally it contains more of CO_2 than the latter. The proportion of nitrogen is always less, about 50%. It is therefore a more concentrated fuel than Siemens gas, and better adapted to the driving of gas-engines. It scarcely costs more to make than ordinary Siemens gas, except where the steam is generated and superheated in special apparatus, as is done in the Dowson producer, which, on the other hand, yields a correspondingly better gas. As is natural, its properties are some way between those of Siemens gas and of water gas; but they approach more nearly the former, both as to costs and as to fuel-value, and also as to the temperatures reached in combustion. This is easily understood if we consider that gas of just the same description can be obtained by mixing one volume of real water gas with the four volumes of Siemens gas made during the blowing-up stage—an operation which is certainly too expensive for practical use. 286

A modification of this gas is the *Mond gas*, which is made, according to Mond's patent, by means of such an excess of steam that most of the nitrogen of the coke is converted into ammonia (Grouven's reaction). Of course much of this steam passes on undecomposed, and the quantity of the gas is greatly increased by the reaction $C + 2H_2O = CO_2 + 2H_2$; hence the fuel-value of this gas is less than that of semi-water gas made in other ways. Against this loss must be set the gain of ammonia which is recovered by means of an arrangement of coolers and scrubbers, and, except at very low prices of ammonia, the profit thus made is probably more than sufficient to cover the extra cost. But as the process requires very large and expensive plant, and its profits would vanish in the case of the value of ammonia becoming much lower (a result which would very probably follow if it were somewhat generally introduced), it cannot be expected to supplant the other descriptions of gaseous fuel to more than a limited extent.

Semi-water gas is especially adapted for the purpose of driving gas-engines on the explosive principle (gas-motors). Ordinary producer-gas is too poor for this purpose in respect of heating power; moreover, owing to the prevalence of carbon monoxide, it does not light quickly enough. These defects are sufficiently overcome in semi-water gas by the larger proportion of hydrogen contained in it. For the purpose in question the gas should be purified from tar and ashes, and should also be cooled down before entering the gas-engine. The Dowson apparatus and others are constructed on this principle.

Air Gas.—By forcing air over or through volatile inflammable liquids a gaseous mixture can be obtained which burns with a bright flame and which can be used for illumination. Its employment for heating purposes is quite exceptional, *e.g.* in chemical laboratories, and we abstain, therefore, from describing any of the numerous appliances, some of them bearing very fanciful names, which have been devised for its manufacture.

(G. L.)

FUENTE OVEJUNA [*Fuenteovejuna*], a town of Spain, in the province of Cordova; near the sources of the river Guadiato, and on the Fuente del Arco-Belmez-Cordova railway. Pop. (1900) 11,777. Fuente Ovejuna is built on a hill, in a well-irrigated district, which, besides producing an abundance of wheat, wine, fruit and honey, also contains argentiferous lead mines and stone quarries. Cattle-breeding is an important local industry, and leather, preserved meat, soap and flour are manufactured. The parish church formerly belonged to the knights of Calatrava (*c.* 1163-1486).

FUENTERRABIA (formerly sometimes written *Fontarabia*; Lat. *Fons Rapidus*), a town of northern Spain, in the province of Guipúzcoa; on the San Sebastian-Bayonne railway; near the Bay of Biscay and on the French frontier. Pop. (1870) about 750; (1900) 4345. Fuenterrabia stands on the slope of a hill on the left bank of the river Bidassoa, and near the point where its estuary begins. Towards the close of the 19th century the town became popular as a summer resort for visitors from the interior of Spain, and, in consequence, its appearance underwent many changes and much of its early prosperity returned. Hotels and villas were built in the new part of the town that sprang up outside the picturesque walled fortress, and there is quite a contrast between the part inside the heavy, half-ruined ramparts, with its narrow, steep streets and curious gable-roofed houses, its fine old church and castle and its massive town hall, and the new suburbs and fishermen’s quarter facing the estuary of the Bidassoa. Many industries flourish on the outskirts of the town, including rope and net manufactures, flour mills, saw mills, mining railways, paper mills.

Fuenterrabia formerly possessed considerable strategic importance, and it has frequently been taken and retaken in wars between France and Spain. The rout of Charlemagne in 778, which has been associated with Fontarabia, by Milton (*Paradise Lost*, i. 587), is generally understood to have taken place not here but at Roncesvalles (*q.v.*), which is nearly 40 m. E.S.E. Unsuccessful attempts to seize Fuenterrabia were made by the French troops in 1476 and again in 1503. In a subsequent campaign (1521) these were more successful, but the fortress was retaken in 1524. The prince of Condé sustained a severe repulse under its walls in 1638, and it was on this occasion that the town received from Philip IV. the rank of city (*muy noble, muy leal, y muy valerosa ciudad*, “most noble, most loyal, and most valiant city”), a privilege which involved some measure of autonomy. After a severe siege, Fuenterrabia surrendered to the duke of Berwick and his French troops in 1719; and in 1794 it again fell into the hands of the French, who so dismantled it that it has never since been reckoned by the Spaniards among their fortified places. It was by the ford opposite Fuenterrabia that the duke of Wellington, on the 8th of October 1813, successfully forced a passage into France in the face of an opposing army commanded by Marshal Soult. Severe fighting also took place here during the Carlist War in 1837.

FUERO, a Spanish term, derived from the Latin *forum*. The Castillan use of the word in the sense of a right, privilege or charter is most probably to be traced to the Roman *conventus juridici*, otherwise known as *jurisdictiones* or *fora*, which in Pliny’s time were already numerous in the Iberian peninsula. In each of these provincial *fora* the Roman magistrate, as is well known, was accustomed to pay all possible deference to the previously established common law of the district; and it was the privilege of every free subject to demand that he should be judged in accordance with the customs and usages of his proper forum. This was especially true in the case of the inhabitants of those towns which were in possession of the *jus italicum*. It is not, indeed, demonstrable, but there are many presumptions, besides some fragments of direct evidence, which make it more than probable that the old administrative arrangements both of the provinces and of the towns, but especially of the latter, remained practically undisturbed at the period of the Gothic occupation of Spain.[1] The Theodosian Code and the Breviary of Alaric alike seem to imply a continuance of the municipal system which had been established by the Romans; nor does the later Lex Visigothorum, though avowedly designed in some points to supersede the Roman law, appear to have contemplated any marked interference with the former *fora*, which were still to a large extent left to be regulated in the administration of justice by unwritten, immemorial, local custom. Little is known of the condition of the subject populations of the peninsula during the Arab occupation; but we are informed that the Christians were, sometimes at least, judged according to their own laws in separate tribunals presided over by Christian judges;[2] and the mere fact of the preservation of the name *alcalde*, an official whose functions corresponded so closely to those of the *judex* or *defensor civitatis*, is fitted to suggest that the old municipal *fora*, if much impaired, were not even then in all cases wholly destroyed. At all events when the word *forum*[3] begins to appear for the first time in documents of the 10th century in the sense of a liberty or privilege, it is generally implied that the thing so named is nothing new. The earliest extant written fuero is probably that which was granted to the province and town of Leon by Alphonso V. in 1020. It emanated from the king in a general council of the kingdom of Leon and Castile, and consisted of two separate parts; in the first 19 chapters were contained a series of statutes which were to be valid for the kingdom at large, while the rest of the document was simply a municipal charter.[4] But in neither portion does it in any sense mark a new legislative departure, unless in so far as it marks the beginning of the era of written charters for towns. The "fuero general" does not profess to supersede the *consuetudines antiquorum jurium* or Chindaswint's codification of these in the Lex Visigothorum; the "fuero municipal" is really for the most part but a resuscitation of usages formerly established, a recognition and definition of liberties and privileges that had long before been conceded or taken for granted. The right of the burgesses to self-government and self-taxation is acknowledged and confirmed, they, on the other hand, being held bound to a constitutional obedience and subjection to the sovereign, particularly to the payment of definite imperial taxes, and the rendering of a certain amount of military service (as the ancient municipia had been). Almost contemporaneous with this fuero of Leon was that granted to Najera (Naxera) by Sancho el Mayor of Navarre (*ob.* 1035), and confirmed, in 1076, by Alphonso VI.[5] Traces of others of perhaps even an earlier date are occasionally to be met with. In the fuero of Cardeña, for example, granted by Ferdinand I. in 1039, reference is made to a previous forum Burgense (Burgos), which, however, has not been preserved, if, indeed, it ever had been reduced to writing at all. The phraseology of that of Sepulveda (1076) in like manner points back to an indefinitely remote antiquity.[6] Among the later fueros of the 11th century, the most important are those of Jaca (1064) and of Logroño (1095). The former of these, which was distinguished by the unusual largeness of its concessions, and by the careful minuteness of its details, rapidly extended to many places in the neighbourhood, while the latter charter was given also to Miranda by Alphonso VI., and was further extended in 1181 by Sancho el Sabio of Navarre to Vitoria, thus constituting one of the earliest written *fora* of the "Provincias Vascongadas." In the course of the 12th and 13th centuries the number of such documents increased very rapidly; that of Toledo especially, granted to the Mozarabic population in 1101, but greatly enlarged and extended by Alphonso VII. (1118) and succeeding sovereigns, was used as a basis for many other Castilian fueros. Latterly the word fuero came to be used in Castile in a wider sense than before, as meaning a general code of laws; thus about the time of Saint Ferdinand the old Lex Visigothorum, then translated for the first time into the vernacular, was called the Fuero Juzgo, a name which was soon retranslated into the barbarous Latin of the period as Forum Judicum;[7] and among the compilations of Alphonso the Learned in like manner were an *Espejo de Fueros* and also the *Fuero de las leyes*, better known perhaps as the *Fuero Real*. The famous code known as the *Ordenamiento Real de Alcalá*, or *Fuero Viejo de Castilla*, dates from a still later period. As the power of the Spanish crown was gradually concentrated and consolidated, royal pragmaticas began to take the place of constitutional laws; the local fueros of the various districts slowly yielded before the superior force of imperialism; and only those of Navarre and the Basque provinces (see Basques) have had sufficient vitality to enable them to survive to comparatively modern times. While actually owning the lordship of the Castilian crown since about the middle of the 14th century, these provinces rigidly insisted upon compliance with their consuetudinary law, and especially with that which provided that the *señor*, before assuming the government, should personally appear before the assembly and swear to maintain the ancient constitutions. Each of the provinces mentioned had distinct sets of fueros, codified at different periods, and varying considerably as to details; the main features, however, were the same in all. Their rights, after having been recognized by successive Spanish sovereigns from Ferdinand the Catholic to Ferdinand VII., were, at the death of the latter in 1833, set aside by the government of Castaños. The result was a civil war, which terminated in a renewed acknowledgment of the fueros by Isabel II. (1839). The provisional government of 1868 also promised to respect them, and similar pledges were given by the governments which succeeded. In consequence, however, of the Carlist rising of 1873-1876, the Basque fueros were finally extinguished in 1876. The history of the *Foraes* of the Portuguese towns, and of the *Fors du Béarn*, is precisely analogous to that of the fueros of Castile. 287

Among the numerous works that more or less expressly deal with this subject, that of Marina (*Ensayo historico-critico sobre la antigua legislacion y principales cuerpos legales de los reynos de Leon y Castilla*) still continues to hold a high place. Reference may also be made to Colmeiro's *Curso de derecho político según la historia de Leon y de Castilla* (Madrid, 1873); to Schäfer's *Geschichte von Spanien*, ii. 418-428, iii. 293 seq.; and to Hallam's *Middle Ages*, c. iv.

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1. The nature of the evidence may be gathered from Savigny, *Gesch. d. röm. Rechts*. See especially i. pp. 154, 259 seq.

2. Compare Lembke u. Schäfer, *Geschichte von Spanien*, i. 314; ii. 117.

3. Or rather *forus*. See Ducange, *s.v.*

4. Cap. xx. begins: "Constituimus etiam ut Legionensis civitas, quae depopulata fuit a Sarracenis in diebus patris mei Veremundi regis, repopulatur *per hos foros subscriptos*."

5. "Mando et concedo et confirmo ut ista civitas cum sua plebe et cum omnibus suis pertinentiis sub tali lege et sub tali foro maneat per saecula cuncta. Amen. Isti sunt fueros quae habuerunt in Naxera in diebus Sanctii regis et

Gartiani regis."

6 "Ego Aldefonsus rex et uxor mea Agnes confirmamus ad Septempublica suo foro quod habuit in tempore antiquo de avolo meo et in tempore comitum Ferrando Gonzalez et comite Garcia Ferdinandez et comite Domno Santio."

7 This Latin is later even than that of Ferdinand, whose words are: "Statuo et mando quod Liber Judicum, quo ego misi Cordubam, translatetur in vulgarem et vocetur forum de Corduba ... et quod per saecula cuncta sit pro foro et nullus sit ausus istud forum aliter appellare nisi forum de Corduba, et jubeo et mando quod omnis morator et populator ... veniet ad judicium et ad forum de Corduba."

FUERTEVENTURA, an island in the Atlantic Ocean, forming part of the Spanish archipelago of the Canary Islands (*q.v.*). Pop. (1900) 11,669; area 665 sq. m. Fuerteventura lies between Lanzarote and Grand Canary. It has a length of 52 m., and an average width of 12 m. Though less mountainous than the other islands, its aspect is barren. There are only two springs of fresh water, and these are confined to one valley. Lava streams and other signs of volcanic action abound, but there has been no igneous activity since the Spaniards took possession. At each extremity of the island are high mountains, which send off branches along the coast so as to enclose a large arid plain. The highest peak reaches 2500 ft. In external appearance, climate and productions, Fuerteventura greatly resembles Lanzarote. An interval of three years without rain has been known. Oliva (pop. 1900, 2464) is the largest town. A smaller place in the centre of the island named Betancuria (586) is the administrative capital. Cabras (1000) on the eastern coast is the chief port. Dromedaries are bred here.

FUGGER, the name of a famous German family of merchants and bankers. The founder of the family was Johann Fugger, a weaver at Graben, near Augsburg, whose son, Johann, settled in Augsburg probably in 1367. The younger Johann added the business of a merchant to that of a weaver, and through his marriage with Clara Widolph became a citizen of Augsburg. After a successful career he died in 1408, leaving two sons, Andreas and Jakob, who greatly extended the business which they inherited from their father. Andreas, called the "rich Fugger," had several sons, among them being Lukas, who was very prominent in the municipal politics of Augsburg and who was very wealthy until he was ruined by the repudiation by the town of Louvain of a great debt owing to him, and Jakob, who was granted the right to bear arms in 1452, and who founded the family of Fugger vom Reh—so called from the first arms of the Fuggers, a roe (*Reh*) or on a field azure—which became extinct on the death of his great-grandson, Ulrich, in 1583. Johann Fugger's son, Jakob, died in 1469, and three of his seven sons, Ulrich (1441-1510), Georg (1453-1506) and Jakob (1459-1525), men of great resource and industry, inherited the family business and added enormously to the family wealth. In 1473 Ulrich obtained from the emperor Frederick III. the right to bear arms for himself and his brothers, and about the same time he began to act as the banker of the Habsburgs, a connexion destined to bring fame and fortune to his house. Under the lead of Jakob, who had been trained for business in Venice, the Fuggers were interested in silver mines in Tirol and copper mines in Hungary, while their trade in spices, wool and silk extended to almost all parts of Europe. Their wealth enabled them to make large loans to the German king, Maximilian I., who pledged to them the county of Kirchberg, the lordship of Weissenhorn and other lands, and bestowed various privileges upon them. Jakob built the castle of Fuggerau in Tirol, and erected the Fuggerei at Augsburg, a collection of 106 dwellings, which were let at low rents to poor people and which still exist. Jakob Fugger and his two nephews, Ulrich (d. 1525) and Hieronymus (d. 1536), the sons of Ulrich, died without direct heirs, and the family was continued by Georg's sons, Raimund (1489-1535) and Anton (1493-1560), under whom the Fuggers attained the summit of their wealth and influence. 288

Jakob Fugger's florins had contributed largely to the election of Charles V. to the imperial throne in 1519, and his nephews and heirs maintained close and friendly relations with the great emperor. In addition to lending him large sums of money, they farmed his valuable quicksilver mines at Almaden, his silver mines at Guadalcanal, the great estates of the military orders which had passed into his hands, and other parts of his revenue as king of Spain; receiving in return several tokens of the emperor's favour. In 1530 Raimund and Anton were granted the imperial dignity of counts of Kirchberg and Weissenhorn, and obtained full possession of these mortgaged properties; in 1534 they were given the right of coining money; and in 1541 received rights of jurisdiction over their lands. During the diet of Augsburg in 1530 Charles V. was the guest of Anton Fugger at his house in the Weinmarkt, and the story relates how the merchant astonished the emperor by lighting a fire of cinnamon with an imperial bond for money due to him. This incident forms the subject of a picture by Carl Becker which is in the National Gallery at Berlin. Continuing their mercantile career, the Fuggers brought the new world within the sphere of their operations, and also carried on an extensive and lucrative business in farming indulgences. Moreover, both brothers found time to acquire landed property, and were munificent patrons of literature and art. When Anton died he is said to have been worth 6,000,000 florins, besides a vast amount of property in Europe, Asia and America; and before this time the total wealth of the family had been estimated at 63,000,000 florins. The Fuggers were devotedly attached to the Roman Catholic Church, which benefited from their liberality. Jakob had been made a count palatine (*Pfalzgraf*) and had received other marks of favour from Pope Leo X., and several members of the family had entered the church; one, Raimund's son, Sigmund, becoming bishop of Regensburg.

In addition to the bishop, three of Raimund Fugger's sons attained some degree of celebrity. Johann Jakob (1516-1575), was the author of *Wahrhaftigen Beschreibung des österreichischen und habsburgischen Nahmens*, which was largely used by S. von Bircken in his *Spiegel der Ehren des Erzhauses Österreich*

(Nuremberg, 1668), and of a *Geheim Ernbuch des Fuggerischen Geschlechtes*. He was also a patron of art, and a distinguished counsellor of Duke Albert IV. of Bavaria. After the death of his son Konstantin, in 1627, this branch of the family was divided into three lines, which became extinct in 1738, 1795 and 1846 respectively. Another of Raimund’s sons was Ulrich (1526-1584), who, after serving Pope Paul III. at Rome, became a Protestant. Hated on this account by the other members of his family, he took refuge in the Rhenish Palatinate; greatly interested in the Greek classics, he occupied himself in collecting valuable manuscripts, which he bequeathed to the university of Heidelberg. Raimund’s other son was Georg (d. 1579), who inherited the countships of Kirchberg and Weissenhorn, and founded a branch of the family which still exists, its present head being Georg, Count Fugger of Kirchberg and Weissenhorn (b. 1850).

Anton Fugger left three sons, Marcus (1529-1597), Johann (d. 1598) and Jakob (d. 1598), all of whom left male issue. Marcus was the author of a book on horse-breeding, *Wie und wo man ein Gestüt von guten edeln Kriegsrossen aufrichten soll* (1578), and of a German translation of the *Historia ecclesiastica* of Nicephorus Callistus. He founded the Nordendorf branch of the family, which became extinct on the death of his grandson, Nicolaus, in 1676. Another grandson of Marcus was Franz Fugger (1612-1664), who served under Wallenstein during the Thirty Years’ War, and was afterwards governor of Ingolstadt. He was killed at the battle of St Gotthard on the 1st of August 1664.

Johann Fugger had three sons, Christoph (d. 1615) and Marcus (d. 1614), who founded the families of Fugger-Glött and Fugger-Kirchheim respectively, and Jakob, bishop of Constance from 1604 until his death in 1626. Christoph’s son, Otto Heinrich (1592-1644), was a soldier of some distinction and a knight of the order of the Golden Fleece. He was one of the most active of the Bavarian generals during the Thirty Years’ War, and acted as governor of Augsburg, where his rule aroused much discontent. The family of Kirchheim died out in 1672. That of Glött was divided into several branches by the sons of Otto Heinrich and of his brother Johann Ernst (d. 1628). These lines, however, have gradually become extinct except the eldest line, represented in 1909 by Karl Ernst, Count Fugger of Glött (b. 1859). Anton Fugger’s third son Jakob, the founder of the family of Wellenburg, had two sons who left issue, but in 1777 the possessions of this branch of the family were again united by Anselm Joseph (d. 1793), Count Fugger of Babenhausen. In 1803 Anselm’s son, Anselm Maria (d. 1821), was made a prince of the Holy Roman Empire, the title of Prince Fugger of Babenhausen being borne by his direct descendant Karl (b. 1861). On the fall of the empire in 1806 the lands of the Fuggers, which were held directly of the empire, were mediatized under Bavaria and Württemberg. The heads of the three existing branches of the Fuggers are all hereditary members of the Bavarian Upper House.

Augsburg has many interesting mementoes of the Fuggers, including the family burial-chapel in the church of St Anna; the Fugger chapel in the church of St Ulrich and St Afra; the Fuggerhaus, still in the possession of one branch of the family; and a statue of Johann Jakob Fugger.

In 1593 a collection of portraits of the Fuggers, engraved by Dominique Custos of Antwerp, was issued at Augsburg. Editions with 127 portraits appeared in 1618 and 1620, the former accompanied by a genealogy in Latin, the latter by one in German. Another edition of this *Pinacotheca Fuggerorum*, published at Vienna in 1754, includes 139 portraits. See *Chronik der Familie Fugger vom Jahre 1599*, edited by C. Meyer (Munich, 1902); A. Geiger, *Jakob Fugger, 1459-1525* (Regensburg, 1895); A. Schulte, *Die Fugger in Rom, 1495-1523* (Leipzig, 1904); R. Ehrenberg, *Das Zeitalter der Fugger* (Jena, 1896); K. Häbler, *Die Geschichte der Fuggerschen Handlung in Spanien* (Weimar, 1897); A. Stauber, *Das Haus Fugger* (Augsburg, 1900); and M. Jansen, *Die Anfänge der Fugger* (Leipzig, 1907).

FUGITIVE SLAVE LAWS, a term applied in the United States to the Statutes passed by Congress in 1793 and 1850 to provide for the return of negro slaves who escaped from one state into another or into a public territory. A fugitive slave clause was inserted in the Articles of Confederation of the New England Confederation of 1643, providing for the return of the fugitive upon the certificate of one magistrate in the jurisdiction out of which the said servant fled—no trial by jury being provided for. This seems to have been the only instance of an inter-colonial provision for the return of fugitive slaves; there were, indeed, not infrequent escapes by slaves from one colony to another, but it was not until after the growth of anti-slavery sentiment and the acquisition of western territory, that it became necessary to adopt a uniform method for the return of fugitive slaves. Such provision was made in the Ordinance of 1787 (for the Northwest Territory), which in Article VI. provided that in the case of “any person escaping into the same [the Northwest Territory] from whom labor or service is lawfully claimed in any one of the original states, such fugitive may be lawfully reclaimed and conveyed to the person claiming his or her labor or service as aforesaid.” An agreement of the
sort was necessary to persuade the slave-holding states to union, and in the Federal Constitution, Article IV., 289
Section II., it is provided that “no person held to service or labor in one state, under the laws thereof, escaping into another, shall, in consequence of any law or regulation therein, be discharged from such service or labor, but shall be delivered up on claim of the party to whom such service or labour may be due.”

The first specific legislation on the subject was enacted on the 12th of February 1793, and like the Ordinance for the Northwest Territory and the section of the Constitution quoted above, did not contain the word “slave”; by its provisions any Federal district or circuit judge or any state magistrate was authorized to decide finally and without a jury trial the status of an alleged fugitive. The measure soon met with strong opposition in the northern states, and Personal Liberty Laws were passed to hamper officials in the execution of the law; Indiana in 1824 and Connecticut in 1828 providing jury trial for fugitives who appealed from an original decision against them. In 1840 New York and Vermont extended the right of trial by jury to fugitives and provided them with attorneys. As early as the first decade of the 19th century individual dissatisfaction with the law of 1793 had taken the form of systematic assistance rendered to negroes escaping from the South to Canada or New England—the so-called “Underground Railroad.”[1] The decision of the Supreme Court of the United States in the case of *Prigg* v. *Pennsylvania* in 1842 (16 Peters 539), that state authorities could not be forced to act in fugitive slave cases, but that national authorities must carry out the national law, was followed by legislation in Massachusetts (1843), Vermont (1843), Pennsylvania (1847) and Rhode Island (1848), forbidding state officials to help enforce the law and refusing the use of state gaols for fugitive slaves. The demand from the South for more effective Federal legislation was voiced in the second fugitive slave law, drafted by Senator J.M. Mason of Virginia, and enacted on the 18th of September 1850 as a part of the Compromise Measures of that year. Special commissioners were to have concurrent jurisdiction with the U.S. circuit and district courts and the inferior courts of Territories in enforcing the law; fugitives could not testify in their own behalf; no trial by jury was provided; penalties were imposed upon marshals who refused to enforce the law or from whom a fugitive should escape, and upon individuals who aided negroes to escape; the marshal might raise a *posse comitatus*; a fee of $10 was paid to the commissioner when his decision favoured the claimant and only $5 when it favoured the fugitive; and both the fact of the escape and the identity of the fugitive were to be determined on purely *ex parte* testimony. The severity of this measure led to gross abuses and defeated its purpose; the number of abolitionists increased, the operations of the Underground Railroad became more efficient, and new Personal Liberty Laws were enacted in Vermont (1850), Connecticut (1854), Rhode Island (1854), Massachusetts (1855), Michigan (1855), Maine (1855 and 1857), Kansas (1858) and Wisconsin (1858). These Personal Liberty Laws forbade justices and judges to take cognizance of claims, extended the *habeas corpus* act and the privilege of jury trial to fugitives, and punished false testimony severely. The supreme court of Wisconsin went so far (1859) as to declare the Fugitive Slave Law unconstitutional. These state laws were one of the grievances officially referred to by South Carolina (in Dec. 1860) as justifying her secession from the Union. Attempts to carry into effect the law of 1850 aroused much bitterness. The arrests of Sims and of Shadrach in Boston in 1851; of "Jerry" M'Henry, in Syracuse, New York, in the same year; of Anthony Burns in 1854, in Boston; and of the two Garner families in 1856, in Cincinnati, with other cases arising under the Fugitive Slave Law of 1850, probably had as much to do with bringing on the Civil War as did the controversy over slavery in the Territories.

With the beginning of the Civil War the legal status of the slave was changed by his master's being in arms. General B.F. Butler, in May 1861, declared negro slaves contraband of war. A confiscation bill was passed in August 1861 discharging from his service or labour any slave employed in aiding or promoting any insurrection against the government of the United States. By an act of the 17th of July 1862 any slave of a disloyal master who was in territory occupied by northern troops was declared *ipso facto* free. But for some time the Fugitive Slave Law was considered still to hold in the case of fugitives from masters in the border states who were loyal to the Union government, and it was not until the 28th of June 1864 that the Act of 1850 was repealed.

See J.F. Rhodes, *History of the United States from the Compromise of 1850*, vols. i. and ii. (New York, 1893); and M.G. M'Dougall, *Fugitive Slaves, 1619-1865* (Boston, 1891).

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1 The precise amount of organization in the Underground Railroad cannot be definitely ascertained because of the exaggerated use of the figure of railroading in the documents of the "presidents" of the road, Robert Purvis and Levi Coffin, and of its many "conductors," and their discussion of the "packages" and "freight" shipped by them. The system reached from Kentucky and Virginia across Ohio, and from Maryland across Pennsylvania and New York, to New England and Canada, and as early as 1817 a group of anti-slavery men in southern Ohio had helped to Canada as many as 1000 slaves. The Quakers of Pennsylvania possibly began the work of the mysterious Underground Railroad; the best known of them was Thomas Garrett (1789-1871), a native of Pennsylvania, who, in 1822, removed to Wilmington, Delaware, where he was convicted in 1848 on four counts under the Fugitive Slave Law and was fined $8000; he is said to have helped 2700 slaves to freedom. The most picturesque figure of the Underground Railroad was Harriet Tubman (c. 1820), called by her friend, John Brown, "General" Tubman, and by her fellow negroes "Moses." She made about a score of trips into the South, bringing out with her 300 negroes altogether. At one time a reward of $40,000 was offered for her capture. She was a mystic, with remarkable clairvoyant powers, and did great service as a nurse, a spy and a scout in the Civil War. Levi Coffin (1798-1877), a native of North Carolina (whose cousin, Vestal Coffin, had established before 1819 a "station" of the Underground near what is now Guilford College, North Carolina), in 1826 settled in Wayne County, Ohio; his home at New Garden (now Fountain City) was the meeting point of three "lines" from Kentucky; and in 1847 he removed to Cincinnati, where his labours in bringing slaves out of the South were even more successful. It has been argued that the Underground Railroad delayed the final decision of the slavery question, inasmuch as it was a "safety valve"; for, without it, the more intelligent and capable of the negro slaves would, it is asserted, have become the leaders of insurrections in the South, and would not have been removed from the places where they could have done most damage. Consult William Still, *The Underground Railroad* (Philadelphia, 1872), a collection of anecdotes by a negro agent of the Pennsylvania Anti-Slavery Society, and of the Philadelphia branch of the Railroad; and the important and scholarly work of Wilbur H. Siebert, *The Underground Railroad from Slavery to Freedom* (New York, 1898).

FUGLEMAN (from the Ger. *Flügelmann*, the man on the *Flügel* or wing), properly a military term for a soldier who is selected to act as "guide," and posted generally on the flanks with the duty of directing the march in the required line, or of giving the time, &c., to the remainder of the unit, which conforms to his movements, in any military exercise. The word is then applied to a ringleader or one who takes the lead in any movement or concerted movement.

FUGUE (Lat. *fuga*, flight), in music, the mutual "pursuit" of voices or parts. It was, up to the end of the 16th century, if not later, the name applied to two art-forms. (A) *Fuga ligata* was the exact reproduction by one or more voices of the statement of a leading part. The reproducing voice (*comes*) was seldom if ever written out, for all differences between it and the *dux* were rigidly systematic; *e.g.* it was an exact inversion, or exactly twice as slow, or to be sung backwards, &c. &c. Hence, a rule or *canon* was given, often in enigmatic form, by which the *comes* was deduced from the *dux*: and so the term *canon* became the appropriate name for the form itself, and is still retained. (B) A composition in which the canonic style was cultivated without canonic restriction was, in the 16th century, called *fuga ricercata* or simply a *ricercare*, a term which is still used by Bach as a title for the fugues in *Das musikalische Opfer*.

The whole conception of fugue, rightly understood, is one of the most important in music, and the reasons why some contrapuntal compositions are called fugues, while others are not, are so trivial, technically as well as aesthetically, that we have preferred to treat the subject separately under the general heading of 290
Contrapuntal Forms, reserving only technical terms for definition here.

(i.) If in the beginning or "exposition" the material with which the opening voice accompanies the answer is faithfully reproduced as the accompaniment to subsequent entries of the subject, it is called a *countersubject* (see Counterpoint, under sub-heading *Double Counterpoint*). Obviously the process may be carried further, the first countersubject going on to a second when the subject enters in the third part and so on. The term is also applied to new subjects appearing later in the fugue in combination (immediate or destined) with the original subject. Cherubini, holding the doctrine that a fugue cannot have more than one subject, insists on applying the term to the less prominent of the subjects of what are commonly called double fugues, *i.e.* fugues which begin with two parts and two subjects simultaneously, and so also with *triple* and *quadruple fugues*.

(ii.) *Episodes* are passages separating the entries of the subject.[1] Episodes are usually developed from the material of the subject and countersubjects; they are very rarely independent, but then conspicuously so.

(iii.) *Stretto*, the overlapping of subject and answer, is a resource the possibilities of which may be exemplified by the setting of the words *omnes generationes* in Bach's *Magnificat* (see Bach).

(iv.) The distinction between *real* and *tonal* fugue, which is still sometimes treated as a thing of great historical and technical importance, is really a mere detail resulting from the fact that a violent oscillation between the keys of tonic and dominant is no part of the function of a fugal exposition, so that the answer is (especially in its first notes and in points that tend to shift the key) not so much a transposition of the subject to the key of the dominant as an adaptation of it from the tonic part to the dominant part of the scale, or vice versa; in short, the answer is as far as possible *on* the dominant, not *in* the dominant. The modifications this principle produces in the answer (which have been happily described as resembling "fore-shortening") are the only distinctive marks of tonal fugue; and the text-books are half filled with the attempt to reduce them from matters of ear to rules of thumb, which rules, however, have the merit (unusual in those of the academic fugue) of being founded on observation of the practice of great masters. But the same principle as often as not produces answers that are exact transpositions of the subject; and so the only kind of real fugue (*i.e.* fugue with an exact answer) that could rightly be contrasted with tonal fugue would be that in which the answer ought to be tonal but is not. It must be admitted that tonal answers are rare in the modal music of the 16th century, though their melodic principles are of yet earlier date; still, though tonal fugue does not become usual until well on in the 17th century, the idea that it is a separate species is manifestly absurd, unless the term simply means "fugue in modern tonality or key," whatever the answer may be.

The term "answer" is usually reserved for those entries of the subject that are placed in what may be called the "complementary" position of the scale, whether they are "tonally" modified or not. Thus the order of entries in the exposition of the first fugue of the *Wohltemp. Klav.* is subject, answer, answer, subject; a departure from the usual rule according to which subject and answer are strictly alternate in the exposition.

In conclusion we may remind the reader of the most accurate as well as the most vivid description ever given of the essentials of a fugue, in the famous lines in *Paradise Lost*, book xi.

"His volant touch,
Instinct through all proportions, low and high,
Fled and pursued transverse the resonant fugue."

It is hard to realize that this description of organ-music was written in no classical period of instrumental polyphony, but just half-way between the death of Frescobaldi and the birth of Bach. Every word is a definition, both retrospective and prophetic; and in "transverse" we see all that Sir Frederick Gore Ouseley expresses in his popular distinction between the "perpendicular" or homophonic style in which harmony is built up in chords, and the "horizontal" or polyphonic style in which it is woven in threads of independent melody.

(D. F. T.)

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1 An episode occurring during the exposition is sometimes called *codetta*, a distinction the uselessness of which at once appears on an analysis of Bach's 2nd fugue in the *Wohltemp. Klav.* (the term codetta is more correctly applied to notes filling in a gap between subject and its first answer, but such a gap is rare in good examples).

FÜHRICH, JOSEPH VON (1800-1876), Austrian painter, was born at Kratzau in Bohemia on the 9th of February 1800. Deeply impressed as a boy by rude pictures adorning the wayside chapels of his native country, his first attempt at composition was a sketch of the Nativity for the festival of Christmas in his father's house. He lived to see the day when, becoming celebrated as a composer of scriptural episodes, his sacred subjects were transferred in numberless repetitions to the roadside churches of the Austrian state, where humble peasants thus learnt to admire modern art reviving the models of earlier ages. Führich has been fairly described as a “Nazarene,” a romantic religious artist whose pencil did more than any other to restore the old spirit of Dürer and give new shape to countless incidents of the gospel and scriptural legends. Without the power of Cornelius or the grace of Overbeck, he composed with great skill, especially in outline. His mastery of distribution, form, movement and expression was considerable. In its peculiar way his drapery was perfectly cast. Essentially creative as a landscape draughtsman, he had still no feeling for colour; and when he produced monumental pictures he was not nearly so successful as when designing subjects for woodcuts. Führich’s fame extended far beyond the walls of the Austrian capital, and his illustrations to Tieck’s *Genofeva*, the Lord’s Prayer, the Triumph of Christ, the Road to Bethlehem, the Succession of Christ according to Thomas à Kempis, the Prodigal Son, and the verses of the Psalter, became well known. His Prodigal Son, especially, is remarkable for the fancy with which the spirit of evil is embodied in a figure constantly recurring, and like that of Mephistopheles exhibiting temptation in a human yet demoniacal shape. Führich became a pupil at the Academy of Prague in 1816. His first inspiration was derived from the prints of Dürer and the Faust of Cornelius, and the first fruit of this turn of study was the Genofeva series. In 1826 he went to Rome, where he added three frescoes to those executed by Cornelius and Overbeck in the Palazzo Massimi. His subjects were taken from the life of Tasso, and are almost solitary examples of his talent in this class of composition. In 1831 he finished the Triumph of Christ now in the Raczynski palace at Berlin. In 1834 he was made custos and in 1841 professor of composition in the Academy of Vienna. After this he completed the monumental pictures of the church of St Nepomuk, and in 1854-1861 the vast series of wall paintings which cover the inside of the Lerchenfeld church at Vienna. In 1872 he was pensioned and made a knight of the order of Franz Joseph; 1875 is the date of his illustrations to the Psalms. He died on the 13th of March 1876.

His autobiography was published in 1875, and a memoir by his son Lucas in 1886.

FUJI (Fuji-san, Fujiyama, Fusiyama), a celebrated mountain of Japan, standing W.S.W. of Tokyo, its base being about 70 m. by rail from that city. It rises to a height of 12,395 ft. and its southern slopes reach the shore of Suruga Bay. It is a cone of beautifully simple form, the more striking to view because it stands isolated; but its summit is not conical, being broken by a crater some 2000 ft. in diameter, for Fuji is a quiescent volcano. Small outbursts of steam are still to be observed at some points. An eruption is recorded so lately as the first decade of the 18th century. The mountain is the resort of great numbers of pilgrims (see also Japan).

FU-KIEN (formerly Min), a south-eastern province of China, bounded N. by the province of Cheh-kiang, S. by that of Kwang-tung, W. by that of Kiang-si and E. by the sea. It occupies an area of 53,480 sq. m. and its population is estimated at 20,000,000. The provincial capital is Fuchow Fu, and it is divided into eleven prefectures, besides that ruled over by the prefect of the capital city. Fu-kien is generally mountainous, being overspread by the Nan-shan ranges, which run a general course of N.E. and S.W. The principal river is the Min, which is formed by the junction, in the neighbourhood of the city of Yen-p’ing Fu, of three rivers, namely, the Nui-si, which takes its rise in the mountains on the western frontier in the prefecture of Kien-ning Fu, the Fuh-tun Ki, the source of which is found in the district of Kwang-tsih in the north-west of the province, and the Ta-shi-ki (Shao Ki), which rises in the mountains in the western district of Ning-hwa. From Yen-p’ing Fu the river takes a south-easterly course, and after passing along the south face of the city of Fuchow Fu, empties itself into the sea about 30 m. below that town. Its upper course is narrow and rocky and abounds in rapids, but as it approaches Fuchow Fu the channel widens and the current becomes slow and even. Its depth is very irregular, and it is navigable only by native boats of a small class. Two other rivers flow into the sea near Amoy, neither of which, however, is navigable for any distance from its mouth owing to the shallows and rapids with which they abound. Thirty-five miles inland from Amoy stands the city of Chang Chow, famous for the bridge which there spans the Kin-lung river. This bridge is 800 ft. long, and consists of granite monoliths stretching from one abutment to another. The soil of the province is, as its name, “Happy Establishment,” indicates, very productive, and the scenery is of a rich and varied character. Most of the hills are covered with verdure, and the less rugged are laid out in terraces. The principal products of the province are tea, of which the best kind is that known as Bohea, which takes its name, by a mispronunciation, from the Wu-e Mountains, in the prefecture of Kien-ning Fu, where it is grown; grains of various kinds, oranges, plantins, lichis, bamboo, ginger, gold, silver, lead, tin, iron, salt (both marine and rock), deers’ horns, beeswax, sugar, fish, birds’ nests, medicine, paper, cloth, timber, &c. Fu-kien has three open ports, Fuchow Fu opened in 1842, Amoy opened to trade in the same year and Funing. The latter port was only opened to foreign trade in 1898, but in 1904 it imported and exported goods to the value of £7668 and £278,160 respectively. 291

FUKUI, a town of Japan in the province of Echizen, Nippon, near the west coast, 20 m. N. by E. of Wakasa Bay. It lies in a volcanic district much exposed to earthquakes, and suffered severely during the disturbances of 1891-1892, when a chasm over 40 m. long was opened across the Neo valley from Fukui to Katabira. But Fukui subsequently revived, and is now in a flourishing condition, with several local industries, especially the manufacture of paper, and an increasing population exceeding 50,000. Fukui has railway communication. There are ruins of a castle of the Daimios of Echizen.

FUKUOKA, a town on the north-west coast of the island of Kiushiu, Japan, in the province of Chikuzen, 90 m. N.N.E. of Nagasaki by rail. Pop. about 72,000. With Hakata, on the opposite side of a small coast stream, it forms a large centre of population, with an increasing export trade and several local industries. Of these the most important is silk-weaving, and Hakata especially is noted for its durable silk fabrics. Fukuoka was formerly the residence of the powerful daimio of Chikuzen, and played a conspicuous part in the medieval history of Japan; the renowned temple of Yeiyas in the district was destroyed by fire during the revolution of 1868. There are several other places of this name in Japan, the most important being Fukuoka in the province of Mutsu, North Nippon, a railway station on the main line from Tokyo to Aimori Ura Bay. Pop. about 5000.

FULA (Fulbe, Fellatah or Peuls), a numerous and powerful African people, spread over an immense region from Senegal nearly to Darfur. Strictly they have no country of their own, and nowhere form the whole of the population, though nearly always the dominant native race. They are most numerous in Upper Senegal and in the countries under French sway immediately south of Senegambia, notably Futa Jallon. Farther east they rule, subject to the control of the French, Segu and Massena, countries on both banks of the upper Niger, to the south-west of Timbuktu. The districts within the great bend of the Niger have a large Fula population. East of that river Sokoto and its tributary emirates are ruled by Fula princes, subject to the control of the British Nigerian administration. Fula are settled in Bornu, Bagirmi, Wadai and the upper Nile Valley,[1] but have no political power in those countries. Their most southerly emirate is Adamawa, the country on both sides of the upper Benue. In this vast region of distribution the Fula populations are most dense towards the west and north, most scattered towards the east and south. Originally herdsmen in the western and central Sudan, they extended their sway east of the Niger, under the leadership of Othman Dan Fodio, during the early years of the 19th century, and having subdued the Hausa states, founded the empire of Sokoto with the vassal emirates of Kano, Gando, Nupe, Adamawa, &c.

The question of the ethnic affinities of the Fula has given rise to an enormous amount of speculation, but the most reasonable theory is that they are a mixture of Berber and Negro. This is now the most generally accepted theory. Certainly there is no reason to connect them with the ancient Egyptians. In the district of Senegal known as Fuladugu or “Fula Land,” where the purest types of the race are found, the people are of a reddish brown or light chestnut colour, with oval faces, ringlety or even smooth hair, never woolly, straight and even aquiline noses, delicately shaped lips and regular features quite differentiating them from the Negro type. Like most conquering races the Fula are, however, not of uniform physique, in many districts approximating to the local type. They nevertheless maintain throughout their widespread territory a certain national solidarity, thanks to common speech, traditions and usages. The ruling caste of the Fula differs widely in character from the herdsmen of the western Sudan. The latter are peaceable, inoffensive and abstemious. They are mainly monogamous, and by rigidly abstaining from foreign marriages have preserved racial purity. The ruling caste in Nigeria, on the other hand, despise their pastoral brethren, and through generations of polygamy with the conquered tribes have become more Negroid in type, black, burly and coarse featured. Love of luxury, pomp and finery is their chief characteristic. Taken as a whole, the Fula race is distinguished by great intelligence, frankness of disposition and strength of character. As soldiers they are renowned almost exclusively as cavalry; and the race has produced several leaders possessed of much strategical skill. Besides the ordinary Negro weapons, they use iron spears with leatherbound handles and swords. They are generally excellent rulers, stern but patient and just. The Nigerian emirs acquired, however, an evil reputation during the 19th century as slave raiders. They have long been devout Mahommedans, and mosques and schools exist in almost all their towns. Tradition says that of old every Fula boy and girl was a scholar; but during the decadence of their power towards the close of the 19th century education was not highly valued. Power seems to have somewhat spoilt this virile race, but such authorities as Sir Frederick Lugard believe them still capable of a great future.

The Fula language has as yet found no place in any African linguistic family. In its rudiments it is akin to the Hamito-Semitic group. It possesses two grammatical genders, not masculine and feminine, but the human and the non-human; the adjective agrees in assonance with its noun, and euphony plays a great part in verbal and nominal inflections. In some ways resembling the Negro dialects, it betrays non-Negroid influences in the use of suffixes. The name of the people has many variations. Fulbe or Fula (sing. Pullo, Peul) is the Mandingan name, Follani the Hausa, Fellatah the Kanuri, Fullan the Arab, and Fulde on the Benue. Like the name Abate, “white,” given them in Kororofa, all these seem to refer to their light reddish hue.

See F. Ratzel, *History of Mankind* (English ed., London, 1896-1898); Sir F. Lugard, “Northern Nigeria,” in *Geographical Journal* (July 1904); Grimai de Guirodon, *Les Puls* (1887); E.A. Brackenbury, *A Short Vocabulary of the Fulani Language* (Zungeru, 1907); the articles Nigeria and Sokoto and authorities there cited.

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[1] Sir Wm. Wallace in a report on Northern Nigeria (“Colonial Office” series, No. 551, 1907) calls attention to the exodus “of thousands of Fulani of all sorts, but mostly Mellawa, from the French Middle Niger,” and states that the majority of the emigrants are settling in the Nile valley.

FULCHER (or Foucher) **OF CHARTRES** (1058-*c.* 1130), French chronicler, was a priest who was present at the council of Clermont in 1095, and accompanied Robert II., duke of Normandy, on the first crusade in 1096. Having spent some time in Italy and taken part in the fighting on the way to the Holy Land, he became chaplain to Baldwin, who was chosen king of Jerusalem in 1100, and lived with Baldwin at Edessa and then at Jerusalem. He accompanied this king on several warlike expeditions, but won more lasting fame by writing his *Historia Hierosolymitana* or *Gesta Francorum Jerusalem expugnantium*, one of the most trustworthy sources for the history of the first crusade. In its final form it is divided into three books, and covers the period between the council of Clermont and 1127, and the author only gives details of events which he himself had witnessed. It was used by William of Tyre. Fulcher died after 1127, probably at Jerusalem. He has been confused with Foucher of Mongervillier (d. 1171), abbot of St-Père-en-Vallée at Chartres, and also with another person of the same name who distinguished himself at the siege of Antioch in 1098.

The *Historia*, but in an incomplete form, was first published by J. Bongars in the *Gesta Dei per Francos* (Hanover, 1611). The best edition is in tome iii. of the *Recueil des historiens des croisades, Historiens occidentaux* (Paris, 1866); and there is a French translation in tome xxiv. of Guizot’s *Collection des mémoires relatifs à l’histoire de France* (Paris, 1823-1835).

See H. von Sybel, *Geschichte des ersten Kreuzzuges* (Leipzig, 1881); and A. Molinier, *Les Sources de l’histoire de France*, tome ii. (Paris, 1902).

FULDA, a town and episcopal see of Germany, in the Prussian province of Hesse-Nassau, between the Rhön and the Vogel-Gebirge, 69 m. N.E. from Frankfort-on-Main on the railway to Bebra. Although irregularly built the town is pleasantly situated, and contains two fine squares, on one of which stands a fine statue of St Boniface. The present cathedral was built at the beginning of the 18th century on the model of St Peter’s at Rome, but it has an ancient crypt, which contains the bones of St Boniface and was restored in 1892. Opposite the cathedral is the former monastery of St Michael, now the episcopal palace. The Michaelskirche, attached to it, is a small round church built, in imitation of the Holy Sepulchre, in 822 and restored in 1853. Of other buildings may be mentioned the Library, with upwards of 80,000 printed books and many valuable MSS., the stately palace with its gardens and orangery, the former Benedictine nunnery (founded 1625, and now used as a seminary), and the Minorite friary (1238) now used as a furniture warehouse. Among the secular buildings are the fine *Schloss*, the *Bibliothek*, the town hall and the post office. There are several schools, a hospital founded in the 13th century, and some new artillery barracks. Many industries are carried on in Fulda. These include weaving and dyeing, the manufacture of linen, plush and other textiles and brewing. There are also railway works in the town. A large trade is done in cattle and grain, many markets being held here. Fine views are obtained from several hills in the neighbourhood, among these being the Frauenberg, the Petersberg and the Kalvarienberg.

Fulda owes its existence to its famous abbey. It became a town in 1208, and during the middle ages there were many struggles between the abbots and the townsfolk. During the Peasants’ War it was captured by the rebels and during the Seven Years’ War by the Hanoverians. It came finally into the possession of Prussia in 1866. From 1734 to 1804 Fulda was the seat of a university, and latterly many assemblies of German bishops have been held in the town.

The great Benedictine abbey of Fulda occupies the place in the ecclesiastical history of Germany which Monte Cassino holds in Italy, St Gall in South Germany, Corvey in Saxony, Tours in France and Iona in Scotland. Founded in 744 at the instigation of St Boniface by his pupil Sturm, who was the first abbot, it became the centre of a great missionary work. It was liberally endowed with land by the princes of the Carolingian house and others, and soon became one of the most famous and wealthy establishments of its kind. About 968 the pope declared that its abbot was primate of all the abbots in Germany and Gaul, and later he became a prince of the Empire. Fulda was specially famous for its school, which was the centre of the theological learning of the early middle ages. Among the teachers here were Alcuin, Hrabanus Maurus, who was abbot from 822 to 842, and Walafrid Strabo. Early in the 10th century the monastery was reformed by introducing monks from Scotland, who were responsible for restoring in its old strictness the Benedictine rule. Later the abbey lost some of its lands and also its high position, and some time before the Reformation the days of its glory were over. Johann von Henneberg, who was abbot from 1529 to 1541, showed some sympathy with the teaching of the reformers, but the Counter-Reformation made great progress here under Abbot Balthasar von Dernbach. Gustavus Adolphus gave the abbey as a principality to William, landgrave of Hesse, but William’s rule only lasted for ten years. In 1752 the abbot was raised to the rank of a bishop, and Fulda ranked as a prince-bishopric. This was secularized in 1802, and in quick succession it belonged to the prince of Orange, the king of France and the grand-duchy of Frankfort. In 1816 the greater part of the principality was ceded by Prussia to Hesse-Cassel, a smaller portion being united with Bavaria. Sharing the fate of Hesse-Cassel, this larger portion was annexed by Prussia in 1866. In 1829 a new bishopric was founded at Fulda.

For the town see A. Hartmann, *Zeitgeschichte von Fulda* (Fulda, 1895); J. Schneider, *Führer durch die Stadt Fulda* (Fulda, 1899); and *Chronik von Fulda und dessen Umgebungen* (1839). For the history of the abbey see Gegenbaur, *Das Kloster Fulda im Karolinger Zeitalter* (Fulda, 1871-1874); Arndt, *Geschichte des Hochstifts Fulda* (Fulda, 1860); and the *Fuldaer Geschichtsblätter* (1902 fol.).

FULGENTIUS, FABIUS PLANCIADES, Latin grammarian, a native of Africa, flourished in the first half of the 6th (or the last part of the 5th) century A.D. He is to be distinguished from Fulgentius, bishop of Ruspe (468-533), to whom he was probably related, and also from the bishop’s pupil and biographer, Fulgentius Ferrandus. Four extant works are attributed to him. (1) *Mythologiarum libri iii.*, dedicated to a certain Catus, a presbyter of Carthage, containing 75 myths briefly told, and then explained in the mystical and allegorical manner of the Stoics and Neoplatonists. For this purpose the author generally invokes the aid of etymologies which, borrowed from the philosophers, are highly absurd. As a Christian, Fulgentius sometimes (but less frequently than might have been expected) quotes the Bible by the side of the philosophers, to give a Christian colouring to the moral lesson. (2) *Expositio Vergilianae continentiae* (*continentia* = contents), a sort of appendix to (1), dedicated to Catus. The poet himself appears to the author and explains the twelve books of the *Aeneid* as a picture of human life. The three words *arma* (= virtus), *vir* (= sapientia), *primus* (= princeps) in the first line represent respectively *substantia corporalis, sensualis, ornans*. Book i. symbolizes the birth and early childhood of man (the shipwreck of Aeneas denotes the peril of birth), book vi. the plunge into the depths of wisdom. (3) *Expositio sermonum antiquorum*, explanations of 63 rare and obsolete words, supported by quotations (sometimes from authors and works that never existed). It is much inferior to the similar work of Nonius, with which it is often edited. (4) *Liber absque litteris de aetatibus mundi et hominis*. In the MS. heading of this work, the name of the author is given as Fabius Claudius Gordianus Fulgentius (Claudius is the name of the father, and Gordianus that of the grandfather of the bishop, to whom some attribute the work). The title *Absque litteris* indicates that one letter of the alphabet is wholly omitted in each successive book (A in bk. i., B in bk. ii.). Only 14 books are preserved. The matter is chiefly taken from sacred history. In addition to these, Fulgentius speaks of early poetical attempts after the manner of Anacreon, and of a work called *Physiologus*, dealing with medical questions, and including a discussion of the mystical signification of the numbers 7 and 9. Fulgentius is a representative of the so-called late African style, taking for his models Apuleius, Tertullian and Martianus Capella. His language is bombastic, affected and incorrect, while the lengthy and elaborate periods make it difficult to understand his meaning.

See the edition of the four works by R. Helm (1898, Teubner series); also M. Zink, *Der Mytholog Fulgentius* 293
(1867); E. Jungmann, “De Fulgentii aetate et scriptis,” in *Acta Societatis Philologae Lipsiensis*, i. (1871); A. Ebert, *Allgemeine Geschichte der Litt. des Mittelalters*, i.; article “Fulgentius” by C.F. Böhr in Ersch and Gruber’s *Allgemeine Encyklopädie*; Teuffel-Schwabe, *History of Roman Literature* (Eng. trans.).

FULGINIAE (mod. *Foligno*), an ancient town of Umbria, Italy, on the later line of the Via Flaminia, 15 m. S. of Nuceria. It appears to have been of comparatively late origin, inasmuch as it had no city walls, but, in imperial times especially, owing to its position on the new line of the Via Flaminia, it must have increased in importance as being the point of departure of roads to Perusia and to Picenum over the pass of Plestia. It appears to have had an amphitheatre, and three bridges over the Topino are attributed to the Roman period. Three miles to the N. lies the independent community of Forum Flaminii, the site of which is marked by the church of S. Giovanni Profiamma, at or near which the newer line of the Via Flaminia rejoined the older. It was no doubt founded by the builder of the road, C. Flaminius, consul in 220 B.C. (See Foligno and Flaminia, Via.)

(T. As.)

FULGURITE (from Lat. *fulgur*, lightning), in petrology, the name given to rocks which have been fused on the surface by lightning, and to the characteristic holes in rocks formed by the same agency. When lightning strikes the naked surfaces of rocks, the sudden rise of temperature may produce a certain amount of fusion, especially when the rocks are dry and the electricity is not readily conducted away. Instances of this have been observed on Ararat and on several mountains in the Alps, Pyrenees, &c. A thin glassy crust, resembling a coat of varnish, is formed; its thickness is usually not more than one-eighth of an inch, and it may be colourless, white or yellow. When examined under the microscope, it usually shows no crystallization, and contains minute bubbles due to the expansion of air or other gases in the fused pellicle. Occasionally small microliths may appear, but this is uncommon because so thin a film would cool with extreme rapidity. The minerals of the rock beneath are in some cases partly fused, but the more refractory often appear quite unaffected. The glass has arisen from the melting of the most fusible ingredients alone.

Another type of fulgurite is commonest in dry sands and takes the shape of vertical tubes which may be nearly half an inch in diameter. Generally they are elliptical in cross section, or flattened by the pressure exerted by the surrounding sand on the fulgurite at a time when it was still very hot and plastic. These tubes are often vertical and may run downwards for several feet through the sand, branching and lessening as they descend. Tubular perforations in hard rocks have been noted also, but these are short and probably follow original cracks. The glassy material contains grains of sand and many small round or elliptical cavities, the long axes of which are radial. Minerals like felspar and mica are fused more readily than quartz, but analysis shows that some fulgurite glasses are very rich in silica, which perhaps was dissolved in the glass rather than simply fused. The central cavity of the tube and the bubbles in its walls point to the expansion of the gases (air, water, &c.) in the sand by sudden and extreme heating. Very fine threads of glass project from the surface of the tube as if fused droplets had been projected outwards with considerable force. Where the quartz grains have been greatly heated but not melted they become white and semi-opaque, but where they are in contact with the glass they usually show partial solution. Occasionally crystallization has begun before the glass solidified, and small microliths, the nature of which is undeterminable, occur in streams and wisps in the clear hyaline matrix.

(J. S. F.)

FULHAM, a western metropolitan borough of London, England, bounded N.W. by Hammersmith, N.E. by Kensington, E. by Chelsea, and S.E., S. and S.W. by the river Thames. Pop. (1901) 137,289. The principal thoroughfares are Fulham Palace Road running S. from Hammersmith, Fulham Road and King’s Road, W. from Chelsea, converging and leading to Putney Bridge over the Thames; North End Road between Hammersmith and Fulham Roads; Lillie Road between South Kensington and Fulham Palace Road; and Wandsworth Bridge Road leading S. from New King’s Road to Wandsworth Bridge. In the north Fulham includes the residential district known as West Kensington, and farther south that of Walham Green. The manor house or palace of the bishops of London stands in grounds, beautifully planted and surrounded by a moat, believed to be a Danish work, near the river west of Putney Bridge. Its oldest portion is the picturesque western quadrangle, built by Bishop Fitzjames (1506-1522). The parish church of All Saints, between the bridge and the grounds, was erected in 1881 from designs by Sir Arthur Blomfield. The fine old monuments from the former building, dating from the 16th to the 18th centuries, are mostly preserved, and in the churchyard are the memorials of several bishops of London and of Theodore Hook (1841). The public recreation grounds include the embankment and gardens between the river and the palace grounds, and there are also two well-known enclosures used for sports within the borough. Of these Hurlingham Park is the headquarters of the Hurlingham Polo Club and a fashionable resort; and Queen’s Club, West Kensington, has tennis and other courts for the use of members, and is also the scene of important football matches, and of the athletic meetings between Oxford and Cambridge Universities, and those between the English and American Universities held in England. In Seagrave Road is the Western fever hospital. The parliamentary borough of Fulham returns one member. The borough council consists of a mayor, 6 aldermen and 36 councillors. Area, 1703.5 acres.

Fulham, or in its earliest form *Fullanham*, is uncertainly stated to signify “the place” either “of fowls” or “of dirt.” The manor is said to have been given to Bishop Erkenwald about the year 691 for himself and his successors in the see of London, and Holinshed relates that the Bishop of London was lodging in his manor place in 1141 when Geoffrey de Mandeville, riding out from the Tower of London, took him prisoner. At the Commonwealth the manor was temporarily out of the bishops’ hands, being sold to Colonel Edmund Harvey. There is no record of the first erection of a parish church, but the first known rector was appointed in 1242, and a church probably existed a century before this. The earliest part of the church demolished in 1881, however, did not date farther back than the 15th century. In 879 Danish invaders, sailing up the Thames, wintered at Fulham and Hammersmith. Near the former wooden Putney Bridge, built in 1729 and replaced in 1886, the earl of Essex threw a bridge of boats across the river in 1642 in order to march his army in pursuit of Charles I., who thereupon fell back on Oxford. Margravine Road recalls the existence of Bradenburg House, a riverside mansion built by Sir Nicholas Crispe in the time of Charles I., used as the headquarters of General Fairfax in 1647 during the civil wars, and occupied in 1792 by the margrave of Bradenburg-Anspach and Bayreuth and his wife, and in 1820 by Caroline, consort of George IV.

FULK, king of Jerusalem (b. 1092), was the son of Fulk IV., count of Anjou, and his wife Bertrada (who ultimately deserted her husband and became the mistress of Philip I. of France). He became count of Anjou in 1109, and considerably added to the prestige of his house. In particular he showed himself a doughty opponent to Henry I. of England, against whom he continually supported Louis VI. of France, until in 1127 Henry won him over by betrothing his daughter Matilda to Fulk’s son Geoffrey Plantagenet. Already in 1120 Fulk had visited the Holy Land, and become a close friend of the Templars. On his return he assigned to the order of the Templars an annual subsidy, while he also maintained two knights in the Holy Land for a year. In 1128 he was preparing to return to the East, when he received an embassy from Baldwin II., king of Jerusalem, who had no male heir to succeed him, offering his daughter Melisinda in marriage, with the right of eventual succession to the kingdom. Fulk readily accepted the offer; and in 1129 he came and was married to Melisinda, receiving the towns of Acre and Tyre as her dower. In 1131, at the age of thirty-nine, he became 294
king of Jerusalem. His reign is not marked by any considerable events: the kingdom which had reached its zenith under Baldwin II., and did not begin to decline till the capture of Edessa in the reign of Baldwin III., was quietly prosperous under his rule. In the beginning of his reign he had to act as regent of Antioch, and to provide a husband, Raymund of Poitou, for the infant heiress Constance. But the great problem with which he had to deal was the progress of the atabeg Zengi of Mosul. In 1137 he was beaten near Barin, and escaping into the fort was surrounded and forced to capitulate. A little later, however, he greatly improved his position by strengthening his alliance with the vizier of Damascus, who also had to fear the progress of Zengi (1140); and in this way he was able to capture the fort of Banias, to the N. of Lake Tiberias. Fulk also strengthened the kingdom on the south; while his butler, Paganus, planted the fortress of Krak to the south of the Dead Sea, and helped to give the kingdom an access towards the Red Sea, he himself constructed Blanche Garde and other forts on the S.W. to overawe the garrison of Ascalon, which was still held by the Mahommedans, and to clear the road towards Egypt. Twice in Fulk’s reign the eastern emperor, John Comnenus, appeared in northern Syria (1137 and 1142); but his coming did not affect the king, who was able to decline politely a visit which the emperor proposed to make to Jerusalem. In 1143 he died, leaving two sons, who both became kings, as Baldwin III. and Amalric I.

Fulk continued the tradition of good statesmanship and sound churchmanship which Baldwin I. and Baldwin II. had begun. William of Tyre speaks of him as a fine soldier, an able politician, and a good son of the church, and only blames him for partiality to his friends, and a forgetfulness of names and faces, which placed him at a disadvantage and made him too dependent on his immediate intimates. Little, perhaps, need be made of these censures: the real fault of Fulk was his neglect to envisage the needs of the northern principalities, and to head a combined resistance to the rising power of Zengi of Mosul.

His reign in Jerusalem is narrated by R. Röhricht (*Geschichte des Königreichs Jerusalem*, Innsbruck, 1898), and has been made the subject of a monograph by G. Dodu (*De Fulconis Hierosolymitani regno*, Paris, 1894).

(E. Br.)

FULK (d. 900), archbishop of Reims, and partisan of Charles the Simple in his struggle with Odo, count of Paris, was elected to the see as archbishop in 883 upon the death of Hincmar. In 887 he was engaged in a struggle with the Normans who invaded his territories. Upon the deposition of Charles the Fat he sided with Charles the Simple in his contest for the West Frankish dominions against Count Odo of Paris, and crowned him king in his own metropolitan church at Reims after most of the nobles had gone over to Odo (893). Upon the death of Odo he succeeded in having Charles recognized as king by a majority of the West Frankish nobility. In 892 he obtained special privileges for his province from Pope Formosus, who promised that thereafter, when the archbishopric became vacant, the revenues should not be enjoyed by anyone while the vacancy existed, but should be reserved for the new incumbent, provided the election took place within the canonical limit of three months. From 898 until his death he held the office of chancellor, which for some time afterwards was regularly filled by the archbishop of Reims. In his efforts to keep the wealthy abbeys and benefices of the church out of the hands of the nobles, he incurred the hatred of Baldwin, count of Flanders, who secured his assassination on the 17th of June 900, a crime which the weak Carolingian monarch left unpunished.

Fulk left some letters, which are collected in Migne, *Patrologia Latina*, vol. cxxxi. 11-14.

FULKE, WILLIAM (1538-1589), Puritan divine, was born in London and educated at Cambridge. After studying law for six years, he became a fellow at St John’s College, Cambridge, in 1564. He took a leading part in the “vestiarian” controversy, and persuaded the college to discard the surplice. In consequence he was expelled from St. John’s for a time, but in 1567 he became Hebrew lecturer and preacher there. After standing unsuccessfully for the headship of the college in 1569, he became chaplain to the earl of Leicester, and received from him the livings of Warley, in Essex, and Dennington in Suffolk. In 1578 he was elected master of Pembroke Hall, Cambridge. As a Puritan controversialist he was remarkably active; in 1580 the bishop of Ely appointed him to defend puritanism against the Roman Catholics, Thomas Watson, ex-bishop of Lincoln (1513-1584), and John Feckenham, formerly abbot of Westminster, and in 1581 he was one of the disputants with the Jesuit, Edmund Campion, while in 1582 he was among the clergy selected by the privy council to argue against any papist. His numerous polemical writings include *A Defense of the sincere true Translations of the holie Scriptures into the English tong* (London, 1583), and confutations of Thomas Stapleton (1535-1598), Cardinal Allen and other Roman Catholic controversialists.

FULK NERRA (*c.* 970-1040), count of Anjou, eldest son of Count Geoffrey I., “Grisegonelle” (Grey Tunic) and Adela of Vermandois, was born about 970 and succeeded his father in the countship of Anjou on the 21st of July 987. He was successful in repelling the attacks of the count of Rennes and laying the foundations of the conquest of Touraine (see Anjou). In this connexion he built a great number of strong castles, which has led in modern times to his being called “the great builder.” He also founded several religious houses, among them the abbeys of Beaulieu, near Loches (*c.* 1007), of Saint-Nicholas at Angers (1020) and of Ronceray at Angers (1028), and, in order to expiate his crimes of violence, made three pilgrimages to the Holy Land (in 1002-1003, *c.* 1008 and in 1039). On his return from the third of these journeys he died at Metz in Lorraine on the 21st of June 1040. By his first marriage, with Elizabeth, daughter of Bouchard le Vénérable, count of Vendôme, he had a daughter, Adela, who married Boon of Nevers and transmitted to her children the countship of Vendôme. Elizabeth having died in 1000, Fulk married Hildegarde of Lorraine, by whom he had a son, Geoffrey Martel (*q.v.*), and a daughter Ermengarde, who married Geoffrey, count of Gâtinais, and was the mother of Geoffrey “le Barbu” (the Bearded) and of Fulk “le Réchin” (see Anjou).

See Louis Halphen, *Le Comté d’Anjou au XI^e siècle* (Paris, 1906). The biography of Fulk Nerra by Alexandre de Salies, *Histoire de Foulques Nerra* (Angers, 1874) is confused and uncritical. A very summary biography is given by Célestin Port, *Dictionnaire historique, géographique et biographique de Maine-et-Loire* (3 vols., Paris-Angers, 1874-1878), vol. ii. pp. 189-192, and there is also a sketch in Kate Norgate, *England under the Angevin Kings* (2 vols., London, 1887), vol. i. ch. iii.

(L. H.*)

FÜLLEBORN, GEORG GUSTAV (1769-1803), German philosopher, philologist and miscellaneous writer, was born at Glogau, Silesia, on the 2nd of March 1769, and died at Breslau on the 6th of February 1803. He was educated at the University of Halle, and was made doctor of philosophy in recognition of his thesis *De Xenophane, Zenone et Gorgia*. He took diaconal orders in 1791, but almost immediately became professor of classics at Breslau. His philosophical works include annotations to Garve's translation of the *Politics* of Aristotle (1799-1800), and a large share in the *Beiträge zur Geschichte der Philosophie* (published in twelve parts between 1791 and 1799), in which he collaborated with Forberg, Reinhold and Niethammer. In philology he wrote *Encyclopaedia philologica sive primae lineae Isagoges in antiquorum studia* (1798; 2nd ed., 1805); *Kurze Theorie des lateinischen Stils* (1793); *Leitfaden der Rhetorik* (1802); and an annotated edition of the *Satires* of Persius. Under the pseudonym "Edelwald Justus" he published several collections of popular tales—*Bunte Blätter* (1795); *Kleine Schriften zur Unterhaltung* (1798); *Nebenstunden* (1799). After his death were published *Taschenbuch für Brunnengäste* (1806) and *Kanzelreden* (1807). He was a frequent contributor to the press, where his writings were very popular.

See Schummel, *Gedächtnisrede* (1803) and *Garve und Fülleborn*; Meusel, *Gelehrtes Teutschland*, vol. ii.

FULLER, ANDREW (1754-1815), English Baptist divine, was born on the 6th of February 1754, at Wicken in Cambridgeshire. In his boyhood and youth he worked on his father's farm. In his seventeenth year he became a member of the Baptist church at Soham, and his gifts as an exhorter met with so much approval that, in the spring of 1775, he was called and ordained as pastor of that congregation. In 1782 he removed to Kettering in Northamptonshire, where he became friendly with some of the most eminent ministers of the denomination. Before leaving Soham he had written the substance of a treatise in which he had sought to counteract the prevailing Baptist hyper-Calvinism which, "admitting nothing spiritually good to be the duty of the unregenerate, and nothing to be addressed to them in a way of exhortation excepting what related to external obedience," had long perplexed his own mind. This work he published, under the title *The Gospel worthy of all Acceptation*, soon after his settlement in Kettering; and although it immediately involved him in a somewhat bitter controversy which lasted for nearly twenty years, it was ultimately successful in considerably modifying the views prevalent among English dissenters. In 1793 he published a treatise, *The Calvinistic and Socinian systems examined and compared as to their moral tendency*, in which he rebutted the accusation of antinomianism levelled by the Socinians against those who over-emphasized the doctrines of free grace. This work, along with another against Deism, entitled *The Gospel its own Witness*, is regarded as the production on which his reputation as a theologian mainly rests. Fuller also published an admirable *Memoir of the Rev. Samuel Pearce*, of Birmingham, and a volume of *Expository Lectures in Genesis*, besides a considerable number of smaller pieces, chiefly sermons and pamphlets, which were issued in a collected form after his death. He was a man of forceful character, more prominent on the practical side of religion than on the devotional, and accordingly not pre-eminently successful in his local ministry. His great work was done in connexion with the Baptist Missionary Society, formed at Kettering in 1792, of which he was secretary until his death on the 7th of May 1815. Both Princeton and Yale, U.S.A., conferred on him the degree of D. D., but he never used it.

Several editions of his collected works have appeared, and a *Memoir*, principally compiled from his own papers, was published about a year after his decease by Dr Ryland, his most intimate friend and coadjutor in the affairs of the Baptist mission. There is also a biography by the Rev. J.W. Morris (1816); and his son prefixed a memoir to an edition of his chief works in Bohn's Standard Library (1852).

FULLER, GEORGE (1822-1884), American figure and portrait painter, was born at Deerfield, Massachusetts, in 1822. At the age of twenty he entered the studio of the sculptor H.K. Brown, at Albany, New York, where he drew from the cast and modelled heads. Having attained some proficiency he went about the country painting portraits, settling at length in Boston, where he studied the works of the earlier Americans, Stuart, Copley and Allston. After three years in that city, and twelve in New York, where in 1857 he was elected a member of the National Academy of Design, he went to Europe for a brief visit and for study. During all this time his work had received little recognition and practically no financial encouragement, and on his return he settled on the family farm at Deerfield, where he continued to work in his own way with no thought of the outside world. In 1876, however, he was forced by pressing needs to dispose of his work, and he sent some pictures to a dealer in Boston, where he met with immediate success, financial and artistic, and for the remaining eight years of his life he never lacked patrons. He died in Boston on the 21st of March 1884. He was a poetic painter, and a dreamer of delicate fancies and quaint, intangible phases of nature, his canvases being usually enveloped in a brown mist that renders the outlines vague. Among his noteworthy canvases are: "The Turkey Pasture," "Romany Girl," "And she was a Witch," "Nydia," "Winifred Dysart" and "The Quadroon."

FULLER, MARGARET, Marchioness Ossoli (1810-1850), American authoress, eldest child of Timothy

Fuller (1778-1835), a lawyer and politician of some eminence, was born at Cambridgeport, Massachusetts, on the 23rd of May 1810. Her education was conducted by her father, who, she states, made the mistake of thinking to “gain time by bringing forward the intellect as early as possible,” the consequence being “a premature development of brain that made her a youthful prodigy by day, and by night a victim of spectral illusions, nightmare and somnambulism.” At six years she began to read Latin, and at a very early age she had selected as her favourite authors Shakespeare, Cervantes and Molière. Soon the great amount of study exacted of her ceased to be a burden, and reading became a habit and a passion. Having made herself familiar with the masterpieces of French, Italian and Spanish literature, she in 1833 began the study of German, and within the year had read some of the masterpieces of Goethe, Körner, Novalis and Schiller.

After her father’s death in 1835 she went to Boston to teach languages, and in 1837 she was chosen principal teacher in the Green Street school, Providence, Rhode Island, where she remained till 1839. From this year until 1844 she stayed at different places in the immediate neighbourhood of Boston, forming an intimate acquaintance with the colonists of Brook Farm, and numbering among her closest friends R.W. Emerson, Nathaniel Hawthorne and W.H. Channing. In 1839 she published a translation of Eckermann’s *Conversations with Goethe*, which was followed in 1842 by a translation of the correspondence between Karoline von Günderode and Bettina von Arnim, entitled *Günderode*. Aided by R.W. Emerson and George Ripley, she in 1840 started *The Dial*, a poetical and philosophical magazine representing the opinions and aims of the New England Transcendentalists. This journal she continued to edit for two years, and while in Boston she also conducted conversation classes for ladies in which philosophical and social subjects were discussed with a somewhat over-accentuated earnestness. These meetings may be regarded as perhaps the beginning of the modern movement in behalf of women’s rights. R.W. Emerson, who had met her as early as 1836, thus describes her appearance: “She was then twenty-six years old. She had a face and frame that would indicate fulness and tenacity of life. She was rather under the middle height; her complexion was fair, with strong fair hair. She was then, as always, carefully and becomingly dressed, and of ladylike self-possession. For the rest her appearance had nothing prepossessing. Her extreme plainness, a trick of incessantly opening and shutting her eyelids, the nasal tone of her voice, all repelled; and I said to myself we shall never get far.” On better acquaintance this unprepossessing exterior seemed, however, to melt away, and her inordinate self-esteem to be lost in the depth and universality of her sympathy. She possessed an almost irresistible power of winning the intellectual and moral confidence of those with whom she came in contact, and “applied herself to her companion as the sponge applies itself to water.” She obtained from each the best they had to give. It was indeed more as a conversationalist than as a writer that she earned the title of the Priestess of Transcendentalism. It was her intimate friends who admired her most. Smart and pungent though she is as a writer, the apparent originality of her views depends more on eccentricity than either intellectual depth or imaginative vigour. In 1844 she removed to New York at the desire of Horace Greeley to write literary criticism for *The Tribune*, and in 1846 she published a selection from her articles on contemporary authors in Europe and America, under the title *Papers on Literature and Art*. The same year she paid a visit to Europe, passing some time in England and France, and finally taking up her residence in Italy. There she was married in December 1847 to the marquis Giovanni Angelo Ossoli, a friend of Mazzini. During 1848-1849 she was present with her husband in Rome, and when the city was besieged she, at the request of Mazzini, took charge of one of the two hospitals while her husband fought on the walls. In May 1850, along with her husband and infant son, she embarked at Leghorn for America, but when they had all but reached their destination the vessel was wrecked on Fire Island beach on the 16th of June, and the 296
Ossolis were among the passengers who perished.

Life Without and Life Within (Boston, 1860) is a collection of essays, poems, &c., supplementary to her *Collected Works*, printed in 1855. See the *Autobiography of Margaret Fuller Ossoli*, with additional memoirs by J.F. Clarke, R.W. Emerson and W.H. Channing (2 vols., Boston, 1852); also *Margaret Fuller (Marchesa Ossoli)*, by Julia Ward Howe (1883), in the “Eminent Women” series; *Margaret Fuller Ossoli* (Boston, 1884), by Thomas Wentworth Higginson in the “American Men of Letters” series, which is based largely on unedited material; and *The Love Letters of Margaret Fuller, 1845-1846* (London and New York, 1903), with an introduction by Julia Ward Howe.

FULLER, MELVILLE WESTON (1833-1910), American jurist, chief justice of the Supreme Court of the United States, was born at Augusta, Maine, on the 11th of February 1833. After graduating at Bowdoin College in 1853 he spent a year at the Harvard Law School, and in 1855 began the practice of law at Augusta, where he was an associate-editor of a Democratic paper, *The Age*, and served in the city council and as city attorney. In 1856 he removed to Chicago, Illinois, where he continued to practise until 1888, rising to a high position at the bar of the Northwest. For some years he was active in Democratic politics, being a member of the Illinois Constitutional Convention in 1862 and of the State House of Representatives from 1863 to 1865. He was a delegate to various National conventions of his party, and in that of 1876 placed Thomas A. Hendricks in nomination for the presidency. In 1888, by President Cleveland’s appointment, he succeeded Morrison R. Waite as chief-justice of the Supreme Court of the United States. In 1899 he was appointed by President McKinley a member of the arbitration commission at Paris to settle the Venezuela-British Guiana boundary dispute.

FULLER, THOMAS (1608-1661), English divine and historian, eldest son of Thomas Fuller, rector of

Aldwincle St Peter's, Northamptonshire, was born at his father's rectory and was baptized on the 19th of June 1608. Dr John Davenant, bishop of Salisbury, was his uncle and godfather. According to Aubrey, Fuller was "a boy of pregnant wit." At thirteen he was admitted to Queens' College, Cambridge, then presided over by Dr John Davenant. His cousin, Edward Davenant, was a tutor in the same college. He was apt and quick in study; and in Lent 1624-1625 he became B.A. and in July 1628 M.A. Being overlooked in an election of fellows of his college, he was removed by Bishop Davenant to Sidney Sussex College, November 1628. In 1630 he received from Corpus Christi College the curacy of St Benet's, Cambridge.

Fuller's quaint and humorous oratory soon attracted attention. He published in 1631 a poem on the subject of David and Bathsheba, entitled *David's Hainous Sinne, Heartie Repentance, Heavie Punishment*. In June of the same year his uncle gave him a prebend in Salisbury, where his father, who died in the following year, held a canonry. The rectory of Broadwindsor, Dorsetshire, then in the diocese of Bristol, was his next preferment (1634); and on the 11th of June 1635 he proceeded B.D. At Broadwindsor he compiled *The Historie of the Holy Warre* (1639), a history of the crusades, and *The Holy State and the Prophane State* (1642). This work describes the holy state as existing in the family and in public life, gives rules of conduct, model "characters" for the various professions and profane biographies. It was perhaps the most popular of all his writings. He was in 1640 elected proctor for Bristol in the memorable convocation of Canterbury, which assembled with the Short Parliament. On the sudden dissolution of the latter he joined those who urged that convocation should likewise dissolve as usual. That opinion was overruled; and the assembly continued to sit by virtue of a royal writ. Fuller has left in his *Church History* a valuable account of the proceedings of this synod, for sitting in which he was fined £200, which, however, was never exacted. His first published volume of sermons appeared in 1640 under the title of *Joseph's party-coloured Coat*, which contains many of his quaint utterances and odd conceits. His grosser mannerisms of style, derived from the divines of the former generation, disappeared for the most part in his subsequent discourses.

About 1640 he had married Eleanor, daughter of Hugh Grove of Chisenbury, Wiltshire. She died in 1641. Their eldest child, John, baptized at Broadwindsor by his father, 6th June 1641, was afterwards rector of Sidney Sussex College, edited the *Worthies of England*, 1662, and became rector of Great Wakering, Essex, where he died in 1687.

At Broadwindsor, early in the year 1641, Thomas Fuller, his curate Henry Sanders, the church wardens, and others, nine persons altogether, certified that their parish, represented by 242 grown-up male persons, had taken the Protestation ordered by the speaker of the Long Parliament. Fuller was not formally dispossessed of his living and prebend on the triumph of the Presbyterian party, but he relinquished both preferments about this time. For a short time he preached with success at the Inns of Court, and thence removed, at the invitation of the master of the Savoy (Dr Balcanqual) and the brotherhood of that foundation, to be lecturer at their chapel of St Mary Savoy. Some of the best discourses of the witty preacher were delivered at the Savoy to audiences which extended into the chapel-yard. In one he set forth with searching and truthful minuteness the hindrances to peace, and urged the signing of petitions to the king at Oxford, and to the parliament, to continue their care in advancing an accommodation. In his *Appeal of Injured Innocence* Fuller says that he was once deputed to carry a petition to the king at Oxford. This has been identified with a petition entrusted to Sir Edward Wardour, clerk of the pells, Dr Dukeson, "Dr Fuller," and four or five others from the city of Westminster and the parishes contiguous to the Savoy. A pass was granted by the House of Lords, on the 2nd of January 1643, for an equipage of two coaches, four or six horses and eight or ten attendants. On the arrival of the deputation at Uxbridge, on the 4th of January, officers of the Parliamentary army stopped the coaches and searched the gentlemen; and they found upon the latter "two scandalous books arraigning the proceedings of the House," and letters with ciphers to Lord Viscount Falkland and the Lord Spencer. Ultimately a joint order of both Houses remanded the party; and Fuller and his friends suffered a brief imprisonment. The Westminster Petition, notwithstanding, reached the king's hands; and it was published with the royal reply (see J.E. Bailey, *Life of Thomas Fuller*, pp. 245 *et seq.*). When it was expected, three months later, that a favourable result would attend the negotiations at Oxford, Fuller preached a sermon at Westminster Abbey, on the 27th of March 1643, on the anniversary of Charles I.'s accession, on the text, "Yea, let him take all, so my Lord the King return in peace." On Wednesday, the 26th of July, he preached on church reformation, satirizing the religious reformers, and maintaining that only the Supreme Power could initiate reforms.

He was now obliged to leave London, and in August 1643 he joined the king at Oxford. He lived in a hired chamber at Lincoln College for 17 weeks. Thence he put forth a witty and effective reply to John Saltmarsh, who had attacked his views on ecclesiastical reform. Fuller subsequently published by royal request a sermon preached on the 10th of May 1644, at St Mary's, Oxford, before the king and Prince Charles, called *Jacob's Vow*.

The spirit of Fuller's preaching, always characterized by calmness and moderation, gave offence to the high royalists, who charged him with lukewarmness in their cause. To silence unjust censures he became chaplain to the regiment of Sir Ralph Hopton. For the first five years of the war, as he said, when excusing the non-appearance of his *Church History*, "I had little list or leisure to write, fearing to be made a history, and shifting daily for my safety. All that time I could not live to study, who did only study to live." After the defeat of Hopton at Cheriton Down, Fuller retreated to Basing House. He took an active part in its defence, and his life with the troops caused him to be afterwards regarded as one of "the great cavalier parsons." In his marches with his regiment round about Oxford and in the west, he devoted much time to the collection of details, from churches, old buildings, and the conversation of ancient gossips, for his *Church-History* and *Worthies of England*. He compiled in 1645 a small volume of prayers and meditations,—the *Good Thoughts in Bad Times*,—which, set up and printed in the besieged city of Exeter, whither he had retired, was called by himself "the first fruits of Exeter press." It was inscribed to Lady Dalkeith, governess to the infant princess, Henrietta Anne (b. 1644), to whose household he was attached as chaplain. The corporation gave him the Bodleian lectureship on the 21st of March 1645/6, and he held it until the 17th of June following, soon after the surrender of the city to the parliament. *The Fear of losing the Old Light* (1646) was his farewell discourse to his Exeter friends. Under the Articles of Surrender Fuller made his composition with the government at London, his "delinquency" being that he had been present in the king's garrisons. In *Andronicus, or the*

Unfortunate Politician (1646), partly authentic and partly fictitious, he satirized the leaders of the Revolution; and for the comfort of sufferers by the war he issued (1647) a second devotional manual, entitled *Good Thoughts in Worse Times*, abounding in fervent aspirations, and drawing moral lessons in beautiful language out of the events of his life or the circumstances of the time. In grief over his losses, which included his library and manuscripts (his “upper and nether millstone”), and over the calamities of the country, he wrote his work on the *Cause and Cure of a Wounded Conscience* (1647). It was prepared at Boughton House in his native county, where he and his son were entertained by Edward Lord Montagu, who had been one of his contemporaries at the university and had taken the side of the parliament.

For the next few years of his life Fuller was mainly dependent upon his dealings with booksellers, of whom he asserted that none had ever lost by him. He made considerable progress in an English translation from the MS. of the *Annales* of his friend Archbishop Ussher. Amongst his benefactors it is curious to find Sir John Danvers of Chelsea, the regicide. Fuller in 1647 began to preach at St Clement’s, Eastcheap, and elsewhere in the capacity of lecturer. While at St Clement’s he was suspended; but speedily recovering his freedom, he preached wherever he was invited. At Chelsea, where also he occasionally officiated, he covertly preached a sermon on the death of Charles I., but he did not break with his Roundhead patrons. James Hay, 2nd earl of Carlisle, made him his chaplain, and presented him in 1648 or 1649 to the curacy of Waltham Abbey. His possession of the living was in jeopardy on the appointment of Cromwell’s “Tryers”; but he evaded their inquisitorial questions by his ready wit. He was not disturbed at Waltham in 1655, when the Protector’s edict prohibited the adherents of the late king from preaching. Lionel, 3rd earl of Middlesex, who lived at Copt Hall, near Waltham, gave him what remained of the books of the lord treasurer his father; and through the good offices of the marchioness of Hertford, part of his own pillaged library was restored to him. Fuller was thus able to prosecute his literary labours, producing successively his descriptive geography of the Holy Land, called *A Pisgah-Sight of Palestine* (1650), and his *Church-History of Britain* (1655), from the birth of Jesus Christ until the year 1648. With the *Church-History* was printed *The History of the University of Cambridge since the Conquest* and *The History of Waltham Abbey*. These works were furthered in no slight degree by his connexion with Sion College, London, where he had a chamber, as well for the convenience of the press as of his city lectureships. The *Church-History* was angrily attacked by Dr P. Heylyn, who, in the spirit of High-Churchmanship, wished, as he said, to vindicate the truth, the church and the injured clergy. About 1652 Fuller married his second wife, Mary Roper, youngest sister of Thomas, Viscount Baltinglass, by whom he had several children. At the Oxford Act of 1657, Robert South, who was *Terrae filius*, lampooned Fuller, whom he described in this *Oratio* as living in London, ever scribbling and each year bringing forth new *folia* like a tree. At length, continues South, the *Church-History* came forth with its 166 dedications to wealthy and noble friends; and with this huge volume under one arm, and his wife (said to be little of stature) on the other, he ran up and down the streets of London, seeking at the houses of his patrons invitations to dinner, to be repaid by his dull jests at table.

His last and best patron was George Berkeley, 1st Earl Berkeley (1628-1698), of Cranford House, Middlesex, whose chaplain he was, and who gave him Cranford rectory (1658). To this nobleman Fuller’s reply to Heylyn’s *Examen Historicum*, called *The Appeal of Injured Innocence* (1659), was inscribed. At the end of the *Appeal* is an epistle “to my loving friend Dr Peter Heylyn,” conceived in the admirable Christian spirit which characterized all Fuller’s dealings with controversialists. “Why should *Peter*,” he asked, “fall out with *Thomas*, both being disciples to the same Lord and Master? I assure you, sir, whatever you conceive to the contrary, I am cordial to the cause of the English Church, and my hoary hairs will go down to the grave in sorrow for her sufferings.”

In *An Alarum to the Counties of England and Wales* (1660) Fuller argued for a free and full parliament—free from force, as he expressed it, as well as from abjurations or previous engagements. *Mixt Contemplations in Better Times* (1660), dedicated to Lady Monk, tendered advice in the spirit of its motto, “Let your moderation be known to all men: the Lord is at hand.” There is good reason to suppose that Fuller was at the Hague immediately before the Restoration, in the retinue of Lord Berkeley, one of the commissioners of the House of Lords, whose last service to his friend was to interest himself in obtaining him a bishopric. *A Panegyrick to His Majesty on his Happy Return* was the last of Fuller’s verse-efforts. On the 2nd of August, by royal letters, he was admitted D.D. at Cambridge. He resumed his lectures at the Savoy, where Samuel Pepys heard him preach; but he preferred his conversation or his books to his sermons. Fuller’s last promotion was that of chaplain in extraordinary to Charles II. In the summer of 1661 he visited the west in connexion with the business of his prebend, which had been restored to him. On Sunday, the 12th of August, while preaching at the Savoy, he was seized with typhus fever, and died at his new lodgings in Covent Garden on the 16th of August. He was buried in Cranford church, where a mural tablet was afterwards set up on the north side of the chancel, with an epitaph which contains a conceit worthy of his own pen, to the effect that while he was endeavouring (viz. in *The Worthies*) to give immortality to others, he himself attained it.

Fuller’s wit and vivacious good-humour made him a favourite with men of both sides, and his sense of humour kept him from extremes. Probably Heylyn and South had some excuse for their attitude towards his very moderate politics. “By his particular temper and management,” said Echard (*Hist. of England*, iii. 71), “he weathered the late great storm with more success than many other great men.” He was known as “a perfect walking library.” The strength of his memory was proverbial, and some amusing anecdotes are connected with it.

His writings were the product of a highly original mind. He had a fertile imagination and a happy faculty of illustration. Antithetic and axiomatic sentences abound in his pages, embodying literally the wisdom of the many in the wit of one. He was “quaint,” and something more. “Wit,” said Coleridge, in a well-known eulogy, “was the stuff and substance of Fuller’s intellect. It was the element, the earthen base, the material which he worked in; and this very circumstance has defrauded him of his due praise for the practical wisdom of the thoughts, for the beauty and variety of the truths, into which he shaped the stuff. Fuller was incomparably the most sensible, the least prejudiced, great man of an age that boasted a galaxy of great men” (*Literary Remains*, vol. ii. (1836), pp. 389-390). This opinion was formed after the perusal of the *Church-History*. That work and *The History of the Worthies of England* are unquestionably Fuller’s greatest efforts. They embody the collections of an entire life; and since his day they have been the delight of many readers. The *Holy State* has taken rank amongst the best books of “characters.” Charles Lamb made some selections from Fuller, and had a profound admiration for the “golden works” of the “dear, fine, silly old angel.” Since Lamb’s time, mainly through the appreciative criticisms of S.T. Coleridge, Robert Southey and others, Fuller’s works have received much attention.

There is an elaborate account of the life and writings of Fuller by William Oldys in the *Biographia Britannica*, vol. iii. (1750), based on Fuller’s own works and the anonymous *Life of ... Dr Thomas Fuller* (1661; reprinted in a volume of selections by A.L.J. Gosset, 1893). The completest account of him is *The Life of Thomas Fuller, with Notices of his Books, his Kinsmen and his Friends* (1874), by J.E. Bailey, who gives a detailed bibliography (pp. 713-762) of his works. *The Worthies of England* was reprinted by John Nichols (1811) and by P.A. Nuttall (1840). His *Collected Sermons* were edited by J.E. Bailey and W.E.A. Axon in 1891. Fuller’s quaint wit lends itself to selection, and there are several modern volumes of extracts from his works.

FULLER, WILLIAM (1670-*c.* 1717), English impostor, was born at Milton in Kent on the 20th of September 1670. His paternity is doubtful, but he was related to the family of Herbert. After 1688 he served James II.’s queen, Mary of Modena, and the Jacobites, seeking at the same time to gain favour with William III.; and after associating with Titus Oates, being imprisoned for debt and pretending to reveal Jacobite plots, the House of Commons in 1692 declared he was an “imposter, cheat and false accuser.” Having stood in the pillory he was again imprisoned until 1695, when he was released; and at this time he took the opportunity to revive the old and familiar story that Mary of Modena was not the mother of the prince of Wales. In 1701 he published his autobiographical *Life of William Fuller* and some *Original Letters of the late King James*. Unable to prove the assertions made in his writings he was put in the pillory, whipped and fined. He died, probably in prison, about 1717. Fuller’s other writings are *Mr William Fuller’s trip to Bridewell, with a full account of his barbarous usage in the pillory; The sincere and hearty confession of Mr William Fuller* (1704); and *An humble appeal to the impartial judgment of all parties in Great Britain* (1716).

He must be distinguished from William Fuller (1608-1675), dean of St Patrick’s (1660), bishop of Limerick (1663), and bishop of Lincoln (1667), the friend of Samuel Pepys; and also from William Fuller (*c.* 1580-1659), dean of Ely and later dean of Durham.

FULLER’S EARTH (Ger. *Walkererde*, Fr. *terre à foulon*, *argile smectique*)—so named from its use by fullers as an absorbent of the grease and oil of cloth,—a clay-like substance, which from its variability is somewhat difficult to define. In colour it is most often greenish, olive-green or greenish-grey; on weathering it changes to a brown tint or it may bleach. As a rule it falls to pieces when placed in water and is not markedly plastic; when dry it adheres strongly to the tongue; since, however, these properties are possessed by many clays that do not exhibit detergent qualities, the only test of value lies in the capacity to absorb grease or clarify oil. Fuller’s earth has a specific gravity of 1.7-2.4, and a shining streak; it is usually unctuous to the touch. Microscopically, it consists of minute irregular-shaped particles of a mineral that appears to be the result of a chloritic or talcose alteration of a felspar. The small size of most of the grains, less than .07 mm., makes their determination almost impossible. Chemical analysis shows that the peculiar properties of this earth are due to its physical rather than its chemical nature.

The following analyses of the weathered and unweathered condition of the earth from Nutfield, Surrey, represent the composition of one of the best known varieties:—

Blue Earth (dried at 100° C.).

Insoluble residue	69.96	Insoluble residue—	
Fe_2O_3	2.48	SiO_2	62.81
Al_2O_3	3.46	Al_2O_3	3.46
CaO	5.87	Fe_2O_3	1.30
MgO	1.41	CaO	1.53
P_2O_5	0.27	MgO	0.86
SO_3	0.05		———
NaCl	0.05		69.96
K_2O	0.74		———
H_2O (combined)	15.57		
	———		
	99.86		
	———		

Yellow Earth (dried at 100° C.).

Insoluble residue	76.13	Insoluble residue—	
Fe_2O_3	2.41	SiO_2	59.37

Al_2O_3	1.77	Al_2O_3	10.05
CaO	4.31	Fe_2O_3	3.86
MgO	1.05	CaO	1.86
P_2O_5	0.14	MgO	1.04
SO_3	0.07		———
NaCl	0.14		76.18
K_2O	0.84		———
H_2O (combined)	13.19		
	———		
	100.05		
	———		

(Analysis by P.G. Sanford, *Geol. Mag.*, 1889, 6, pp. 456, 526.)

Of other published analyses, not a few show a lower silica content (44%, 50%), along with a higher proportion of alumina (11%, 23%).

Fuller’s earth may occur on any geological horizon; at Nutfield in Surrey, England, it is in the Cretaceous formations; at Midford near Bath it is of Jurassic age; at Bala, North Wales, it occurs in Ordovician strata; in Saxony it appears to be the decomposition product of a diabasic rock. In America it is found in California in rocks ranging from Cretaceous to Pleistocene age; in S. Dakota, Custer county and elsewhere a yellow, gritty earth of Jurassic age is worked; in Florida and Georgia occurs a brittle, whitish earth of Oligocene age. Other deposits are worked in Arkansas, Texas, Colorado, Massachusetts and South Carolina.

Fuller’s earth is either mined or dug in the open according to local circumstances. It is then dried in the sun or by artificial heat and transported in small lumps in sacks. In other cases it is ground to a fine powder after being dried; or it is first roughly ground and made into a slurry with water, which is allowed to carry off the finer from the coarser particles and deposit them in a creamy state in suitable tanks. After consolidation this fine material is dried artificially on drying floors, broken into lumps, and packed for transport. The use of fuller’s earth for cleansing wool and cloth has greatly decreased, but the demand for the material is as great or greater than it ever was. It is now used very largely in the filtration of mineral oils, and also for decolourizing certain vegetable oils. It is employed in the formation of certain soaps and cleansing preparations.

The term “Fuller’s Earth” has a special significance in geology, for it was applied by W. Smith in 1799 to certain clays in the neighbourhood of Bath, and the use of the expression is still retained by English geologists, either in this form or in the generalized “Fullonian.” The Fullonian lies at the base of the Great Oolite or Bathonian series, but its palaeontological characters place it between that series and the underlying Inferior Oolite. The zonal fossils are *Perisphinctes arbustigerus* and *Macrocephalus subcontractus* with *Ostrea acuminata*, *Rhynchonella concinna* and *Goniomya angulifera*. The formation is in part the equivalent of the “Vesulien” of J. Marcou (Vesoul in Haute-Saône). In Dorsetshire and Somersetshire, where it is best developed, it is represented by an Upper Fuller’s Earth Clay, the Fuller’s Earth Rock (an impersistent earthy limestone, usually fossiliferous), and the Lower Fuller’s Earth Clay. Commercial fuller’s earth has been obtained only from the Upper Clay. In eastern Gloucestershire and northern Oxfordshire the Fuller’s Earth passes downwards without break into the Inferior Oolite; northward it dies out about Chipping Norton in Oxfordshire and passes laterally into the Stonesfield Slates series; in the midland counties it may perhaps be represented by the “Upper Estuarine Series.” In parts of Dorsetshire the clays have been used for brickmaking and the limestone (rock) for local buildings.

See H.B. Woodward, “Jurassic Rocks of Great Britain,” vol. iv. (1894), *Mem. Geol. Survey* (London).

[J. A. H.]

FULLERTON, LADY GEORGIANA CHARLOTTE (1812-1885), English novelist and philanthropist, youngest daughter of the 1st Earl Granville, was born at Tixall Hall in Staffordshire on the 23rd of September 1812. In 1833 she married Alexander George Fullerton, then an Irish officer in the guards. After living in Paris for some eight years she and her husband accompanied Lord Granville to Cannes and thence to Rome. In 1843 her husband entered the Roman Catholic church, and in the following year Lady Georgiana Fullerton 299
published her first novel, *Ellen Middleton*, which attracted W.E. Gladstone’s attention in the *English Review*. In 1846 she entered the Roman Catholic church. The death of her only son in 1854 plunged her in grief, and she continued to wear mourning until the end of her life. In 1856 she became one of the third order of St Francis, and thenceforward devoted herself to charitable work. In conjunction with Miss Taylor she founded the religious community known as “The Poor Servants of the Mother of God Incarnate,” and she also took an active part in bringing to England the sisters of St Vincent of Paul. Her philanthropic work is described in Mrs Augustus Craven’s work *Lady Georgiana Fullerton, sa vie et ses œuvres* (Paris, 1888), which was translated into English by Henry James Coleridge. She died at Bournemouth on the 19th of January 1885. Among her other novels were *Grantley Manor* (1847), *Lady Bird* (1852), and *Too Strange not to be True* (1864).

FULMAR, from the Gaelic *Fulmaire*, the *Fulmarus glacialis* of modern ornithologists, one of the largest of the petrels (*Procellariidae*) of the northern hemisphere, being about the size of the common gull (*Larus canus*) and not unlike it in general coloration, except that its primaries are grey instead of black. This bird, which ranges over the North Atlantic, is seldom seen on the European side below lat. 53° N., but on the American side comes habitually to lat. 45° or even lower. In the Pacific it is represented by a scarcely separable form, *F. glupischa*. It has been commonly believed to have two breeding-places in the British Islands, namely, St Kilda and South Barra; but, according to Robert Gray (*Birds of the West of Scotland*, p. 499), it has abandoned the latter since 1844, though still breeding in Skye. Northward it established itself about 1838 on Myggenaes Holm, one of the Faeroes, while it has several stations off the coast of Iceland and Spitsbergen, as well as at Bear Island. Its range towards the pole seems to be only bounded by open water, and it is the constant attendant upon all who are employed in the whale and seal fisheries, showing the greatest boldness in approaching boats and ships, and feeding on the offal obtained from them. By British seamen it is commonly called the "molly mawk"[1] (corrupted from *Mallemuck*), and is extremely well known to them, its flight, as it skims over the waves, first with a few beats of the wings and then gliding for a long way, being very peculiar. It only visits the land to deposit its single white egg, which is laid on a rocky ledge, where a shallow nest is made in the turf and lined with a little dried grass. Many of its breeding-places are a most valuable property to those who live near them and take the eggs and young, which, from the nature of the locality, are only to be had at a hazardous risk of life. In St Kilda a large number of the young are killed in one week of August, the only time when, by the custom of the community, they are allowed to be taken. These, after the oil is extracted from them, serve the islanders with food for the winter. The oil has been chemically analysed and found to be a fish-oil, and to possess nearly all the qualities of that obtained from the liver of the cod, with a lighter specific gravity. It, however, has an extremely strong scent, which is said by those who have visited St Kilda to pervade every thing and person on the island, and is certainly retained by an egg or skin of the bird for many years. Whenever a live example is seized in the hand it ejects a considerable quantity of this oil from its mouth.

1 A name misapplied in the southern hemisphere to *Diomedea melanophrys*, one of the albatrosses.

FULMINIC ACID, HCNO or $H_2C_2N_2O_2$, an organic acid isomeric with cyanic and cyanuric acids; its salts, termed fulminates, are very explosive and are much employed as detonators. The free acid, which is obtained by treating the salts with acids, is an oily liquid smelling like prussic acid; it is very explosive, and the vapour is poisonous to about the same degree as that of prussic acid. The first fulminate prepared was the "fulminating silver" of L.G. Brugnatelli, who found in 1798 that if silver be dissolved in nitric acid and the solution added to spirits of wine, a white, highly explosive powder was obtained. This substance is to be distinguished from the black "fulminating silver" obtained by C.L. Berthollet in 1788 by acting with ammonia on precipitated silver oxide. The next salt to be obtained was the mercuric salt, which was prepared in 1799 by Edward Charles Howard, who substituted mercury for silver in Brugnatelli's process. A similar method is that of J. von Liebig (1823), who heated a mixture of alcohol, nitric acid and mercuric nitrate; the salt is largely manufactured by processes closely resembling the last. A laboratory method is to mix solutions of sodium nitromethane, $CH_2 : NO(ONa)$, and mercuric chloride, a yellow basic salt being formed at the same time. Mercuric fulminate is less explosive than the silver salt, and forms white needles (with $\frac{1}{2}H_2O$) which are tolerably soluble in water. The use of mercuric fulminate as a detonator dates from about 1814, when the explosive cap was invented. It is still the commonest detonator, but it is now usually mixed with other substances; the British service uses for percussion caps 6 parts of fulminate, 6 of potassium chlorate and 4 of antimony sulphide, and for time fuses 4 parts of fulminate, 6 of potassium chlorate and 4 of antimony sulphide, the mixture being damped with a shellac varnish; for use in blasting, a home office order of 1897 prescribes a mixture of 4 parts of fulminate and 1 of potassium chlorate. In 1900 Bielefeldt found that a fulminate placed on top of an aromatic nitro compound, such as trinitrotoluene, formed a useful detonator; this discovery has been especially taken advantage of in Germany, in which country detonators of this nature are being largely employed. Tetranitromethylaniline (tetryl) has also been employed (Brit. Pat. 13340 of 1905). It has been proposed to replace fulminate by silver azoimide (Wöhler & Matter, Brit. Pat. 4468 of 1908), and by lead azoimide (Hyronimus, Brit. Pat. 1819 of 1908).

The constitution of fulminic acid has been investigated by many experimenters, but apparently without definitive results. The researches of Liebig (1823), Liebig and Gay-Lussac (1824), and of Liebig again in 1838 showed the acid to be isomeric with cyanic acid, and probably $(HCNO)_2$, since it gave mixed and acid salts. Kekulé, in 1858, concluded that it was nitroacetonitrile, $NO_2{\cdot}CH_2{\cdot}CN$, a view opposed by Steiner (1883), E. Divers and M. Kawakita (1884), R. Scholl (1890), and by J.U. Nef (1894), who proposed the formulae:

$$\begin{array}{cccc} \begin{matrix} C : N{\cdot}OH \\ \dot{C} : N{\cdot}OH, \end{matrix} & O\!<\!\begin{matrix} N : CH \\ N : \dot{C}{\cdot}OH, \end{matrix} & \begin{matrix} CH : N{\cdot}O \\ \dot{C}H : N{\cdot}O, \end{matrix} & C : N{\cdot}OH. \\ \text{Steiner,} & \text{Divers,} & \text{Scholl,} & \text{Nef.} \end{array}$$

The formulae of Kekulé, Divers and Armstrong have been discarded, and it remains to be shown whether Nef's carbonyloxime formula (or the bimolecular formula of Steiner) or Scholl's glyoxime peroxide formula is correct. There is some doubt as to the molecular formula of fulminic acid. The existence of double salts, and the observations of L. Wöhler and K. Theodorovits (*Ber.*, 1905, 38, p. 345), that only compounds containing two carbon atoms yielded fulminates, points to $(HCNO)_2$; on the other hand, Wöhler (*loc. cit.* p. 1351) found that cryoscopic and electric conductivity measurements showed sodium fulminate to be NaCNO. Nef based his formula, which involves bivalent carbon, on many reactions; in particular, that silver fulminate with hydrochloric acid gave salts of formylchloridoxime, which with water gave hydroxylamine and formic acid, thus

$$C:NO\,OAg \rightarrow HC{<}^{NOAg}_{Cl} \rightarrow HC{<}^{N\cdot OH}_{OH} \rightarrow H\cdot CO_2H + H_2N\cdot OH,$$

and also on the production from sodium nitromethane and mercuric chloride, thus $CH_2 : NO{\cdot}Ohg \rightarrow H_2O + C : NOhg$(hg = ½Hg). H. Wieland and F.C. Palazzo (1907) support this formula, finding that methyl nitrolic acid, $NO_2{\cdot}CH : N{\cdot}OH$, yielded under certain conditions fulminic acid, and vice versa (Palazzo, 1907). M.Z. Jowitschitsch (*Ann.*, 1906, 347, p. 233) inclines to Scholl's formula; he found that the synthetic silver salt of glyoxime peroxide resembled silver fulminate in yielding hydroxylamine with hydrochloric acid, but differed in being less explosive, and in being soluble in nitric acid. H. Wieland and his collaborators regard "glyoxime peroxide" as an oxide of furazane (*q.v.*), and have shown that a close relationship exists between the nitrile oxides, furoxane, and fulminic acid (see *Ann. Rep.*, London Chem. Soc., 1909, p. 84). *Fulminuric acid*, $(HCNO)_3$, obtained by Liebig by boiling mercuric fulminate with water, was synthesized in 1905 by C. Ulpiani and L. Bernardini (*Gazetta*, iii. 35, p. 7), who regard it as $NO_2{\cdot}CH(CN){\cdot}CO{\cdot}NH_2$. It deflagrates at 145°, and forms a characteristic cuprammonium salt.

The early history of mercuric fulminate and a critical account of its application as a detonator is given in *The Rise and Progress of the British Explosives Industry* (International Congress of Applied Chemistry, 1909). The manufacture and modern aspects are treated in Oscar Guttmann, *The Manufacture of Explosives*, and *Manufacture of Explosives, Twenty Years' Progress* (1909).

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FULTON, ROBERT (1765-1815), American engineer, was born in 1765 in Little Britain (now Fulton, Lancaster county), Pa. His parents were Irish, and so poor that they could afford him only a very scanty education. At an early age he was bound apprentice to a jeweller in Philadelphia, but subsequently adopted portrait and landscape painting as his profession. In his twenty-second year, with the object of studying with his countryman, Benjamin West, he went to England, and there became acquainted with the duke of Bridgewater, Earl Stanhope and James Watt. Partly by their influence he was led to devote his attention to engineering, especially in connexion with canal construction; he obtained an English patent in 1794 for superseding canal locks by inclined planes, and in 1796 he published a *Treatise on the Improvement of Canal Navigation*. He then took up his residence in Paris, where he projected the first panorama ever exhibited in that city, and constructed a submarine boat, the "Nautilus," which was tried in Brest harbour in 1801 before a commission appointed by Napoleon I., and by the aid of which he was enabled to blow up a small vessel with a torpedo. It was at Paris also in 1803 that he first succeeded in propelling a boat by steam-power, thus realizing a design which he had conceived ten years previously. Returning to America he continued his experiments with submarine explosives, but failed to convince either the English, French or United States governments of the adequacy of his methods. With steam navigation he had more success. In association with Robert R. Livingston (*q.v.*), who in 1798 had been granted the exclusive right to navigate the waters of New York state with steam-vessels, he constructed the "Clermont," which, engined by Boulton & Watt of Birmingham, began to ply on the Hudson between New York and Albany in 1807. The privilege obtained by Livingston in 1798 was granted jointly to Fulton and Livingston in 1803, and by an act passed in 1808 the monopoly was secured to them and their associates for a period depending on the number of steamers constructed, but limited to a maximum of thirty years. In 1814-1815, on behalf of the United States government, he constructed the "Fulton," a vessel of 38 tons with central paddle-wheels, which was the first steam warship. He died at New York on the 24th of February 1815. Among Fulton's inventions were machines for spinning flax, for making ropes, and for sawing and polishing marble.

See C.D. Colden, *Life of Robert Fulton* (New York, 1817); Robert H. Thurston, *History of the Growth of the Steam-Engine* (New York, 1878); George H. Preble, *Chronological History of Steam Navigation* (Philadelphia, 1883); and Mrs A.C. Sutcliffe, *Robert Fulton and the Clermont* (New York, 1909).

FULTON, a city and the county-seat of Callaway county, Missouri, U.S.A., 25 m. N.E. of Jefferson City. Pop. (1890) 4314; (1900) 4883 (1167 negroes); (1910) 5228. It is served by the Chicago & Alton railway. The city has an important stock market and manufactures fire-brick and pottery. At Fulton are the Westminster College (Presbyterian, founded in 1853), the Synodical College for Young Women (Pres., founded in 1871), the William Woods College for Girls (Christian Church, 1890), and the Missouri school for the deaf (1851). Here, too, is a state hospital for the insane (1847), the first institution of the kind in Missouri. The place was laid out as a town in 1825 and named Volney, but in honour of Robert Fulton the present name was adopted a little later. Fulton was incorporated in 1859.

FULTON, a city of Oswego county, New York, U.S.A., on the right bank of the Oswego river, about 10 m. S.

by E. of Oswego. Pop. (1900) 5281; (1905, state census) 8847; (1910) 10,480. Fulton is served by the Delaware, Lackawanna & Western, the New York Central & Hudson River, and the New York, Ontario & Western railways, by electric railway to Oswego and Syracuse and by the Oswego Canal. The city has a Carnegie library. Ample water-power is furnished by the Oswego river, which here flows in a series of rapids, and the manufactures are many in kind. On the 3rd of July 1756, on an island (afterward called Battle Island) 4 m. N. of the present city of Fulton, a British force of about 300 under Captain John Bradstreet (1711-1774) defeated an attacking force of French and Indians (numbering about 700) under De Villiers. Soon after this, Bradstreet built a fort within the present limits of Fulton. The first civilian settler came in 1793, and the first survey (which included only a part of the subsequent village) was made in 1815. Fulton was incorporated as a village in 1835, and in April 1902 was combined with the village of Oswego Falls (pop. in 1900, 2925) and was chartered as a city.

FUM, or Funj Hwang, one of the four symbolical creatures which in Chinese mythology are believed to keep watch and ward over the Celestial Empire. It was begotten by fire, was born in the Hill of the Sun’s Halo, and its body bears inscribed on it the five cardinal virtues. It has the breast of a goose, the hindquarters of a stag, a snake’s neck, a fish’s tail, a fowl’s forehead, a duck’s down, the marks of a dragon, the back of a tortoise, the face of a swallow, the beak of a cock, is about six cubits high, and perches only on the woo-tung tree. The appearance of Fum heralds an age of universal virtue. Its figure is that which is embroidered on the dresses of some mandarins.

FUMARIC AND MALEIC ACIDS, two isomeric unsaturated acids of composition $C_4H_4O_4$. *Fumaric acid* is found in fumitory (*Fumaria officinalis*), in various fungi (*Agaricus piperatus*, &c.), and in Iceland moss. It is obtained by heating malic acid alone to 150° C., or by heating it with hydrochloric acid (V. Dessaignes, *Jahresb.*, 1856, p. 463) or with a large quantity of hydrobromic acids (A. Kekulé, *Ann.*, 1864, 130, p. 21). It may also be obtained by boiling monobromsuccinic acid with water; by the action of dichloracetic acid and water on silver malonate (T. Komnenos, *Ann.*, 1883, 218, p. 169); by the cyanide synthesis from acetylene di-iodide; and by heating maleic acid to 210° C. (Z. Skraup, *Monats. f. Chemie*, 1891, 12, p. 112). It crystallizes in small prisms or needles, and is practically insoluble in cold water. It sublimes to some extent at about 200° C., being partially converted into maleic anhydride and water, the reaction becoming practically quantitative if dehydrating agents be used. Reducing agents (zinc and caustic alkali, hydriodic acid, sodium amalgam, &c.) convert it into succinic acid. Bromine converts it into dibromsuccinic acid. Potassium permanganate oxidizes it to racemic acid (A. Kekulé and R. Anschutz, *Ber.*, 1881, 14, p. 713). By long-continued heating with caustic soda at 100° C. it is converted into inactive malic acid.

Maleic acid is obtained by distilling malic or fumaric acids; by heating fumaric acid with acetyl chloride to 100° C; or by the hydrolysis of trichlorphenomalic acid (β-trichloraceto-acrylic acid) [A. Kekulé, *Ann.*, 1884, 223, p. 185]. It crystallizes in monoclinic prisms, which are easily soluble in water, melt at 130° C., and boil at 160° C., decomposing into water and maleic anhydride. When heated with concentrated hydrobromic or hydriodic acids, it is converted into fumaric acid. It yields an anilide; oxidation converts it into mesotartaric acid. Maleic anhydride is obtained by distilling fumaric acid with phosphorus pentoxide. It forms triclinic crystals which melt at 60° C. and boil at 196° C.

Both acids are readily esterified by the action of alkyl halides on their silver salts, and the maleic ester is readily transformed into the fumaric ester by warming with iodine, the same result being obtained by esterification of maleic acid in alcoholic solution by means of hydrochloric acid. Both acids yield acetylene by the electrolysis of aqueous solutions of their alkali salts, and on reduction both yield succinic acid, whilst by the addition of hydrobromic acid they both yield monobromsuccinic acid (R. Fittig, *Ann.*, 1877, 188, p. 98). From these results it follows that the two acids are structurally identical, and the isomerism has consequently to be explained on other grounds. This was accomplished by W. Wislicenus [”Über die räumliche Anordnung der Atome,” &c., *Trans, of the Saxon Acad. of Sciences* (Math. Phys. Section), 1887, p. 14] by an extension of the van’t Hoff hypothesis (see Stereo-Isomerism). The formulae of the acids are written thus:

$$\begin{array}{l}HC\cdot CO_2H\\ \ddot{H}C\cdot CO_2H\end{array}\ \text{Maleic acid.}\qquad \begin{array}{r}HC\cdot CO_2H\\ HO_2C\cdot \ddot{C}\cdot H\end{array}\ \text{Fumaric acid.}$$

These account for maleic acid readily yielding an anhydride, whereas fumaric acid does not, and for the behaviour of the acids towards bromine, fumaric acid yielding ordinary dibromsuccinic acid, and maleic acid the isomeric isodibromsuccinic acid.

FUMAROLE, a vent from which volcanic vapours issue, named indirectly from the Lat. *fumariolum*, a smoke-hole. The vapours from fumaroles were studied first by R.W. Bunsen, on his visit to Iceland, and afterwards by H. Sainte-Claire Deville and other chemists and geologists in France, who examined the vapours from Santorin, Etna, &c. The hottest vapours issue from dry fumaroles, at temperatures of at least 500° C., and consist chiefly of anhydrous chlorides, notably sodium chloride. The acid fumaroles yield vapours of lower temperature (300° to 400°) containing much water vapour, with hydrogen chloride and sulphur dioxide. The alkaline fumaroles are still cooler, though above 100°, and evolve ammonium chloride with other vapours. Cold fumaroles, below 100°, discharge principally aqueous vapour, with carbon dioxide, and perhaps hydrogen sulphide. The fumaroles of Mont Pelé in Martinique during the eruption of 1902 were examined by A. Lacroix, and the vapours analysed by H. Moissan, who found that they consisted chiefly of water vapour, with hydrogen chloride, sulphur, carbon dioxide, carbon monoxide, methane, hydrogen, nitrogen, oxygen and argon. These vapours issued at a temperature of about 400°. Armand Gautier has pointed out that these gases are practically of the same composition as those which he obtained on heating granite and certain other rocks. (See Volcano).

FUMIGATION (from Lat. *fumigare*, to smoke), the process of producing smoke or fumes, as by burning sulphur, frankincense, tobacco, &c., whether as a ceremony of incantation, or for perfuming a room, or for purposes of disinfection or destruction of vermin. In medicine the term has been used of the exposure of the body, or a portion of it, to fumes such as those of nitre, sal-ammoniac, mercury, &c.; fumigation, by the injection of tobacco smoke into the great bowel, was a recognized procedure in the 18th century for the resuscitation of the apparently drowned. “Fumigated” or “fumed” oak is oak which has been darkened by exposure to ammonia vapour.

FUMITORY, in botany, the popular name for the British species of *Fumaria*, a genus of small, branched, often climbing annual herbs with much-divided leaves and racemes of small flowers. The flowers are tubular with a spurred base, and in the British species are pink to purplish in colour. They are weeds of cultivation growing in fields and waste places. *F. capreolata* climbs by means of twisting petioles. In past times fumitory was in esteem for its reputed cholagogue and other medicinal properties; and in England, boiled in water, milk or whey, it was used as a cosmetic. The root of the allied species (*Corydalis cava* or *tuberosa*) is known as *radix aristolochia*, and has been used medicinally for various cutaneous and other disorders, in doses of 10 to 30 grains. Some eleven alkaloids have been isolated from it. The herbage of *Fumaria officinalis* and *F. racemosa* is used in China under the name of *Tsze-hwa-ti-ting* as an application for glandular swellings, carbuncles and abscesses, and was formerly valued in jaundice, and in cases of accidental swallowing of the beard of grain (see F. Porter Smith, *Contrib. towards the Mat. Medica ... of China*, p. 99, 1871). The name fumitory, Latin *fumus terrae*, has been supposed to be derived from the fact that its juice irritates the eyes like smoke (see Fuchs, *De historia stirpium*, p. 338, 1542); but *The Grete Herball*, cap. clxix., 1529, fol., following the *De simplici medicina* of Platearius, fo. xciii. (see in *Nicolai Praepositi dispensatorium ad aromatarios*, 1536), says: “It is called Fumus terre fume or smoke of the erthe bycause it is engendred of a cours fumosyte rysynge frome the erthe in grete quantyte lyke smoke: this grosse or cours fumosyte of the erthe wyndeth and wryeth out: and by workynge of the ayre and sonne it turneth into this herbe.”

FUNCHAL, the capital of the Portuguese archipelago of the Madeiras; on the south coast of Madeira, in 32° 37′ N. and 16° 54′ W. Pop. (1900) 20,850. Funchal is the see of a bishop, in the archiepiscopal province of Lisbon; it is also the administrative centre of the archipelago, and the residence of the governor and foreign consuls. The city has an attractive appearance from the sea. Its whitewashed houses, in their gardens full of tropical plants, are built along the curving shore of Funchal Bay, and on the lower slopes of an amphitheatre of mountains, which form a background 4000 ft. high. Numerous country houses (*quintas*), with terraced gardens, vineyards and sugar-cane plantations occupy the surrounding heights. Three mountain streams traverse the city through deep channels, which in summer are dry, owing to the diversion of the water for irrigation. A small fort, on an isolated rock off shore, guards the entrance to the bay, and a larger and more powerfully armed fort crowns an eminence inland. The chief buildings include the cathedral, Anglican and Presbyterian churches, hospitals, opera-house, museum and casino. There are small public gardens and a meteorological observatory. In the steep and narrow streets, which are lighted by electricity, wheeled traffic is impossible; sledges drawn by oxen, and other primitive conveyances are used instead (see Madeira). In winter the fine climate and scenery attract numerous invalids and other visitors, for whose accommodation there are good hotels; many foreigners engaged in the coal and wine trades also reside here permanently. The majority of these belong to the British community, which was first established here in the 18th century. Funchal is the headquarters of Madeiran industry and commerce (see Madeira). It has no docks and no facilities for landing passengers or goods; vessels are obliged to anchor in the roadstead, which, however, is sheltered from every wind except the south. Funchal is connected by cable with Carcavellos (for Lisbon), Porthcurnow (for Falmouth, England) and St Vincent in the Cape Verde Islands (for Pernambuco, Brazil).

FUNCTION,[1] in mathematics, a variable number the value of which depends upon the values of one or more other variable numbers. The theory of functions is conveniently divided into (I.) Functions of Real Variables, wherein real, and only real, numbers are involved, and (II.) Functions of Complex Variables, wherein complex or imaginary numbers are involved.

I. Functions of Real Variables

1. *Historical.*—The word function, defined in the above sense, was introduced by Leibnitz in a short note of date 1694 concerning the construction of what we now call an "envelope" (*Leibnizens mathematische Schriften*, edited by C.I. Gerhardt, Bd. v. p. 306), and was there used to denote a variable length related in a defined way to a variable point of a curve. In 1698 James Bernoulli used the word in a special sense in connexion with some isoperimetric problems (Joh. Bernoulli, *Opera*, t. i. p. 255). He said that when it is a question of selecting from an infinite set of like curves that one which best fulfils some function, then of two curves whose intersection determines the thing sought one is always the "line of the function" (*Linea functionis*). In 1718 John Bernoulli (*Opera*, t. ii. p. 241) defined a "function of a variable magnitude" as a quantity made up in any way of this variable magnitude and constants; and in 1730 (Opera, t. iii. p. 174) he noted a distinction between "algebraic" and "transcendental" functions. By the latter he meant integrals of algebraic functions. The notation $f(x)$ for a function of a variable x was introduced by Leonhard Euler in 1734 (*Comm. Acad. Petropol.* t. vii. p. 186), in connexion with the theorem of the interchange of the order of differentiations. The notion of functionality or functional relation of two magnitudes was thus of geometrical origin; but a function soon came to be regarded as an analytical expression, not necessarily an algebraic expression, containing the variable or variables. Thus we may have rational integral algebraic functions such as $ax^2 + bx + c$, or rational algebraic functions which are not integral, such as

$$\frac{a_1x^n + a_2x^{n-1} + \dots + a_n}{b_1x^m + b_2x^{m-1} + \dots + b_m},$$

or irrational algebraic functions, such as $\sqrt{x}$, or, more generally the algebraic functions that are determined implicitly by an algebraic equation, as, for instance,

$$f_n(x, y) + f_{n-1}(x, y) + \dots + f_0 = 0$$

where $f_n(x, y)$, ... mean homogeneous expressions in x and y having constant coefficients, and having the degrees indicated by the suffixes, and f_0 is a constant. Or again we may have trigonometrical functions, such as $\sin x$ and $\tan x$, or inverse trigonometrical functions, such as $\sin^{-1}x$, or exponential functions, such as e^x and a^x, or logarithmic functions, such as $\log x$ and $\log(1 + x)$. We may have these functional symbols combined in various ways, and thus there arises a great number of functions. Further we may have functions of more than one variable, as, for instance, the expression $xy/(x^2 + y^2)$, in which both x and y are regarded as variable. Such functions were introduced into analysis somewhat unsystematically as the need for them arose, and the later developments of analysis led to the introduction of other classes of functions. 302

2. *Graphic Representation.*—In the case of a function of one variable x, any value of x and the corresponding value y of the function can be the co-ordinates of a point in a plane. To any value of x there corresponds a point N on the axis of x, in accordance with the rule that x is the abscissa of N. The corresponding value of y determines a point P in accordance with the rule that x is the abscissa and y the ordinate of P. The ordinate y gives the value of the function which corresponds to that value of the variable x which is specified by N; and it may be described as "the value of the function at N." Since there is a one-to-one correspondence of the points N and the numbers x, we may also describe the ordinate as "the value of the function at x." In simple cases the aggregate of the points P which are determined by any particular function (of one variable) is a curve, called the "graph of the function" (see § 14). In like manner a function of two variables defines a surface.

3. *The Variable.*—Graphic methods of representation, such as those just described, enabled mathematicians to deal with irrational values of functions and variables at the time when there was no theory of irrational numbers other than Euclid's theory of incommensurables. In that theory an irrational number was the ratio of two incommensurable geometric magnitudes. In the modern theory of number irrational numbers are defined in a purely arithmetical manner, independent of the measurement of any quantities or magnitudes, whether geometric or of any other kind. The definition is effected by means of the system of *ordinal* numbers (see Number). When this formal system is established, the theory of measurement may be founded upon it; and, in particular, the co-ordinates of a point are defined as numbers (not lengths), which are assigned in accordance with a rule. This rule involves the measurement of lengths. The theory of functions can be developed without any reference to graphs, or co-ordinates or lengths. The process by which analysis has been freed from any consideration of measurable quantities has been called the "arithmetization of analysis." In the theory so developed, the variable upon which a function depends is always to be regarded as a number, and the corresponding value of the function is also a number. Any reference to points or co-ordinates is to be regarded as a picturesque mode of expression, pointing to a possible application of the theory to geometry. The development of "arithmetized analysis" in the 19th century is associated with the name of Karl Weierstrass.

All possible values of a variable are numbers. In what follows we shall confine our attention to the case where the numbers are real. When complex numbers are introduced, instead of real ones, the theory of functions receives a wide extension, which is accompanied by appropriate limitations (see below, II. Functions of Complex Variables). The set of all real numbers forms a *continuum*. In fact the notion of a one-dimensional continuum first becomes precise in virtue of the establishment of the system of real numbers.

4. *Domain of a Variable.—Theory of Aggregates.*—The notion of a "variable" is that of a number to which we may assign at pleasure any one of the values that belong to some chosen set, or *aggregate*, of numbers; and this set, or aggregate, is called the "domain of the variable." This domain may be an "interval," that is to say it may consist of two terminal numbers, all the numbers between them and no others. When this is the case

the number is said to be “continuously variable.” When the domain consists of all real numbers, the variable is said to be “unrestricted.” A domain which consists of all the real numbers which exceed some fixed number may be described as an “interval unlimited towards the right”; similarly we may have an interval “unlimited towards the left.”

In more complicated cases we must have some rule or process for assigning the aggregate of numbers which constitute the domain of a variable. The methods of definition of particular types of aggregates, and the theorems relating to them, form a branch of analysis called the “theory of aggregates” (*Mengenlehre, Théorie des ensembles, Theory of sets of points*). The notion of an “aggregate” in general underlies the system of ordinal numbers. An aggregate is said to be “infinite” when it is possible to effect a one-to-one correspondence of all its elements to some of its elements. For example, we may make all the integers correspond to the even integers, by making 1 correspond to 2, 2 to 4, and generally n to 2n. The aggregate of positive integers is an infinite aggregate. The aggregates of all rational numbers and of all real numbers and of points on a line are other examples of infinite aggregates. An aggregate whose elements are real numbers is said to “extend to infinite values” if, after any number N, however great, is specified, it is possible to find in the aggregate numbers which exceed N in absolute value. Such an aggregate is always infinite. The “neighbourhood of a number (or point) a for a positive number h” is the aggregate of all numbers (or points) x for which the absolute value of $x - a$ denoted by $|x - a|$, does not exceed h.

5. *General Notion of Functionality.*—A function of one variable was for a long time commonly regarded as the ordinate of a curve; and the two notions (1) that which is determined by a curve supposed drawn, and (2) that which is determined by an analytical expression supposed written down, were not for a long time clearly distinguished. It was for this reason that Fourier’s discovery that a single analytical expression is capable of representing (in different parts of an interval) what would in his time have been called different functions so profoundly struck mathematicians (§ 23). The analysts who, in the middle of the 19th century, occupied themselves with the theory of the convergence of Fourier’s series were led to impose a restriction on the character of a function in order that it should admit of such representation, and thus the door was opened for the introduction of the general notion of functional dependence. This notion may be expressed as follows: We have a variable number, y, and another variable number, x, a domain of the variable x, and a rule for assigning one or more definite values to y when x is any point in the domain; then y is said to be a “function” of the variable x, and x is called the “argument” of the function. According to this notion a function is, as it were, an indefinitely extended table, like a table of logarithms; to each point in the domain of the argument there correspond values for the function, but it remains arbitrary what values the function is to have at any such point.

For the specification of any particular function two things are requisite: (1) a statement of the values of the variable, or of the aggregate of points, to which values of the function are to be made to correspond, *i.e.* of the “domain of the argument”; (2) a rule for assigning the value or values of the function that correspond to any point in this domain. We may refer to the second of these two essentials as “the rule of calculation.” The relation of functions to analytical expressions may then be stated in the form that the rule of calculation is: “Give the function the value of the expression at any point at which the expression has a determinate value,” or again more generally, “Give the function the value of the expression at all points of a definite aggregate included in the domain of the argument.” The former of these is the rule of those among the earlier analysts who regarded an analytical expression and a function as the same thing, and their usage may be retained without causing confusion and with the advantage of brevity, the analytical expression serving to specify the domain of the argument as well as the rule of calculation, *e.g.* we may speak of “the function $1/x$.” This function is defined by the analytical expression $1/x$ at all points except the point $x = 0$. But in complicated cases separate statements of the domain of the argument and the rule of calculation cannot be dispensed with. In general, when the rule of calculation is determined as above by an analytical expression at any aggregate of points, the function is said to be “represented” by the expression at those points.

When the rule of calculation assigns a single definite value for a function at each point in the domain of the argument the function is “uniform” or “one-valued.” In what follows it is to be understood that all the functions considered are one-valued, and the values assigned by the rule of calculation real. In the most 303
important cases the domain of the argument of a function of one variable is an interval, with the possible exception of isolated points.

6. *Limits.*—Let $f(x)$ be a function of a variable number x; and let a be a point such that there are points of the domain of the argument x in the neighbourhood of a for any number h, however small. If there is a number L which has the property that, after any positive number ε, however small, has been specified, it is possible to find a positive number h, so that $|L - f(x)| < \varepsilon$ for all points x of the domain (other than a) for which $|x - a| < h$, then L is the “limit of $f(x)$ at the point a.” The condition for the existence of L is that, after the positive number ε has been specified, it must be possible to find a positive number h, so that $|f(x') - f(x)| < \varepsilon$ for all points x and x′ of the domain (other than a) for which $|x - a| < h$ and $|x' - a| < h$.

It is a fundamental theorem that, when this condition is satisfied, there exists a perfectly definite number L which is the limit of $f(x)$ at the point a as defined above. The limit of $f(x)$ at the point a is denoted by $\mathrm{Lt}_{x=a}f(x)$, or by $\lim_{x=a}f(x)$.

If $f(x)$ is a function of one variable x in a domain which extends to infinite values, and if, after ε has been specified, it is possible to find a number N, so that $|f(x') - f(x)| < \varepsilon$ for all values of x and x′ which are in the domain and exceed N, then there is a number L which has the property that $|f(x) - L| < \varepsilon$ for all such values of x. In this case $f(x)$ has a limit L at $x = \infty$. In like manner $f(x)$ may have a limit at $x = -\infty$. This statement includes the case where the domain of the argument consists exclusively of positive integers. The values of the function then form a “sequence,” $u_1, u_2, \dots u_n, \dots$, and this sequence can have a limit at $n = \infty$.

The principle common to the above definitions and theorems is called, after P. du Bois Reymond, “the general principle of convergence to a limit.”

It must be understood that the phrase “$x = \infty$” does not mean that x takes some particular value which is infinite. There is no such value. The phrase always refers to a limiting process in which, as the process is carried out, the variable number x increases without limit: it may, as in the above example of a sequence, increase by taking successively the values of all the integral numbers; in other cases it may increase by taking the values that belong to any domain which "extends to infinite values."

A very important type of limits is furnished by *infinite series*. When a sequence of numbers u_1, u_2, ... u_n, ... is given, we may form a new sequence s_1, s_2, ... s_n, ... from it by the rules $s_1 = u_1$, $s_2 = u_1 + u_2$, ... $s_n = u_1 + u_2 + ... + u_n$ or by the equivalent rules $s_1 = u$, $s_n - s_{n-1} = u_n (n = 2, 3, ...)$. If the new sequence has a limit at $n = \infty$, this limit is called the "sum of the infinite series" $u_1 + u_2 + ...$, and the series is said to be "convergent" (see Series).

A function which has not a limit at a point a may be such that, if a certain aggregate of points is chosen out of the domain of the argument, and the points x in the neighbourhood of a are restricted to belong to this aggregate, then the function has a limit at a. For example, sin(1/x) has limit zero at 0 if x is restricted to the aggregate 1/π, 1/2π, ... 1/nπ, ... or to the aggregate 1/2π, 2/5π, ... $n/(n^2 + 1)\pi$, ..., but if x takes all values in the neighbourhood of 0, sin (1/x) has not a limit at 0. Again, there may be a limit at a if the points x in the neighbourhood of a are restricted by the condition that x − a is positive; then we have a "limit on the right" at a; similarly we may have a "limit on the left" at a point. Any such limit is described as a "limit for a restricted domain." The limits on the left and on the right are denoted by ƒ(a − 0) and ƒ(a + 0).

The limit L of ƒ(x) at a stands in no necessary relation to the value of ƒ(x) at a. If the point a is in the domain of the argument, the value of ƒ(x) at a is assigned by the rule of calculation, and may be different from L. In case ƒ(a) = L the limit is said to be "attained." If the point a is not in the domain of the argument, there is no value for ƒ(x) at a. In the case where ƒ(x) is defined for all points in an interval containing a, except the point a, and has a limit L at a, we may arbitrarily annex the point a to the domain of the argument and assign to ƒ(a) the value L; the function may then be said to be "extrinsically defined." The so-called "indeterminate forms" (see Infinitesimal Calculus) are examples.

7. *Superior and Inferior Limits; Infinities.*—The value of a function at every point in the domain of its argument is finite, since, by definition, the value can be assigned, but this does not necessarily imply that there is a number N which exceeds all the values (or is less than all the values). It may happen that, however great a number N we take, there are among the values of the function numbers which exceed N (or are less than −N).

If a number can be found which is greater than every value of the function, then either (α) there is one value of the function which exceeds all the others, or (β) there is a number S which exceeds every value of the function but is such that, however small a positive number ε we take, there are values of the function which exceed S − ε. In the case (α) the function has a greatest value; in case (β) the function has a "superior limit" S, and then there must be a point a which has the property that there are points of the domain of the argument, in the neighbourhood of a for any h, at which the values of the function differ from S by less than ε. Thus S is the limit of the function at a, either for the domain of the argument or for some more restricted domain. If a is in the domain of the argument, and if, after omission of a, there is a superior limit S which is in this way the limit of the function at a, if further ƒ(a) = S, then S is the greatest value of the function: in this case the greatest value is a limit (at any rate for a restricted domain) which is attained; it may be called a "superior limit which is attained." In like manner we may have a "smallest value" or an "inferior limit," and a smallest value may be an "inferior limit which is attained."

All that has been said here may be adapted to the description of greatest values, superior limits, &c., of a function in a restricted domain contained in the domain of the argument. In particular, the domain of the argument may contain an interval; and therein the function may have a superior limit, or an inferior limit, which is attained. Such a limit is a *maximum* value or a *minimum* value of the function.

Again, if, after any number N, however great, has been specified, it is possible to find points of the domain of the argument at which the value of the function exceeds N, the values of the function are said to have an "infinite superior limit," and then there must be a point a which has the property that there are points of the domain, in the neighbourhood of a for any h, at which the value of the function exceeds N. If the point a is in the domain of the argument the function is said to "tend to become infinite" at a; it has of course a finite value at a. If the point a is not in the domain of the argument the function is said to "become infinite" at a; it has of course no value at a. In like manner we may have a (negatively) infinite inferior limit. Again, after any number N, however great, has been specified and a number h found, so that all the values of the function, at points in the neighbourhood of a for h, exceed N in absolute value, all these values may have the same sign; the function is then said to become, or to tend to become, "determinately (positively or negatively) infinite"; otherwise it is said to become or to tend to become, "indeterminately infinite."

All the infinities that occur in the theory of functions are of the nature of variable finite numbers, with the single exception of the infinity of an infinite aggregate. The latter is described as an "actual infinity," the former as "improper infinities." There is no "actual infinitely small" corresponding to the actual infinity. The only "infinitely small" is zero. All "infinite values" are of the nature of superior and inferior limits which are not attained.

8. *Increasing and Decreasing Functions*.—A function ƒ(x) of one variable x, defined in the interval between a and b, is "increasing throughout the interval" if, whenever x and x′ are two numbers in the interval and x′ > x, then ƒ(x′) > ƒ(x); the function "never decreases throughout the interval" if, x′ and x being as before, ƒ(x′) > ƒ(x). Similarly for decreasing functions, and for functions which never increase throughout an interval. A function which either never increases or never diminishes throughout an interval is said to be "monotonous throughout" the interval. If we take in the above definition b > a, the definition may apply to a function under the restriction that x′ is not b and x is not a; such a function is "monotonous within" the interval. In this case we have the theorem that the function (if it never decreases) has a limit on the left at b and a limit on the right at a, and these are the superior and inferior limits of its values at all points within the interval (the ends excluded); the like holds *mutatis mutandis* if the function never increases. If the function is monotonous throughout the interval, ƒ(b) is the greatest (or least) value of ƒ(x) in the interval; and if ƒ(b) is the limit of ƒ(x) on the left at b, such a greatest (or least) value is an example of a superior (or inferior) limit which is attained. In these cases the function tends continually to its limit.

These theorems and definitions can be extended, with obvious modifications, to the cases of a domain which

is not an interval, or extends to infinite values. By means of them we arrive at sufficient, but not necessary, criteria for the existence of a limit; and these are frequently easier to apply than the general principle of convergence to a limit (§ 6), of which principle they are particular cases. For example, the function represented by $x \log (1/x)$ continually diminishes when $1/e > x > 0$ and x diminishes towards zero, and it 304
never becomes negative. It therefore has a limit on the right at $x = 0$. This limit is zero. The function represented by $x \sin (1/x)$ does not continually diminish towards zero as x diminishes towards zero, but is sometimes greater than zero and sometimes less than zero in any neighbourhood of $x = 0$, however small. Nevertheless, the function has the limit zero at $x = 0$.

9. *Continuity of Functions.*—A function $f(x)$ of one variable x is said to be continuous at a point a if (1) $f(x)$ is defined in an interval containing a; (2) $f(x)$ has a limit at a; (3) $f(a)$ is equal to this limit. The limit in question must be a limit for continuous variation, not for a restricted domain. If $f(x)$ has a limit on the left at a and $f(a)$ is equal to this limit, the function may be said to be "continuous to the left" at a; similarly the function may be "continuous to the right" at a.

A function is said to be "continuous throughout an interval" when it is continuous at every point of the interval. This implies continuity to the right at the smaller end-value and continuity to the left at the greater end-value. When these conditions at the ends are not satisfied the function is said to be continuous "within" the interval. By a "continuous function" of one variable we always mean a function which is continuous throughout an interval.

The principal properties of a continuous function are:

1. The function is practically constant throughout sufficiently small intervals. This means that, after any point a of the interval has been chosen, and any positive number ε, however small, has been specified, it is possible to find a number h, so that the difference between any two values of the function in the interval between $a - h$ and $a + h$ is less than ε. There is an obvious modification if a is an end-point of the interval.

2. The continuity of the function is "uniform." This means that the number h which corresponds to any ε as in (1) may be the same at all points of the interval, or, in other words, that the numbers h which correspond to ε for different values of a have a positive inferior limit.

3. The function has a greatest value and a least value in the interval, and these are superior and inferior limits which are attained.

4. There is at least one point of the interval at which the function takes any value between its greatest and least values in the interval.

5. If the interval is unlimited towards the right (or towards the left), the function has a limit at ∞ (or at $-\infty$).

10. *Discontinuity of Functions.*—The discontinuities of a function of one variable, defined in an interval with the possible exception of isolated points, may be classified as follows:

(1) The function may become infinite, or tend to become infinite, at a point.

(2) The function may be undefined at a point.

(3) The function may have a limit on the left and a limit on the right at the same point; these may be different from each other, and at least one of them must be different from the value of the function at the point.

(4) The function may have no limit at a point, or no limit on the left, or no limit on the right, at a point.

In case a function $f(x)$, defined as above, has no limit at a point a, there are four limiting values which come into consideration. Whatever positive number h we take, the values of the function at points between a and $a + h$ (a excluded) have a superior limit (or a greatest value), and an inferior limit (or a least value); further, as h decreases, the former never increases and the latter never decreases; accordingly each of them tends to a limit. We have in this way two limits on the right—the inferior limit of the superior limits in diminishing neighbourhoods, and the superior limit of the inferior limits in diminishing neighbourhoods. These are denoted by $\overline{f(a + 0)}$ and $\underline{f(a + 0)}$, and they are called the "limits of indefiniteness" on the right. Similar limits on the left are denoted by $\overline{f(a - 0)}$ and $\underline{f(a - 0)}$. Unless $f(x)$ becomes, or tends to become, infinite at a, all these must exist, any two of them may be equal, and at least one of them must be different from $f(a)$, if $f(a)$ exists. If the first two are equal there is a limit on the right denoted by $f(a + 0)$; if the second two are equal, there is a limit on the left denoted by $f(a - 0)$. In case the function becomes, or tends to become, infinite at a, one or more of these limits is infinite in the sense explained in § 7; and now it is to be noted that, *e.g.* the superior limit of the inferior limits in diminishing neighbourhoods on the right of a may be negatively infinite; this happens if, after any number N, however great, has been specified, it is possible to find a positive number h, so that all the values of the function in the interval between a and $a + h$ (a excluded) are less than $-N$; in such a case $f(x)$ tends to become negatively infinite when x decreases towards a; other modes of tending to infinite limits may be described in similar terms.

11. *Oscillation of Functions.*—The difference between the greatest and least of the numbers $f(a)$, $\overline{f(a + 0)}$, $\underline{f(a + 0)}$, $\overline{f(a - 0)}$, $\underline{f(a - 0)}$, when they are all finite, is called the "oscillation" or "fluctuation" of the function $f(x)$ at the point a. This difference is the limit for $h = 0$ of the difference between the superior and inferior limits of the values of the function at points in the interval between $a - h$ and $a + h$. The corresponding difference for points in a finite interval is called the "oscillation of the function in the interval." When any of the four limits of indefiniteness is infinite the oscillation is infinite in the sense explained in § 7.

For the further classification of functions we divide the domain of the argument into partial intervals by means of points between the end-points. Suppose that the domain is the interval between a and b. Let intermediate points $x_1, x_2 \dots x_{n-1}$, be taken so that $b > x_{n-1} > x_{n-2} \dots > x_1 > a$. We may devise a rule by which, as n increases indefinitely, all the differences $b - x_{n-1}$, $x_{n-1} - x_{n-2}$, ... $x_1 - a$ tend to zero as a limit. The interval is then said to be divided into "indefinitely small partial intervals."

A function defined in an interval with the possible exception of isolated points may be such that the interval can be divided into a set of finite partial intervals within each of which the function is monotonous (§ 8). When this is the case the sum of the oscillations of the function in those partial intervals is finite, provided the function does not tend to become infinite. Further, in such a case the sum of the oscillations will remain below a fixed number for any mode of dividing the interval into indefinitely small partial intervals. A class of functions may be defined by the condition that the sum of the oscillations has this property, and such functions are said to have "restricted oscillation." Sometimes the phrase "limited fluctuation" is used. It can be proved that any function with restricted oscillation is capable of being expressed as the sum of two monotonous functions, of which one never increases and the other never diminishes throughout the interval. Such a function has a limit on the right and a limit on the left at every point of the interval. This class of functions includes all those which have a finite number of maxima and minima in a finite-interval, and some which have an infinite number. It is to be noted that the class does not include all continuous functions.

12. *Differentiable Function.*—The idea of the differentiation of a continuous function is that of a process for measuring the rate of growth; the increment of the function is compared with the increment of the variable. If $f(x)$ is defined in an interval containing the point a, and $a - k$ and $a + k$ are points of the interval, the expression

$$\frac{f(a + h) - f(a)}{h} \qquad (1)$$

represents a function of h, which we may call $\varphi(h)$, defined at all points of an interval for h between $-k$ and k except the point 0. Thus the four limits $\overline{\varphi(+0)}$, $\underline{\varphi(+0)}$, $\overline{\varphi(-0)}$, $\underline{\varphi(-0)}$ exist, and two or more of them may be equal. When the first two are equal either of them is the "progressive differential coefficient" of $f(x)$ at the point a; when the last two are equal either of them is the "regressive differential coefficient" of $f(x)$ at a; when all four are equal the function is said to be "differentiable" at a, and either of them is the "differential coefficient" of $f(x)$ at a, or the "first derived function" of $f(x)$ at a. It is denoted by $df(x) / dx$ or by $f'(x)$. In this case $\varphi(h)$ has a definite limit at $h = 0$, or is determinately infinite at $h = 0$ (§ 7). The four limits here in question are called, after Dini, the "four derivates" of $f(x)$ at a. In accordance with the notation for derived functions they may be denoted by

$$\overline{f' + (a)},\ \underline{f' + (a)},\ \overline{f' - (a)},\ \underline{f' - (a)}.$$

A function which has a finite differential coefficient at all points of an interval is continuous throughout the interval, but if the differential coefficient becomes infinite at a point of the interval the function may or may not be continuous throughout the interval; on the other hand a function may be continuous without being differentiable. This result, comparable in importance, from the point of view of the general theory of functions, with the discovery of Fourier's theorem, is due to G.F.B. Riemann; but the failure of an attempt made by Ampère to prove that every continuous function must be differentiable may be regarded as the first step in the theory. Examples of analytical expressions which represent continuous functions that are not differentiable have been given by Riemann, Weierstrass, Darboux and Dini (see § 24). The most important theorem in regard to differentiable functions is the "theorem of intermediate value." (See Infinitesimal Calculus.)

13. *Analytic Function.*—If $f(x)$ and its first n differential coefficients, denoted by $f'(x)$, $f''(x)$, ... $f^{(n)}(x)$, are continuous in the interval between a and $a + h$, then 305

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n,$$

where R_n may have various forms, some of which are given in the article Infinitesimal Calculus. This result is known as "Taylor's theorem."

When Taylor's theorem leads to a representation of the function by means of an infinite series, the function is said to be "analytic" (cf. § 21).

14. *Ordinary Function.*—The idea of a curve representing a continuous function in an interval is that of a line which has the following properties: (1) the co-ordinates of a point of the curve are a value x of the argument and the corresponding value y of the function; (2) at every point the curve has a definite tangent; (3) the interval can be divided into a finite number of partial intervals within each of which the function is monotonous; (4) the property of monotony within partial intervals is retained after interchange of the axes of co-ordinates x and y. According to condition (2) y is a continuous and differentiable function of x, but this condition does not include conditions (3) and (4): there are continuous partially monotonous functions which are not differentiable, there are continuous differentiable functions which are not monotonous in any interval however small; and there are continuous, differentiable and monotonous functions which do not satisfy condition (4) (cf. § 24). A function which can be represented by a curve, in the sense explained above, is said to be "ordinary," and the curve is the graph of the function (§2). All analytic functions are ordinary, but not all ordinary functions are analytic.

15. *Integrable Function.*—The idea of integration is twofold. We may seek the function which has a given function as its differential coefficient, or we may generalize the question of finding the area of a curve. The first inquiry leads directly to the indefinite integral, the second directly to the definite integral. Following the second method we define "the definite integral of the function $f(x)$ through the interval between a and b" to be the limit of the sum

$$\sum_1^n f(x'_r)\,(x_r - x_{r-1})$$

when the interval is divided into ultimately indefinitely small partial intervals by points $x_1, x_2, \ldots x_{n-1}$. Here x'_r denotes any point in the rth partial interval, x_0 is put for a, and x_n for b. It can be shown that the limit in question is finite and independent of the mode of division into partial intervals, and of the choice of the points such as x'_r, provided (1) the function is defined for all points of the interval, and does not tend to become infinite at any of them; (2) for any one mode of division of the interval into ultimately indefinitely small partial intervals, the sum of the products of the oscillation of the function in each partial interval and the difference of the end-values of that partial interval has limit zero when n is increased indefinitely. When these conditions are satisfied the function is said to be "integrable" in the interval. The numbers a and b which limit the interval are usually called the "lower and upper limits." We shall call them the "nearer and further end-values." The above definition of integration was introduced by Riemann in his memoir on trigonometric series (1854). A still more general definition has been given by Lebesgue. As the more general definition cannot be made intelligible without the introduction of some rather recondite notions belonging to the theory of aggregates, we shall, in what follows, adhere to Riemann's definition.

We have the following theorems:—

1. Any continuous function is integrable.

2. Any function with restricted oscillation is integrable.

3. A discontinuous function is integrable if it does not tend to become infinite, and if the points at which the oscillation of the function exceeds a given number σ, however small, can be enclosed in partial intervals the sum of whose breadths can be diminished indefinitely.

These partial intervals must be a set chosen out of some complete set obtained by the process used in the definition of integration.

4. The sum or product of two integrable functions is integrable.

As regards integrable functions we have the following theorems:

1. If S and I are the superior and inferior limits (or greatest and least values) of $f(x)$ in the interval between a and b, $\int_a^b f(x)dx$ is intermediate between $S(b - a)$ and $I(b - a)$.

2. The integral is a continuous function of each of the end-values.

3. If the further end-value b is variable, and if $\int_a^x f(x)dx = F(x)$, then if $f(x)$ is continuous at b, $F(x)$ is differentiable at b, and $F'(b) = f(b)$.

4. In case $f(x)$ is continuous throughout the interval $F(x)$ is continuous and differentiable throughout the interval, and $F'(x) = f(x)$ throughout the interval.

5. In case $f'(x)$ is continuous throughout the interval between a and b,

$$\int_a^b f'(x)dx = f(b) - f(a).$$

6. In case $f(x)$ is discontinuous at one or more points of the interval between a and b, in which it is integrable,

$$\int_a^x f(x)dx$$

is a function of x, of which the four derivates at any point of the interval are equal to the limits of indefiniteness of $f(x)$ at the point.

7. It may be that there exist functions which are differentiable throughout an interval in which their differential coefficients are not integrable; if, however, $F(x)$ is a function whose differential coefficient, $F'(x)$, is integrable in an interval, then

$$F(x) = \int_a^x F'(x)dx + \text{const.},$$

where a is a fixed point, and x a variable point, of the interval. Similarly, if any one of the four derivates of a function is integrable in an interval, all are integrable, and the integral of either differs from the original function by a constant only.

The theorems (4), (6), (7) show that there is some discrepancy between the indefinite integral considered as the function which has a given function as its differential coefficient, and as a definite integral with a variable end-value.

We have also two theorems concerning the integral of the product of two integrable functions $f(x)$ and $\varphi(x)$; these are known as "the first and second theorems of the mean." The first theorem of the mean is that, if $\varphi(x)$ is one-signed throughout the interval between a and b, there is a number M intermediate between the superior and inferior limits, or greatest and least values, of $f(x)$ in the interval, which has the property expressed by the equation

$$M \int_a^b \varphi(x)dx = \int_a^b f(x)\varphi(x)dx$$

The second theorem of the mean is that, if $f(x)$ is monotonous throughout the interval, there is a number ξ between a and b which has the property expressed by the equation

$$\int_a^b f(x)\ \varphi(x)dx = f(a) \int_a^\xi \varphi(x)dx + f(b) \int_\xi^b \varphi(x)dx.$$

(*See* Fourier's Series.)

16. *Improper Definite Integrals.*—We may extend the idea of integration to cases of functions which are not defined at some point, or which tend to become infinite in the neighbourhood of some point, and to cases where the domain of the argument extends to infinite values. If c is a point in the interval between a and b at which $f(x)$ is not defined, we impose a restriction on the points x'_r of the definition: none of them is to be the point c. This comes to the same thing as defining $\int_a^b f(x)dx$ to be

$$\text{Lt}_{\varepsilon=0}\int_a^{c-\varepsilon} f(x)dx + \text{Lt}_{\varepsilon'=0}\int_{c+\varepsilon'}^b f(x)dx, \tag{1}$$

where, to fix ideas, b is taken > a, and ε and ε′ are positive. The same definition applies to the case where $f(x)$ becomes infinite, or tends to become infinite, at c, provided both the limits exist. This definition may be otherwise expressed by saying that a partial interval containing the point c is omitted from the interval of integration, and a limit taken by diminishing the breadth of this partial interval indefinitely; in this form it applies to the cases where c is a or b.

Again, when the interval of integration is unlimited to the right, or extends to positively infinite values, we have as a definition

$$\int_a^\infty f(x)dx = \mathrm{Lt}_{h=\infty} \int_a^h f(x)dx,$$

provided this limit exists. Similar definitions apply to 306

$$\int_a^{-\infty} f(x)dx, \text{ and to } \int_{-\infty}^{\infty} f(x)dx.$$

All such definite integrals as the above are said to be "improper." For example, $\int_0^\infty \sin x / x \, dx$ is improper in two ways. It means

$$\mathrm{Lt}_{h=\infty}\ \mathrm{Lt}_{\varepsilon=0} \int_\varepsilon^h \sin x/x \, dx,$$

in which the positive number ε is first diminished indefinitely, and the positive number h is afterwards increased indefinitely.

The "theorems of the mean" (§ 15) require modification when the integrals are improper (see Fourier's Series).

When the improper definite integral of a function which becomes, or tends to become, infinite, exists, the integral is said to be "convergent." If $f(x)$ tends to become infinite at a point c in the interval between a and b, and the expression (1) does not exist, then the expression $\int_a^b f(x)dx$, which has no value, is called a "divergent integral, "and it may happen that there is a definite value for

$$\mathrm{Lt} \left\{ \int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon'}^b f(x)\,dx \right\}$$

provided that ε and ε' are connected by some definite relation, and both, remaining positive, tend to limit zero. The value of the above limit is then called a "principal value" of the divergent integral. Cauchy's principal value is obtained by making $\varepsilon' = \varepsilon$, *i.e.* by taking the omitted interval so that the infinity is at its middle point. A divergent integral which has one or more principal values is sometimes described as "semi-convergent."

17. *Domain of a Set of Variables.*—The numerical continuum of n dimensions (C_n) is the aggregate that is arrived at by attributing simultaneous values to each of n variables $x_1, x_2, \dots x_n$, these values being any real numbers. The elements of such an aggregate are called "points," and the numbers $x_1, x_2 \dots x_n$ the "co-ordinates" of a point. Denoting in general the points $(x_1, x_2, \dots x_n)$ and $(x'_1, x'_2 \dots x'_n)$ by x and x', the sum of the differences $|x_1 - x'_1| + |x_2 - x'_2| + \dots + |x_n - x'_n|$ may be denoted by $|x - x'|$ and called the "difference of the two points." We can in various ways choose out of the continuum an aggregate of points, which may be an infinite aggregate, and any such aggregate can be the "domain" of a "variable point." The domain is said to "extend to an infinite distance" if, after any number N, however great, has been specified, it is possible to find in the domain points of which one or more co-ordinates exceed N in absolute value. The "neighbourhood" of a point a for a (positive) number h is the aggregate constituted of all the points x, which are such that the "difference" denoted by $|x - a| < h$. If an infinite aggregate of points does not extend to an infinite distance, there must be at least one point a, which has the property that the points of the aggregate which are in the neighbourhood of a for any number h, however small, themselves constitute an infinite aggregate, and then the point a is called a "limiting point" of the aggregate; it may or may not be a point of the aggregate. An aggregate of points is "perfect" when all its points are limiting points of it, and all its limiting points are points of it; it is "connected" when, after taking any two points a, b of it, and choosing any positive number ε, however small, a number m and points $x', x'', \dots x^{(m)}$ of the aggregate can be found so that all the differences denoted by $|x' - a|, |x'' - x'|, \dots |b - x^{(m)}|$ are less than ε. A perfect connected aggregate is a *continuum*. This is G. Cantor's definition.

The definition of a continuum in C_n leaves open the question of the number of dimensions of the continuum, and a further explanation is necessary in order to define arithmetically what is meant by a "homogeneous part" H_n of C_n. Such a part would correspond to an interval in C_1, or to an area bounded by a simple closed contour in C_2; and, besides being perfect and connected, it would have the following properties: (1) There are points of C_n, which are not points of H_n; these form a complementary aggregate H'_n. (2) There are points "within" H_n; this means that for any such point there is a neighbourhood consisting exclusively of points of H_n. (3) The points of H_n which do not lie "within" H_n are limiting points of H'_n; they are not points of H'_n, but the neighbourhood of any such point for any number h, however small, contains points within H_n and points of H'_n: the aggregate of these points is called the "boundary" of H_n. (4) When any two points a, b within H_n are taken, it is possible to find a number ε and a corresponding number m, and to choose points $x', x'', \dots x^{(m)}$, so that the neighbourhood of a for ε contains x', and consists exclusively of points within H_n, and similarly for x' and x'', x'' and x''', $\dots x^{(m)}$ and b. Condition (3) would exclude such an aggregate as that of the points within and upon two circles external to each other and a line joining a point on one to a point on the other, and condition (4) would exclude such an aggregate as that of the points within and upon two circles which touch externally.

18. Functions of Several Variables.—A function of several variables differs from a function of one variable in that the argument of the function consists of a set of variables, or is a variable point in a C_n when there are n variables. The function is definable by means of the domain of the argument and the rule of calculation. In the most important cases the domain of the argument is a homogeneous part H_n of C_n with the possible exception of isolated points, and the rule of calculation is that the value of the function in any assigned part of the domain of the argument is that value which is assumed at the point by an assigned analytical expression. The limit of a function at a point a is defined in the same way as in the case of a function of one variable.

We take a positive fraction ε and consider the neighbourhood of a for h, and from this neighbourhood we exclude the point a, and we also exclude any point which is not in the domain of the argument. Then we take x and x' to be any two of the retained points in the neighbourhood. The function f has a limit at a if for any positive ε, however small, there is a corresponding h which has the property that $|f(x') - f(x)| < \varepsilon$, whatever points x, x' in the neighbourhood of a for h we take (a excluded). For example, when there are two variables x_1, x_2, and both are unrestricted, the domain of the argument is represented by a plane, and the values of the function are correlated with the points of the plane. The function has a limit at a point a, if we can mark out on the plane a region containing the point a within it, and such that the difference of the values of the function which correspond to any two points of the region (neither of the points being a) can be made as small as we please in absolute value by contracting all the linear dimensions of the region sufficiently. When the domain of the argument of a function of n variables extends to an infinite distance, there is a "limit at an infinite distance" if, after any number ε, however small, has been specified, a number N can be found which is such that $|f(x') - f(x)| < \epsilon$, for all points x and x′ (of the domain) of which one or more co-ordinates exceed N in absolute value. In the case of functions of several variables great importance attaches to limits for a restricted domain. The definition of such a limit is verbally the same as the corresponding definition in the case of functions of one variable (§ 6). For example, a function of x_1 and x_2 may have a limit at ($x_1 = 0$, $x_2 = 0$) if we first diminish x_1 without limit, keeping x_2 constant, and afterwards diminish x_2 without limit. Expressed in geometrical language, this process amounts to approaching the origin along the axis of x_2. The definitions of superior and inferior limits, and of maxima and minima, and the explanations of what is meant by saying that a function of several variables becomes infinite, or tends to become infinite, at a point, are almost identical verbally with the corresponding definitions and explanations in the case of a function of one variable (§ 7). The definition of a continuous function (§ 9) admits of immediate extension; but it is very important to observe that a function of two or more variables may be a continuous function of each of the variables, when the rest are kept constant, without being a continuous function of its argument. For example, a function of x and y may be defined by the conditions that when $x = 0$ it is zero whatever value y may have, and when $x \neq 0$ it has the value of $\sin \{4 \tan^{-1} (y/x)\}$. When y has any particular value this function is a continuous function of x, and, when x has any particular value this function is a continuous function of y; but the function of x and y is discontinuous at ($x = 0$, $y = 0$).

19. *Differentiation and Integration.*—The definition of partial differentiation of a function of several variables presents no difficulty. The most important theorems concerning differentiable functions are the "theorem of the total differential," the theorem of the interchangeability of the order of partial differentiations, and the extension of Taylor's theorem (see INFINITESIMAL CALCULUS).

With a view to the establishment of the notion of integration through a domain, we must define the "extent" of the domain. Take first a domain consisting of the point a and all the points x for which $|x - a| < \frac{1}{2}h$, where h is a chosen positive number; the extent of this domain is h^n, n being the number of variables; such a domain may be described as "square," and the number h may be called its "breadth"; it is a homogeneous part of the 307
numerical continuum of n dimensions, and its boundary consists of all the points for which $|x - a| = \frac{1}{2}h$. Now the points of any domain, which does not extend to an infinite distance, may be assigned to a finite number m of square domains of finite breadths, so that every point of the domain is either within one of these square domains or on its boundary, and so that no point is within two of the square domains; also we may devise a rule by which, as the number m increases indefinitely, the breadths of all the square domains are diminished indefinitely. When this process is applied to a homogeneous part, H, of the numerical continuum C_n, then, at any stage of the process, there will be some square domains of which all the points belong to H, and there will generally be others of which some, but not all, of the points belong to H. As the number m is increased indefinitely the sums of the extents of both these categories of square domains will tend to definite limits, which cannot be negative; when the second of these limits is zero the domain H is said to be "measurable," and the first of these limits is its "extent"; it is independent of the rule adopted for constructing the square domains and contracting their breadths. The notion thus introduced may be adapted by suitable modifications to continua of lower dimensions in C_n.

The integral of a function $f(x)$ through a measurable domain H, which is a homogeneous part of the numerical continuum of n dimensions, is defined in just the same way as the integral through an interval, the extent of a square domain taking the place of the difference of the end-values of a partial interval; and the condition of integrability takes the same form as in the simple case. In particular, the condition is satisfied when the function is continuous throughout the domain. The definition of an integral through a domain may be adapted to any domain of measurable extent. The extensions to "improper" definite integrals may be made in the same way as for a function of one variable; in the particular case of a function which tends to become infinite at a point in the domain of integration, the point is enclosed in a partial domain which is omitted from the integration, and a limit is taken when the extent of the omitted partial domain is diminished indefinitely; a divergent integral may have different (principal) values for different modes of contracting the extent of the omitted partial domain. In applications to mathematical physics great importance attaches to convergent integrals and to principal values of divergent integrals. For example, any component of magnetic force at a point within a magnet, and the corresponding component of magnetic induction at the same point are expressed by different principal values of the same divergent integral. Delicate questions arise as to the possibility of representing the integral of a function of n variables through a domain H_n, as a repeated integral, of evaluating it by successive integrations with respect to the variables one at a time and of interchanging the order of such integrations. These questions have been discussed very completely by C. Jordan, and we may quote the result that all the transformations in question are valid when the function is continuous throughout the domain.

20. *Representation of Functions in General.*—We have seen that the notion of a function is wider than the notion of an analytical expression, and that the same function may be "represented" by one expression in one part of the domain of the argument and by some other expression in another part of the domain (§ 5). Thus there arises the general problem of the representation of functions. The function may be given by specifying the domain of the argument and the rule of calculation, or else the function may have to be determined in accordance with certain conditions; for example, it may have to satisfy in a prescribed domain an assigned differential equation. In either case the problem is to determine, when possible, a single analytical expression which shall have the same value as the function at all points in the domain of the argument. For the representation of most functions for which the problem can be solved recourse must be had to limiting processes. Thus we may utilize infinite series, or infinite products, or definite integrals; or again we may represent a function of one variable as the limit of an expression containing two variables in a domain in which one variable remains constant and another varies. An example of this process is afforded by the expression $\mathrm{Lt}_{y = \infty} xy / (x^2y + 1)$, which represents a function of x vanishing at $x = 0$ and at all other values of x having the value of $1/x$. The method of series falls under this more general process (cf. § 6). When the terms $u_1, u_2, \ldots$ of a series are functions of a variable x, the sum s_n of the first n terms of the series is a function of x

and n; and, when the series is convergent, its sum, which is $Lt_{n} = \infty\ s_n$, can represent a function of x. In most cases the series converges for some values of x and not for others, and the values for which it converges form the “domain of convergence.” The sum of the series represents a function in this domain.

The apparently more general method of representation of a function of one variable as the limit of a function of two variables has been shown by R. Baire to be identical in scope with the method of series, and it has been developed by him so as to give a very complete account of the possibility of representing functions by analytical expressions. For example, he has shown that Riemann’s totally discontinuous function, which is equal to 1 when x is rational and to 0 when x is irrational, can be represented by an analytical expression. An infinite process of a different kind has been adapted to the problem of the representation of a continuous function by T. Brodén. He begins with a function having a graph in the form of a regular polygon, and interpolates additional angular points in an ordered sequence without limit. The representation of a function by means of an infinite product falls clearly under Baire’s method, while the representation by means of a definite integral is analogous to Brodén’s method. As an example of these two latter processes we may cite the Gamma function [$\Gamma(x)$] defined for positive values of x by the definite integral

$$\int_0^\infty e^{-t}\, t^{x-1}\, dt,$$

or by the infinite product

$$Lt_{n=\infty}\ n^x/x(1 + x)(1 + ½x) \dots \left(1 + \frac{x}{n-1}\right).$$

The second of these expressions avails for the representation of the function at all points at which x is not a negative integer.

21. *Power Series.*—Taylor’s theorem leads in certain cases to a representation of a function by an infinite series. We have under certain conditions (§ 13)

$$f(x) = f(a) + \sum_{r=1}^{n-1} \frac{(x-a)^r}{r!} f^{(r)}(a) + R_n;$$

and this becomes

$$f(x) = f(a) + \sum_{r=1}^{\infty} \frac{(x-a)^r}{r!} f^{(r)}(a),$$

provided that (α) a positive number k can be found so that at all points in the interval between a and a + k (except these points) f(x) has continuous differential coefficients of all finite orders, and at a has progressive differential coefficients of all finite orders; (β) Cauchy’s form of the remainder R_n, viz. $[(x - a) / (n - 1)!]\ (1 - \theta)^{n-1}\ f^n\ \{a + \theta(x - a)\}$, has the limit zero when n increases indefinitely, for all values of θ between 0 and 1, and for all values of x in the interval between a and a + k, except possibly a + k. When these conditions are satisfied, the series (1) represents the function at all points of the interval between a and a + k, except possibly a + k, and the function is “analytic” (§ 13) in this domain. Obvious modifications admit of extension to an interval between a and a − k, or between a − k and a + k. When a series of the form (1) represents a function it is called “the Taylor’s series for the function.”

Taylor’s series is a power series, *i.e.* a series of the form

$$\sum_{n=0}^{\infty} a_n\, (x - a)^n.$$

As regards power series we have the following theorems:

1. If the power series converges at any point except a there is a number k which has the property that the series converges absolutely in the interval between a − k and a + k, with the possible exception of one or both end-points.

2. The power series represents a continuous function in its domain of convergence (the end-points may have to be excluded).

3. This function is analytic in the domain, and the power series representing it is the Taylor’s series for the function.

The theory of power series has been developed chiefly from the point of view of the theory of functions of complex variables.

22. *Uniform Convergence.*—We shall suppose that the domain of convergence of an infinite series of functions is an interval with the possible exception of isolated points. Let f(x) be the sum of the series at any point x of the domain, and $f_n(x)$ the sum of the first n + 1 terms. The condition of convergence at a point a is that, after any positive number ε, however small, has been specified, it must be possible to find a number n so that $|f_m(a) - f_p(a)| < \varepsilon$ for all values of m and p which exceed n. The sum, f(a), is the limit of the sequence of numbers $f_n(a)$ at n = ∞. The convergence is said to be “uniform” in an interval if, after specification of ε, the same number n suffices at all points of the interval to make $|f(x) - f_m(x)| < \varepsilon$ for all values of m which exceed n. The numbers n corresponding to any ε, however small, are all finite, but, when ε is less than some fixed finite number, they may have an infinite superior limit (§ 7); when this is the case there must be at least one point, a, of the interval which has the property that, whatever number N we take, ε can be taken so small that, at some point in the neighbourhood of a, n must be taken > N to make $|f(x) - f_m(x)| < \varepsilon$ when m > n; then the series does not converge uniformly in the neighbourhood of a. The distinction may be otherwise expressed thus: Choose a first and ε afterwards, then the number n is finite; choose ε first and allow a to vary, then the number n becomes a function of a, which may tend to become infinite, or may remain below a fixed number; if such a fixed number exists, however small ε may be, the convergence is uniform. 308

For example, the series sin x − ½ sin 2x + ⅓ sin 3x − ... is convergent for all real values of x, and, when π > x > −π its sum is ½x; but, when x is but a little less than π, the number of terms which must be taken in order to bring the sum at all near to the value of ½x is very large, and this number tends to increase indefinitely as x approaches π. This series does not converge uniformly in the neighbourhood of x = π. Another example is afforded by the series

$$\sum_{n=0}^{\infty} \frac{nx}{n^2x^2+1} - \frac{(n+1)x}{(n+1)^2x^2+1},$$

of which the remainder after n terms is $nx/(n^2x^2+1)$. If we put $x = 1/n$, for any value of n, however great, the remainder is ½; and the number of terms required to be taken to make the remainder tend to zero depends upon the value of x when x is near to zero—it must, in fact, be large compared with $1/x$. The series does not converge uniformly in the neighbourhood of $x = 0$.

As regards series whose terms represent continuous functions we have the following theorems:

(1) If the series converges uniformly in an interval it represents a function which is continuous throughout the interval.

(2) If the series represents a function which is discontinuous in an interval it cannot converge uniformly in the interval.

(3) A series which does not converge uniformly in an interval may nevertheless represent a function which is continuous throughout the interval.

(4) A power series converges uniformly in any interval contained within its domain of convergence, the end-points being excluded.

(5) If $\sum_{r=0}^{\infty} f_r(x) = f(x)$ converges uniformly in the interval between a and b

$$\int_a^b f(x)dx = \sum_{r=0}^{b} \int_a^b f_r(x)dx,$$

or a series which converges uniformly may be integrated term by term.

(6) If $\sum_{r=0}^{\infty} f'_r(x)$ converges uniformly in an interval, then $\sum_{r=0}^{\infty} f_r(x)$ converges in the interval, and represents a continuous differentiable function, $\varphi(x)$; in fact we have

$$\varphi'(x) = \sum_{r=0}^{\infty} f'_r(x),$$

or a series can be differentiated term by term if the series of derived functions converges uniformly.

A series whose terms represent functions which are not continuous throughout an interval may converge uniformly in the interval. If $\sum_{r=0}^{\infty} f_r(x) = f(x)$, is such a series, and if all the functions $f_r(x)$ have limits at a, then $f(x)$ has a limit at a, which is $\sum_{r=0}^{\infty} \mathrm{Lt}_{x=a} f_r(x)$. A similar theorem holds for limits on the left or on the right.

23. Fourier's Series.—An extensive class of functions admit of being represented by series of the form

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right),$$

and the rule for determining the coefficients a_n, b_n of such a series, in order that it may represent a given function $f(x)$ in the interval between $-c$ and c, was given by Fourier, viz. we have

$$a_0 = \frac{1}{2c} \int_{-c}^{c} f(x)dx, \quad a_n = \frac{1}{c} \int_{-c}^{c} f(x)\cos \frac{n\pi x}{c} dx, \quad b_n = \frac{1}{c} \int_{-c}^{c} \sin \frac{n\pi x}{c} dx.$$

The interval between $-c$ and c may be called the "periodic interval," and we may replace it by any other interval, *e.g.* that between 0 and 1, without any restriction of generality. When this is done the sum of the series takes the form

$$\mathrm{Lt}_{n=\infty} \int_0^1 \sum_{r=-n}^{r=n} f(z) \cos \{2r\pi(z-x)\}dz,$$

and this is

$$\mathrm{Lt}_{n=\infty} \int_0^1 f(z) \frac{\sin \{(2n+1)(z-x)\pi\}}{\sin \{(z-x)\pi\}} dz. \qquad \text{(ii.)}$$

Fourier's theorem is that, if the periodic interval can be divided into a finite number of partial intervals within each of which the function is ordinary (§ 14), the series represents the function within each of those partial intervals. In Fourier's time a function of this character was regarded as completely arbitrary.

By a discussion of the integral (ii.) based on the Second Theorem of the Mean (§ 15) it can be shown that, if $f(x)$ has restricted oscillation in the interval (§ 11), the sum of the series is equal to $\frac{1}{2}\{f(x+0) + f(x-0)\}$ at any point x within the interval, and that it is equal to $\frac{1}{2}\{f(+0) + f(1-0)\}$ at each end of the interval. (See the article Fourier's Series.) It therefore represents the function at any point of the periodic interval at which the function is continuous (except possibly the end-points), and has a definite value at each point of discontinuity. The condition of restricted oscillation includes all the functions contemplated in the statement of the theorem and some others. Further, it can be shown that, in any partial interval throughout which $f(x)$ is continuous, the series converges uniformly, and that no series of the form (i), with coefficients other than those determined by Fourier's rule, can represent the function at all points, except points of discontinuity, in the same periodic interval. The result can be extended to a function $f(x)$ which tends to become infinite at a finite number of points a of the interval, provided (1) $f(x)$ tends to become determinately infinite at each of the points a, (2) the improper definite integral of $f(x)$ through the interval is convergent, (3) $f(x)$ has not an infinite number of discontinuities or of maxima or minima in the interval.

24. *Representation of Continuous Functions by Series.*—If the series for $f(x)$ formed by Fourier's rule converges at the point a of the periodic interval, and if $f(x)$ is continuous at a, the sum of the series is $f(a)$; but it has been proved by P. du Bois Reymond that the function may be continuous at a, and yet the series formed by Fourier's rule may be divergent at a. Thus some continuous functions do not admit of representation by Fourier's series. All continuous functions, however, admit of being represented with arbitrarily close approximation in either of two forms, which may be described as "terminated Fourier's series" and

“terminated power series,” according to the two following theorems:

(1) If $f(x)$ is continuous throughout the interval between 0 and 2π, and if any positive number ε however small is specified, it is possible to find an integer n, so that the difference between the value of $f(x)$ and the sum of the first n terms of the series for $f(x)$, formed by Fourier’s rule with periodic interval from 0 to 2π, shall be less than ε at all points of the interval. This result can be extended to a function which is continuous in any given interval.

(2) If $f(x)$ is continuous throughout an interval, and any positive number ε however small is specified, it is possible to find an integer n and a polynomial in x of the nth degree, so that the difference between the value of $f(x)$ and the value of the polynomial shall be less than ε at all points of the interval.

Again it can be proved that, if $f(x)$ is continuous throughout a given interval, polynomials in x of finite degrees can be found, so as to form an infinite series of polynomials whose sum is equal to $f(x)$ at all points of the interval. Methods of representation of continuous functions by infinite series of rational fractional functions have also been devised.

Particular interest attaches to continuous functions which are not differentiable. Weierstrass gave as an example the function represented by the series $\sum_0^\infty a^n \cos(b^n x\pi)$, where a is positive and less than unity, and b is an odd integer exceeding $(1 + \frac{3}{2}\pi)/a$. It can be shown that this series is uniformly convergent in every interval, and that the continuous function $f(x)$ represented by it has the property that there is, in the 309
neighbourhood of any point x_0, an infinite aggregate of points x', having x_0 as a limiting point, for which $\{f(x') - f(x_0)\} / (x' - x_0)$ tends to become infinite with one sign when $x' - x_0$ approaches zero through positive values, and infinite with the opposite sign when $x' - x_0$ approaches zero through negative values. Accordingly the function is not differentiable at any point. The definite integral of such a function $f(x)$ through the interval between a fixed point and a variable point x, is a continuous differentiable function $F(x)$, for which $F'(x) = f(x)$; and, if $f(x)$ is one-signed throughout any interval $F(x)$ is monotonous throughout that interval, but yet $F(x)$ cannot be represented by a curve. In any interval, however small, the tangent would have to take the same direction for infinitely many points, and yet there is no interval in which the tangent has everywhere the same direction. Further, it can be shown that all functions which are everywhere continuous and nowhere differentiable are capable of representation by series of the form $\Sigma a_n \varphi_n(x)$, where Σa_n is an absolutely convergent series of numbers, and $\varphi_n(x)$ is an analytic function whose absolute value never exceeds unity.

25. *Calculations with Divergent Series.*—When the series described in (1) and (2) of § 24 diverge, they may, nevertheless, be used for the approximate numerical calculation of the values of the function, provided the calculation is not carried beyond a certain number of terms. Expansions in series which have the property of representing a function approximately when the expansion is not carried too far are called “asymptotic expansions.” Sometimes they are called “semi-convergent series”; but this term is avoided in the best modern usage, because it is often used to describe series whose convergence depends upon the order of the terms, such as the series $1 - \frac{1}{2} + \frac{1}{3} - \dots$

In general, let $f_0(x) + f_1(x) + \dots$ be a series of functions which does not converge in a certain domain. It may happen that, if any number ε, however small, is first specified, a number n can afterwards be found so that, at a point a of the domain, the value $f(a)$ of a certain function $f(x)$ is connected with the sum of the first $n + 1$ terms of the series by the relation $|f(a) - \sum_{r=0}^{n} f_r(a)| < \varepsilon$. It must also happen that, if any number N, however great, is specified, a number $n'(>n)$ can be found so that, for all values of m which exceed n', $|\sum_{r=0}^{m} f_r(a)| > N$. The divergent series $f_0(x) + f_1(x) + \dots$ is then an asymptotic expansion for the function $f(x)$ in the domain.

The best known example of an asymptotic expansion is Stirling’s formula for $n!$ when n is large, viz.

$$n! = \sqrt{(2\pi)}\ \tfrac{1}{2} n^{n+\frac{1}{2}} e^{-n+\theta/12n},$$

where θ is some number lying between 0 and 1. This formula is included in the asymptotic expansion for the Gamma function. We have in fact

$$\log\{\Gamma(x)\} = (x - \tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + \overline{\omega}(x),$$

where $\overline{\omega}(x)$ is the function defined by the definite integral

$$\overline{\omega}(x) = \int_0^\infty \{(1 - e^{-t})^{-1} - t^{-1} - \tfrac{1}{2}\}\, t^{-1} e^{-tx}\, dt.$$

The multiplier of e^{-tx} under the sign of integration can be expanded in the power series

$$\frac{B_1}{2!} - \frac{B_2}{4!} t^2 + \frac{B_3}{6!} t^4 - \dots,$$

where $B_1, B_2, \dots$ are “Bernoulli’s numbers” given by the formula

$$B_m = 2.2m!\,(2\pi)^{-2m} \sum_{r=1}^{\infty} (r^{-2m}).$$

When the series is integrated term by term, the right-hand member of the equation for $\overline{\omega}(x)$ takes the form

$$\frac{B_1}{1\cdot 2}\,\frac{1}{x} - \frac{B_2}{3\cdot 4}\,\frac{1}{x^3} + \frac{B_3}{5\cdot 6}\,\frac{1}{x^5} - \dots,$$

This series is divergent; but, if it is stopped at any term, the difference between the sum of the series so terminated and the value of $\overline{\omega}(x)$ is less than the last of the retained terms. Stirling’s formula is obtained by retaining the first term only. Other well-known examples of asymptotic expansions are afforded by the descending series for Bessel’s functions. Methods of obtaining such expansions for the solutions of linear differential equations of the second order were investigated by G.G. Stokes (*Math. and Phys. Papers*, vol. ii. p. 329), and a general theory of asymptotic expansions has been developed by H. Poincaré. A still more general theory of divergent series, and of the conditions in which they can be used, as above, for the purposes of approximate calculation has been worked out by É. Borel. The great merit of asymptotic expansions is that they admit of addition, subtraction, multiplication and division, term by term, in the same way as absolutely convergent series, and they admit also of integration term by term; that is to say, the results of such operations are asymptotic expansions for the sum, difference, product, quotient, or integral, as the case may be.

26. *Interchange of the Order of Limiting Operations.*—When we require to perform any limiting operation upon a function which is itself represented by the result of a limiting process, the question of the possibility of interchanging the order of the two processes always arises. In the more elementary problems of analysis it generally happens that such an interchange is possible; but in general it is not possible. In other words, the performance of the two processes in different orders may lead to two different results; or the performance of them in one of the two orders may lead to no result. The fact that the interchange is possible under suitable restrictions for a particular class of operations is a theorem to be proved.

Among examples of such interchanges we have the differentiation and integration of an infinite series term by term (§ 22), and the differentiation and integration of a definite integral with respect to a parameter by performing the like processes upon the subject of integration (§ 19). As a last example we may take the limit of the sum of an infinite series of functions at a point in the domain of convergence. Suppose that the series $\sum_0^\infty f_r(x)$ represents a function (ƒx) in an interval containing a point a, and that each of the functions $f_r(x)$ has a limit at a. If we first put x=a, and then sum the series, we have the value ƒ(a); if we first sum the series for any x, and afterwards take the limit of the sum at x = a, we have the limit of ƒ(x) at a; if we first replace each function $f_r(x)$ by its limit at a, and then sum the series, we may arrive at a value different from either of the foregoing. If the function ƒ(x) is continuous at a, the first and second results are equal; if the functions $f_r(x)$ are all continuous at a, the first and third results are equal; if the series is uniformly convergent, the second and third results are equal. This last case is an example of the interchange of the order of two limiting operations, and a sufficient, though not always a necessary, condition, for the validity of such an interchange will usually be found in some suitable extension of the notion of uniform convergence.

Authorities.—Among the more important treatises and memoirs connected with the subject are: R. Baire, *Fonctions discontinues* (Paris, 1905); O. Biermann, *Analytische Functionen* (Leipzig, 1887); É. Borel, *Théorie des fonctions* (Paris, 1898) (containing an introductory account of the Theory of Aggregates), and *Séries divergentes* (Paris, 1901), also *Fonctions de variables réelles* (Paris, 1905); T.J. I'A. Bromwich, *Introduction to the Theory of Infinite Series* (London, 1908); H.S. Carslaw, *Introduction to the Theory of Fourier's Series and Integrals* (London, 1906); U. Dini, *Functionen e. reellen Grösse* (Leipzig, 1892), and *Serie di Fourier* (Pisa, 1880); A. Genocchi u. G. Peano, *Diff.- u. Int.-Rechnung* (Leipzig, 1899); J. Harkness and F. Morley, *Introduction to the Theory of Analytic Functions* (London, 1898); A. Harnack, *Diff. and Int. Calculus* (London, 1891); E.W. Hobson, *The Theory of Functions of a real Variable and the Theory of Fourier's Series* (Cambridge, 1907); C. Jordan, *Cours d'analyse* (Paris, 1893-1896); L. Kronecker, *Theorie d. einfachen u. vielfachen Integrale* (Leipzig, 1894); H. Lebesgue, *Leçons sur l'intégration* (Paris, 1904); M. Pasch, *Diff.- u. Int.-Rechnung* (Leipzig, 1882); E. Picard, *Traité d'analyse* (Paris, 1891); O. Stolz, *Allgemeine Arithmetik* (Leipzig, 1885), and *Diff.- u. Int.-Rechnung* (Leipzig, 1893-1899); J. Tannery, *Théorie des fonctions* (Paris, 1886); W.H. and G.C. Young, *The Theory of Sets of Points* (Cambridge, 1906); Brodén, "Stetige Functionen e. reellen Veränderlichen," *Crelle*, Bd. cxviii.; G. Cantor, A series of memoirs on the "Theory of Aggregates" and on "Trigonometric series" in *Acta Math*. tt. ii., vii., and *Math. Ann*. Bde. iv.-xxiii.; Darboux, "Fonctions discontinues," *Ann. Sci. École normale sup.* (2), t. iv.; Dedekind, *Was sind u. was sollen d. Zahlen*? (Brunswick, 1887), and *Stetigkeit u. irrationale Zahlen* (Brunswick, 1872); Dirichlet, "Convergence des séries trigonométriques," *Crelle*, Bd. iv.; P. Du Bois Reymond, *Allgemeine Functionentheorie* (Tübingen, 1882), and many memoirs in *Crelle* and in *Math. Ann.*; Heine, "Functionenlehre," *Crelle*, Bd. lxxiv.; J. Pierpont, *The Theory of Functions of a real Variable* (Boston, 1905); F. Klein, "Allgemeine Functionsbegriff," *Math. Ann*. Bd. xxii.; W.F. Osgood, "On Uniform Convergence," *Amer. J. of Math*. vol. xix.; Pincherle, "Funzioni analitiche secondo Weierstrass," *Giorn. di mat*. t. xviii.; Pringsheim, "Bedingungen d. Taylorschen Lehrsatzes," *Math. Ann*. Bd. xliv.; Riemann, "Trigonometrische Reihe," *Ges. Werke* (Leipzig, 1876); Schoenflies, "Entwickelung d. Lehre v. d. Punktmannigfaltigkeiten," *Jahresber. d. deutschen Math.-Vereinigung*, Bd. viii.; Study, Memoir on "Functions with Restricted Oscillation," *Math. Ann*. Bd. xlvii.; Weierstrass, Memoir on "Continuous Functions that are not Differentiable," *Ges. math. Werke*, Bd. ii. p. 71 (Berlin, 1895), and on the "Representation of Arbitrary Functions," ibid. Bd. iii. p. 1; W.H. Young, "On Uniform and Non-uniform Convergence," *Proc. London Math. Soc.* (Ser. 2) t. 6. Further information and very full references will be found in the articles by Pringsheim, Schoenflies and Voss in the *Encyclopädie der math. Wissenschaften*, Bde. i., ii. (Leipzig, 1898, 1899).

(A. E. H. L.)

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II—Functions of Complex Variables

In the preceding section the doctrine of functionality is discussed with respect to real quantities; in this section the theory when complex or imaginary quantities are involved receives treatment. The following abstract explains the arrangement of the subject matter: (§ 1), *Complex numbers*, states what a complex number is; (§ 2), *Plotting of simple expressions involving complex numbers*, illustrates the meaning in some simple cases, introducing the notion of conformal representation and proving that an algebraic equation has complex, if not real, roots; (§ 3), *Limiting operations*, defines certain simple functions of a complex variable which are obtained by passing to a limit, in particular the exponential function, and the generalized logarithm, here denoted by λ(z); (§ 4), *Functions of a complex variable in general*, after explaining briefly what is to be understood by a region of the complex plane and by a path, and expounding a logical principle of some importance, gives the accepted definition of a function of a complex variable, establishes the existence of a complex integral, and proves Cauchy's theorem relating thereto; (§ 5), *Applications*, considers the differentiation and integration of series of functions of a complex variable, proves Laurent's theorem, and establishes the expansion of a function of a complex variable as a power series, leading, in (§ 6), *Singular points*, to a definition of the region of existence and singular points of a function of a complex variable, and thence, in (§ 7), *Monogenic Functions*, to what the writer believes to be the simplest definition of a function of a complex variable, that of Weierstrass; (§ 8), *Some elementary properties of single valued functions*, first discusses the meaning of a pole, proves that a single valued function with only poles is rational, gives Mittag-Leffler's theorem, and Weierstrass's theorem for the primary factors of an integral function, stating generalized forms for these, leading to the theorem of (§ 9), *The construction of a monogenic function with a given region of existence*, with which is connected (§10), *Expression of a monogenic function by rational* *functions in a given region*, of which the method is applied in (§ 11), *Expression of* $(1 - z)^{-1}$ *by polynomials*, to a definite example, used here to obtain (§ 12), *An expansion of an arbitrary function by means of a series of polynomials, over a star region*, also obtained in the original manner of Mittag-Leffler; (§ 13), *Application of Cauchy's theorem to the determination of definite integrals*, gives two examples of this method; (§ 14), *Doubly Periodic Functions*, is introduced at this stage as furnishing an excellent example of the preceding principles. The reader who wishes to approach the matter from the point of view of Integral Calculus should first consult the section (§ 20) below, dealing with *Elliptic Integrals*; (§ 15), *Potential Functions, Conformal representation in general*, gives a sketch of the connexion of the theory of potential functions with the theory of conformal representation, enunciating the Schwarz-Christoffel theorem for the representation of a polygon, with the application to the case of an equilateral triangle; (§ 16), *Multiple-valued Functions, Algebraic Functions*, deals for the most part with algebraic functions, proving the residue theorem, and establishing that an algebraic function has a definite Order; (§ 17), *Integrals of Algebraic Functions*, enunciating Abel's theorem; (§ 18), *Indeterminateness of Algebraic Integrals*, deals with the periods associated with an algebraic integral, establishing that for an elliptic integral the number of these is two; (§ 19), *Reversion of an algebraic integral*, mentions a problem considered below in detail for an elliptic integral; (§ 20), *Elliptic Integrals*, considers the algebraic reduction of any elliptic integral to one of three standard forms, and proves that the function obtained by reversion is single-valued; (§ 21), *Modular Functions*, gives a statement of some of the more elementary properties of some functions of great importance, with a definition of Automorphic Functions, and a hint of the connexion with the theory of linear differential equations; (§ 22), *A property of integral functions, deduced from the theory of modular functions*, proves that there cannot be more than one value not assumed by an integral function, and gives the basis of the well-known expression of the modulus of the elliptic functions in terms of the ratio of the periods; (§ 23), *Geometrical applications of Elliptic Functions*, shows that any plane curve of deficiency unity can be expressed by elliptic functions, and gives a geometrical proof of the addition theorem for the function $\mathfrak{P}(u)$; (§ 24), *Integrals of Algebraic Functions in connexion with the theory of plane curves*, discusses the generalization to curves of any deficiency; (§ 25), *Monogenic Functions of several independent variables*, describes briefly the beginnings of this theory, with a mention of some fundamental theorems: (§ 26), *Multiply-Periodic Functions and the Theory of Surfaces*, attempts to show the nature of some problems now being actively pursued.

Beside the brevity necessarily attaching to the account here given of advanced parts of the subject, some of the more elementary results are stated only, without proof, as, for instance: the monogeneity of an algebraic function, no reference being made, moreover, to the cases of differential equations whose integrals are monogenic; that a function possessing an algebraic addition theorem is necessarily an elliptic function (or a particular case of such); that any area can be conformally represented on a half plane, a theorem requiring further much more detailed consideration of the meaning of *area* than we have given; while the character and properties, including the connectivity, of a Riemann surface have not been referred to. The theta functions are referred to only once, and the principles of the theory of Abelian Functions have been illustrated only by the developments given for elliptic functions.

§ 1. *Complex Numbers.*—Complex numbers are numbers of the form $x + iy$, where x, y are ordinary real numbers, and i is a symbol imagined capable of combination with itself and the ordinary real numbers, by way of addition, subtraction, multiplication and division, according to the ordinary commutative, associative and distributive laws; the symbol i is further such that $i^2 = -1$.

Taking in a plane two rectangular axes Ox, Oy, we assume that every point of the plane is definitely associated with two real numbers x, y (its co-ordinates) and conversely; thus any point of the plane is associated with a single complex number; in particular, for every point of the axis Ox, for which $y = O$, the associated number is an ordinary real number; the complex numbers thus include the real numbers. The axis Ox is often called the real axis, and the axis Oy the imaginary axis. If P be the point associated with the complex variable $z = x + iy$, the distance OP be called r, and the positive angle less than 2π between Ox and OP be called θ, we may write $z = r (\cos \theta + i \sin \theta)$; then r is called the modulus or absolute value of z and often denoted by $|z|$ and θ is called the phase or amplitude of z, and often denoted by ph (z); strictly the phase is ambiguous by additive multiples of 2π. If $z' = x' + iy'$ be represented by P′, the complex argument $z' + z$ is represented by a point P″ obtained by drawing from P′ a line equal to and parallel to OP; the geometrical representation involves for its validity certain properties of the plane; as, for instance, the equation $z' + z = z + z'$ involves the possibility of constructing a parallelogram (with OP″ as diagonal). It is important constantly to bear in mind, what is capable of easy algebraic proof (and geometrically is Euclid's proposition III. 7), that the modulus of a sum or difference of two complex numbers is generally less than (and is never greater than) the sum of their moduli, and is greater than (or equal to) the difference of their moduli; the former statement thus holds for the sum of any number of complex numbers. We shall write $E(i\theta)$ for $\cos \theta + i \sin \theta$; it is at once verified that $E(i\alpha)$. $E(i\beta) = E[i(\alpha + \beta)]$, so that the phase of a product of complex quantities is obtained by addition of their respective phases.

§ 2. *Plotting and Properties of Simple Expressions involving a Complex Number.*—If we put $\zeta = (z-i)/(z + i)$, and, putting $\zeta = \xi + i\eta$, take a new plane upon which ξ, η are rectangular co-ordinates, the equations $\xi = (x^2 + y^2 - 1)/[x^2 + (y + 1)^2]$, $\eta = -2xy/[x^2 + (y + i)^2]$ will determine, corresponding to any point of the first plane, a point of the second plane. There is the one exception of $z = -i$, that is, $x = 0$, $y = -1$, of which the corresponding point is at infinity. It can now be easily proved that as z describes the real axis in its plane the point ζ describes once a circle of radius unity, with centre at $\zeta = 0$, and that there is a definite correspondence of point to point between points in the z-plane which are above the real axis and points of the ζ-plane which are interior to this circle; in particular $z = i$ corresponds to $\zeta = 0$.

Moreover, ζ being a rational function of z, both ξ and η are continuous differentiable functions of x and y, save when ζ is infinite; writing $\zeta = f(x, y) = f(z - iy, y)$, the fact that this is really independent of y leads at once to $\partial f/\partial x + i\partial f/\partial y = 0$, and hence to

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial x'}, \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x'}, \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0;$$

so that ξ is not any arbitrary function of x, y, and when ξ is known η is determinate save for an additive constant. Also, in virtue of these equations, if ζ, ζ' be the values of ζ corresponding to two near values of z, say z and z′, the ratio $(\zeta' - \zeta)/(z' - z)$ has a definite limit when $z' = z$, independent of the ultimate phase of $z' - z$, this limit being therefore equal to $\partial\zeta/\partial x$, that is, $\partial\xi/\partial x + i\partial\eta)/\partial x$. Geometrically this fact is interpreted by saying that if two curves in the z-plane intersect at a point P, at which both the differential coefficients $\partial\xi/\partial x$, $\partial\eta/\partial x$ are not zero, and P′, P″ be two points near to P on these curves respectively, and the corresponding points of the ζ-plane be Q, Q′, Q″, then (1) the ratios PP″/PP′, QQ″/QQ′ are ultimately equal, (2) the angle P′PP″ is equal to Q′QQ″, (3) the rotation from PP′ to PP″ is in the same sense as from QQ′ to QQ″, it being understood that the axes of ξ, η in the one plane are related as are the axes of x, y. Thus any diagram of the z-plane becomes a diagram of the ζ-plane with the same angles; the magnification, however, which is equal to $[(\partial\xi/\partial x)^2 + (\partial\xi/\partial y)^2]^{1/2}$ varies from point to point. Conversely, it appears subsequently that the expression of any copy of a diagram (say, a map) which preserves angles requires the intervention of the complex variable.

As another illustration consider the case when ζ is a polynomial in z,

$$\zeta = p_0z^n + p_1z^{n-1} + \dots + p_n;$$

H being an arbitrary real positive number, it can be shown that a radius R can be found such for every $|z| > R$ we have $|\zeta| > H$; consider the lower limit of $|\zeta|$ for $|z| < R$; as $\xi^2 + \eta^2$ is a real continuous function of x, y for $|z| < R$, there is a point (x, y), say (x_0, y_0), at which $|\zeta|$ is least, say equal to ρ, and therefore within a circle in the ζ-plane whose centre is the origin, of radius ρ, there are no points ζ representing values corresponding to $|z| < R$. But if ζ_0 be the value of ζ corresponding to (x_0, y_0), and the expression of $\zeta - \zeta_0$ near $z_0 = x_0 + iy_0$, in terms of $z - z_0$, be $A(z - z_0)^m + B(z - z_0)^{m+1} + \dots$, where A is not zero, to two points near to (x_0, y_0), say (x_1, y_1) or z_1 and $z_2 = z_0 + (z_1 - z_0)(\cos \pi/m + i \sin \pi/m)$, will correspond two points near to ζ_0, say ζ_1, and $2\zeta_0 - \zeta'_1$, situated so that ζ_0 is between them. One of these must be within the circle (ρ). We infer then that $\rho = 0$, and have proved that every polynomial in z vanishes for some value of z, and can therefore be written as a product of factors of the form $z - \alpha$, where α denotes a complex number. This proposition alone suffices to suggest the importance of complex numbers.

§ 3. *Limiting Operations*.—In order that a complex number $\zeta = \xi + i\eta$ may have a limit it is necessary and sufficient that each of ξ and η has a limit. Thus an infinite series $w_0 + w_1 + w_2 + \dots$, whose terms are complex numbers, is convergent if the real series formed by taking the real parts of its terms and that formed by the imaginary terms are both convergent. The series is also convergent if the real series formed by the moduli of its terms is convergent; in that case the series is said to be absolutely convergent, and it can be shown that its sum is unaltered by taking the terms in any other order. Generally the necessary and sufficient condition of convergence is that, for a given real positive ε, a number m exists such that for every $n > m$, and every positive p, the batch of terms $w_n + w_{n+1} + \dots + w_{n+p}$ is less than ε in absolute value. If the terms depend upon a complex variable z, the convergence is called *uniform* for a range of values of z, when the inequality holds, for the same ε and m, for all the points z of this range.

The infinite series of most importance are those of which the general term is a_nz^n, wherein a_n is a constant, and z is regarded as variable, n = 0, 1, 2, 3, ... Such a series is called a power series, if a real and positive number M exists such that for $z = z_0$ and every n, $|a_nz_0{}^n| < M$, a condition which is satisfied, for instance, if the series converges for $z = z_0$, then it is at once proved that the series converges absolutely for every z for which $|z| < |z_0|$, and converges uniformly over every range $|z| < r'$ for which $r' < |z_0|$. To every power series there belongs then a circle of convergence within which it converges absolutely and uniformly; the function of z represented by it is thus continuous within the circle (this being the result of a general property of uniformly convergent series of continuous functions); the sum for an interior point z is, however, continuous with the sum for a point z_0 on the circumference, as z approaches to z_0 provided the series converges for $z = z_0$, as can be shown without much difficulty. Within a common circle of convergence two power series $\Sigma\, a_nz^n$, $\Sigma\, b_nz^n$ can be multiplied together according to the ordinary rule, this being a consequence of a theorem for absolutely convergent series. If r_1 be less than the radius of convergence of a series $\Sigma\, a_n z^n$ and for $|z| = r_1$, the sum of the series be in absolute value less than a real positive quantity M, it can be shown that for $|z| = r_1$ every term is also less than M in absolute value, namely, $|a_n| < Mr_1{}^{-n}$. If in every arbitrarily small neighbourhood of z=0 there be a point for which two converging power series $\Sigma a_n z^n$, Σb_nz^n agree in value, then the series are identical, or $a_n = b_n$; thus also if Σa_nz^n vanish at $z = 0$ there is a circle of finite radius about $z = 0$ as centre within which no other points are found for which the sum of the series is zero. Considering a power series $f(z) = \Sigma a_nz^n$ of radius of convergence R, if $|z_0| < R$ and we put $z = z_0 + t$ with $|t| < R\text{-}|z_0|$, the resulting series $\Sigma a_n(z_0 + t)^n$ may be regarded as a double series in z_0 and t, which, since $|z_0| + t < R$, is absolutely convergent; it may then be arranged according to powers of t. Thus we may write $f(z) = \Sigma\, A_nt^n$; hence $A_0 = f(z_0)$, and we have $[f(z_0 + t) - f(z_0)]/t = \Sigma_{n=1} A_nt^{n-1}$, wherein the continuous series on the right reduces to A_1 for $t = 0$; thus the ratio on the left has a definite limit when $t = 0$, equal namely to A_1 or $\Sigma na_nz_0{}^{n-1}$. In other words, the original series may legitimately be differentiated at any interior point z_0 of its circle of convergence. Repeating this process we find $f(z_0 + t) = \Sigma t^nf^{(n)}(z_0)/n!$, where $f^{(n)}(z_0)$ is the nth differential coefficient. Repeating for this power series, in t, the argument applied about $z = 0$ for Σa_nz^n, we infer that for the series f(z) every point which reduces it to zero is an isolated point, and of such points only a finite number lie within a circle which is within the circle of convergence of f(z).

Perhaps the simplest possible power series is $e^z = \exp(z) = 1 + z^2/2! + z^3/3! + \dots$ of which the radius of convergence is infinite. By multiplication we have $\exp(z)\cdot\exp(z^1) = \exp(z + z^1)$. In particular when x, y are real, and $z = x + iy$, $\exp(z) = \exp(x)\exp(iy)$. Now the functions

$$U_0 = \sin y,\ V_0 = 1 - \cos y,\ U_1 = y - \sin y,$$

$$V_1 = \tfrac{1}{2}y^2 - 1 + \cos y,\ U_2 = \tfrac{1}{6}y^3 - y + \sin y,\ V_2 = \tfrac{1}{24}y^4 - \tfrac{1}{2}y^2 + 1 - \cos y,\ \dots$$

all vanish for $y = 0$, and the differential coefficient of any one after the first is the preceding one; as a function (of a real variable) is increasing when its differential coefficient is positive, we infer, for y positive, that each of these functions is positive; proceeding to a limit we hence infer that

$$\cos y = 1 - \tfrac{1}{2}y^2 + \tfrac{1}{24}y^4 - \dots,\qquad \sin y = y - \tfrac{1}{6}y^3 + \tfrac{1}{120}y^5 - \dots,$$

for positive, and hence, for all values of y. We thus have $\exp(iy) = \cos y + i \sin y$, and $\exp(z) = \exp(x)\cdot(\cos y + i \sin y)$. In other words, the modulus of exp (z) is exp (x) and the phase is y. Hence also

$$\exp(z + 2\pi i) = \exp(x) [\cos (y + 2\pi) + i \sin(y + 2\pi)],$$

which we express by saying that exp (z) has the period $2\pi i$, and hence also the period $2k\pi i$, where k is an arbitrary integer. From the fact that the constantly increasing function exp (x) can vanish only for $x = 0$, we at once prove that exp (z) has no other periods.

Taking in the plane of z an infinite strip lying between the lines $y = 0$, $y = 2\pi$ and plotting the function $\zeta = \exp (z)$ upon a new plane, it follows at once from what has been said that every complex value of ζ arises when z takes in turn all positions in this strip, and that no value arises twice over. The equation $\zeta = \exp(z)$ thus defines z, regarded as depending upon ζ, with only an additive ambiguity $2k\pi i$, where k is an integer. We write $z = \lambda(\zeta)$; when ζ is real this becomes the logarithm of ζ; in general $\lambda(\zeta) = \log |\zeta| + i \text{ ph } (\zeta) + 2k\pi i$, where k is an integer; and when ζ describes a closed circuit surrounding the origin the phase of ζ increases by 2π, or k increases by unity. Differentiating the series for ζ we have $d\zeta/dz = \zeta$, so that z, regarded as depending upon ζ, is also differentiable, with $dz/d\zeta = \zeta^{-1}$. On the other hand, consider the series $\zeta - 1 - \frac{1}{2}(\zeta - 1)^2 + \frac{1}{3}(\zeta - 1)^3 - \dots$; it converges when $\zeta = 2$ and hence converges for $|\zeta - 1| < 1$; its differential coefficient is, however, $1 - (\zeta - 1) + (\zeta - 1)^2 - \dots$, that is, $(1 + \zeta - 1)^{-1}$. Wherefore if $\varphi(\zeta)$ denote this series, for $|\zeta - 1| < 1$, the difference $\lambda(\zeta) - \varphi(\zeta)$, regarded as a function of ξ and η, has vanishing differential coefficients; if we take the value of $\lambda(\zeta)$ which vanishes when $\zeta = 1$ we infer thence that for $|\zeta - 1| < 1$, $\lambda(\zeta) = \Sigma_{n=1} [(-1)^{(n-1)}/n\ (\zeta - 1)^n]$. It is to be remarked that it is impossible for ζ while subject to $|\zeta - 1| < 1$ to make a circuit about the origin. For values of ζ for which $|\zeta - 1| \nless 1$, we can also calculate $\lambda(\zeta)$ with the help of infinite series, utilizing the fact that $\lambda(\zeta\zeta') = \lambda(\zeta) + \lambda(\zeta')$.

The function $\lambda(\zeta)$ is required to define ζ^a when ζ and a are complex numbers; this is defined as $\exp [a\lambda(\zeta)]$, that is as $\Sigma_{n=0}\ a^n [\lambda(\zeta)]^n/n!$. When a is a real integer the ambiguity of $\lambda(\zeta)$ is immaterial here, since $\exp [a\lambda(\zeta) + 2ka\pi i] = \exp [a\lambda(\zeta)]$; when a is of the form $1/q$, where q is a positive integer, there are q values possible for $\zeta^{1/q}$, of the form $\exp [1/q\ \lambda(\zeta)] \exp (2k\pi i/q)$, with $k = 0, 1, \dots q - 1$, all other values of k leading to one of these; the qth power of any one of these values is ζ; when $a = p/q$, where p, q are integers without common factor, q being positive, we have $\zeta^{p/q} = (\zeta^{1/q})^p$. The definition of the symbol ζ^a is thus a generalization of the ordinary definition of a power, when the numbers are real. As an example, let it be required to find the meaning of i^i; the number i is of modulus unity and phase $\frac{1}{2}\pi$; thus $\lambda(i) = i (\frac{1}{2}\pi + 2k\pi)$; thus

$$i^i = \exp (-\tfrac{1}{2}\pi - 2k\pi) = \exp (-\tfrac{1}{2}\pi) \exp (-2k\pi),$$

is always real, but has an infinite number of values.

The function exp (z) is used also to define a generalized form of the cosine and sine functions when z is 312
complex; we write, namely, $\cos z = \frac{1}{2}[\exp (iz) + \exp (-iz)]$ and $\sin z = -\frac{1}{2}i [\exp (iz) - \exp(-iz)]$. It will be found that these obey the ordinary relations holding when z is real, except that their moduli are not inferior to unity. For example, $\cos i = 1 + 1/2! + 1/4! + \dots$ is obviously greater than unity.

§4. *Of Functions of a Complex Variable in General*.—We have in what precedes shown how to generalize the ordinary rational, algebraic and logarithmic functions, and considered more general cases, of functions expressible by power series in z. With the suggestions furnished by these cases we can frame a general definition. So far our use of the plane upon which z is represented has been only illustrative, the results being capable of analytical statement. In what follows this representation is vital to the mode of expression we adopt; as then the properties of numbers cannot be ultimately based upon spatial intuitions, it is necessary to indicate what are the geometrical ideas requiring elucidation.

Consider a square of side a, to whose perimeter is attached a definite direction of description, which we take to be counter-clockwise; another square, also of side a, may be added to this, so that there is a side common; this common side being erased we have a composite region with a definite direction of perimeter; to this a third square of the same size may be attached, so that there is a side common to it and one of the former squares, and this common side may be erased. If this process be continued any number of times we obtain a region of the plane bounded by one or more polygonal closed lines, no two of which intersect; and at each portion of the perimeter there is a definite direction of description, which is such that the region is on the left of the describing point. Similarly we may construct a region by piecing together triangles, so that every consecutive two have a side in common, it being understood that there is assigned an upper limit for the greatest side of a triangle, and a lower limit for the smallest angle. In the former method, each square may be divided into four others by lines through its centre parallel to its sides; in the latter method each triangle may be divided into four others by lines joining the middle points of its sides; this halves the sides and preserves the angles. When we speak of a *region* of the plane in general, unless the contrary is stated, we shall suppose it capable of being generated in this latter way by means of a finite number of triangles, there being an upper limit to the length of a side of the triangle and a lower limit to the size of an angle of the triangle. We shall also require to speak of a *path* in the plane; this is to be understood as capable of arising as a limit of a polygonal path of finite length, there being a definite direction or sense of description at every point of the path, which therefore never meets itself. From this the meaning of a closed path is clear. The boundary points of a region form one or more closed paths, but, in general, it is only in a limiting sense that the interior points of a closed path are a region.

There is a logical principle also which must be referred to. We frequently have cases where, about every interior or boundary, point z_0 of a certain region a circle can be put, say of radius r_0, such that for all points z of the region which are interior to this circle, for which, that is, $|z - z_0| < r_0$, a certain property holds. Assuming that to r_0 is given the value which is the upper limit for z_0, of the possible values, we may call the points $|z - z_0| < r_0$, the neighbourhood belonging to or *proper* to z_0, and may speak of the property as the property (z, z_0). The value of r_0 will in general vary with z_0; what is in most cases of importance is the question whether the lower limit of r_0 for all positions is zero or greater than zero. (A) This lower limit is certainly greater than zero provided the property (z, z_0) is of a kind which we may call extensive; such, namely, that if it holds, for some position of z_0 and all positions of z, within a certain region, then the property (z, z_1) holds within a circle of radius R about any interior point z_1 of this region for all points z for which the circle $|z - z_1| = R$ is within the region. Also in this case r_0 varies continuously with z_0. (B) Whether the property is of this extensive character or not we can prove that the region can be divided into a finite number of sub-regions such that, for every one of these, the property holds, (1) for *some* point z_0 within or upon the boundary of the sub-region, (2) for *every* point z within or upon the boundary of the sub-region.

We prove these statements (A), (B) in reverse order. To prove (B) let a region for which the property (z, z_0) holds for all points z and some point z_0 of the region, be called *suitable*: if each of the triangles of which the region is built up be suitable, what is desired is proved; if not let an unsuitable triangle be subdivided into four, as before explained; if one of these subdivisions is unsuitable let it be again subdivided; and so on. Either the process terminates and then what is required is proved; or else we obtain an indefinitely continued sequence of unsuitable triangles, each contained in the preceding, which converge to a point, say ζ; after a certain stage all these will be interior to the proper region of ζ; this, however, is contrary to the supposition that they are all unsuitable.

We now make some applications of this result (B). Suppose a definite finite real value attached to every interior or boundary point of the region, say $f(x, y)$. It may have a finite upper limit H for the region, so that no point (x, y) exists for which $f(x, y) > H$, but points (x, y) exist for which $f(x, y) > H - \varepsilon$, however small ε may be; if not we say that its upper limit is infinite. There is then at least one point of the region such that, for points of the region within a circle about this point, the upper limit of $f(x, y)$ is H, however small the radius of the circle be taken; for if not we can put about every point of the region a circle within which the upper limit of $f(x, y)$ is less than H; then by the result (B) above the region consists of a finite number of sub-regions within each of which the upper limit is less than H; this is inconsistent with the hypothesis that the upper limit for the whole region is H. A similar statement holds for the lower limit. A case of such a function $f(x, y)$ is the radius r_0 of the neighbourhood proper to any point z_0, spoken of above. We can hence prove the statement (A) above.

Suppose the property (z, z_0) extensive, and, if possible, that the lower limit of r_0 is zero. Let then ζ be a point such that the lower limit of r_0 is zero for points z_0 within a circle about ζ however small; let r be the radius of the neighbourhood proper to ζ; take z_0 so that $|z_0-\zeta| < \tfrac{1}{2}r$; the property (z, z_0), being extensive, holds within a circle, centre z_0, of radius $r - |z_0 - \zeta|$, which is greater than $|z_0 - \zeta|$, and increases to r as $|z_0 - \zeta|$ diminishes; this being true for all points z_0 near ζ, the lower limit of r_0 is not zero for the neighbourhood of ζ, contrary to what was supposed. This proves (A). Also, as is here shown that $r_0 \geqq r - |z_0 - \zeta|$, may similarly be shown that $r \geqq r_0 - |z_0 - \zeta|$. Thus r_0 differs arbitrarily little from r when $|z_0 - \zeta|$ is sufficiently small; that is, r_0 varies continuously with z_0. Next suppose the function $f(x, y)$, which has a definite finite value at every point of the region considered, to be continuous but not necessarily real, so that about every point z_0, within or upon the boundary of the region, η being an arbitrary real positive quantity assigned beforehand, a circle is possible, so that for all points z of the region interior to this circle, we have $|f(x, y) - f(x_0, y_0)| < \tfrac{1}{2}\eta$, and therefore (x', y') being any other point interior to this circle, $|f(x', y') - f(x, y)| < \eta$. We can then apply the result (A) obtained above, taking for the neighbourhood proper to any point z_0 the circular area within which, for any two points (x, y), (x', y'), we have $|f(x', x') - f(x, y)| < \eta$. This is clearly an extensive property. Thus, a number r is assignable, greater than zero, such that, for any two points (x, y), (x', y') within a circle $|z - z_0| = r$ about any point z_0, we have $|f(x', y') - f(x, y)| < \eta$, and, in particular, $|f(x, y) - f(x_0, y_0)| < \eta$, where η is an arbitrary real positive quantity agreed upon beforehand.

Take now any path in the region, whose extreme points are z_0, z, and let $z_1, \dots z_{n-1}$ be intermediate points of the path, in order; denote the continuous function $f(x, y)$ by $f(z)$, and let f_r denote any quantity such that $|f_r - f(z_r)| \leqq |f(z_{r+1}) - f(z_r)|$; consider the sum

$$(z_1 - z_0)f_0 + (z_2 - z_1)f_1 + \dots + (z - z_{n-1})f_{n-1}.$$

By the definition of a path we can suppose, n being large enough, that the intermediate points $z_1, \dots z_{n-1}$ are so taken that if z_i, z_{i+1} be any two points intermediate, in order, to z_r and z_{r+1}, we have $|z_{i+i}-z_i| < |z_{r+1} - z_r|$; we can thus suppose $|z_1 - z_0|$, $|z_2 - z_1|$, $\dots$ $|z - z_{n-1}|$all to converge constantly to zero. This being so, we can show that the sum above has a definite limit. For this it is sufficient, as in the case of an integral of a function of one real variable, to prove this to be so when the convergence is obtained by taking new points of division intermediate to the former ones. If, however, $z_{r,1}, z_{r,2}, \dots z_{r,m-1}$ be intermediate in order to z_r and z_{r+1}, and $|f_{r,i} - f(z_{r,i})| < |f(z_{r,i+1}) - f(z_{r,i})|$, the difference between $\Sigma(z_{r+1} - z_r)f_r$ and

$$\sum \{ (z_{r,1}-z_r)f_{r,0} + (z_{r,2} - z_{r,1})f_{r,1} + \dots + (z_{r+1} - z_{r,m-1})f_{r,m-1} \},$$

which is equal to

$$\sum_r \sum_i (z_{r,i+1} - z_{r,i})(f_{r,i} - f_r),$$

is, when $|z_{r+1} - z_r|$ is small enough, to ensure $|f(z_{r+1}) - f(z_r)| < \eta$, less in absolute value than

$$\sum 2\eta \sum |z_{r,i+1} - z_{r,i}|,$$

which, if S be the upper limit of the perimeter of the polygon from which the path is generated, is $< 2\eta S$, and is therefore arbitrarily small.

The limit in question is called $\int_{z_0}^{z} f(z)dz$. In particular when $f(z) = 1$, it is obvious from the definition that its value is $z - z_0$; when $f(z) = z$, by taking $f_r = \tfrac{1}{2}(z_{r+1} - z_r)$, it is equally clear that its value is $\tfrac{1}{2}(z^2 - z_0^2)$; these results will be applied immediately.

Suppose now that to every interior and boundary point z_0 of a certain region there belong two definite finite numbers $f(z_0)$, $F(z_0)$, such that, whatever real positive quantity η may be, a real positive number ε exists for which the condition

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - F(z_0) \right| < \eta,$$

which we describe as the condition (z, z_0), is satisfied for every point z, within or upon the boundary of the region, satisfying the limitation $|z - z_0| < \varepsilon$. Then $f(z_0)$ is called a differentiable function of the complex variable z_0 over this region, its differential coefficient being $F(z_0)$. The function $f(z_0)$ is thus a continuous function of the real variables x_0, y_0, where $z_0 = x_0 + iy_0$, over the region; it will appear that $F(z_0)$ is also continuous and in fact also a differentiable function of z_0.

Supposing η to be retained the same for all points z_0 of the region, and σ_0 to be the upper limit of the possible values of ε for the point z_0, it is to be presumed that σ_0 will vary with z_0, and it is not obvious as yet that the lower limit of the values of σ_0 as z_0 varies over the region may not be zero. We can, however, show that the region can be divided into a finite number of sub-regions for each of which the condition (z, z_0), above, is satisfied for all points z, within or upon the boundary of this sub-region, for an appropriate position of z_0, within or upon the boundary of this sub-region. This is proved above as result (B).

Hence it can be proved that, for a differentiable function $f(z)$, the integral $\int_{z1}^{z} f(z)dz$ has the same value by whatever path within the region we pass from z_1 to z. This we prove by showing that when taken round a closed path in the region the integral $\int f(z)dz$ vanishes. Consider first a triangle over which the condition (z, z_0) holds, for some position of z_0 and every position of z, within or upon the boundary of the triangle. Then as

$$f(z) = f(z_0) + (z - z_0) F(z_0) + \eta\theta(z - z_0), \text{ where } |\theta| < 1,$$

we have

$$\int f(z)dz = [f(z_0) - z_0 F(z_0)] \int dz + F(z_0) \int zdz + \eta\int\theta(z - z_0)dz,$$

which, as the path is closed, is $\eta \int\theta(z - z_0)dz$. Now, from the theorem that the absolute value of a sum is less than the sum of the absolute values of the terms, this last is less, in absolute value, than ηap, where a is the greatest side of the triangle and p is its perimeter; if Δ be the area of the triangle, we have $\Delta = ½ab \sin C > (\alpha/\pi)$ ba, where α is the least angle of the triangle, and hence $a(a + b + c) < 2a(b + c) < 4\pi\Delta/\alpha$; the integral $\int f(z)dz$ round the perimeter of the triangle is thus $< 4\pi\eta\Delta/\alpha$. Now consider any region made up of triangles, as before explained, in each of which the condition (z, z_0) holds, as in the triangle just taken. The integral $\int f(z)dz$ round the boundary of the region is equal to the sum of the values of the integral round the component triangles, and thus less in absolute value than $4\pi\eta K/\alpha$, where K is the whole area of the region, and α is the smallest angle of the component triangles. However small η be taken, such a division of the region into a finite number of component triangles has been shown possible; the integral round the perimeter of the region is thus arbitrarily small. Thus it is actually zero, which it was desired to prove. Two remarks should be added: (1) The theorem is proved only on condition that the closed path of integration belongs to the region at every point of which the conditions are satisfied. (2) The theorem, though proved only when the region consists of triangles, holds also when the boundary points of the region consist of one or more closed paths, no two of which meet.

Hence we can deduce the remarkable result that the value of $f(z)$ at any interior point of a region is expressible in terms of the value of $f(z)$ at the boundary points. For consider in the original region the function $f(z)/(z - z_0)$, where z_0 is an interior point: this satisfies the same conditions as $f(z)$ except in the immediate neighbourhood of z_0. Taking out then from the original region a small regular polygonal region with z_0 as centre, the theorem holds for the remaining portion. Proceeding to the limit when the polygon becomes a circle, it appears that the integral $\int dzf(z)/(z - z_0)$ round the boundary of the original region is equal to the same integral taken counter-clockwise round a small circle having z_0 as centre; on this circle, however, if $z - z_0 = rE(i\theta)$, $dz/(z - z_0) = id\theta$, and $f(z)$ differs arbitrarily little from $f(z_0)$ if r is sufficiently small; the value of the integral round this circle is therefore, ultimately, when r vanishes, equal to $2\pi i f(z_0)$. Hence $f(z_0) = 1/2\pi i \int (dtf(t)/(t - z_0)$, where this integral is round the boundary of the original region. From this it appears that

$$F(z_0) = \lim. \frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{2\pi i} \int \frac{dtf(t)}{(t - z_0)^2}$$

also round the boundary of the original region. This form shows, however, that $F(z_0)$ is a continuous, finite, differentiable function of z_0 over the whole interior of the original region.

§ 5. *Applications.*—The previous results have manifold applications.

(1) If an infinite series of differentiable functions of z be uniformly convergent along a certain path lying with the region of definition of the functions, so that $S(2) = u_0(z) + u_1(z) + \dots + u_{n-1}(z) + R_n(z)$, where $|R_n(z)| < \varepsilon$ for all points of the path, we have

$$\int_{z0}^{z} S(z)dz = \int_{z0}^{z} u_0(z)dz + \int_{z0}^{z} u_1(z)dz + \dots + \int_{z0}^{z} u_{n-1}(z)dz + \int_{z0}^{z} R_n(z)dz,$$

wherein, in absolute value, $\int_{z0}^{z} R_n(z)dz < \varepsilon L$, if L be the length of the path. Thus the series may be integrated, and the resulting series is also uniformly convergent.

(2) If $f(x, y)$ be definite, finite and continuous at every point of a region, and over any closed path in the region $\int f(x, y)dz = 0$, then $\psi(z) = \int_{z0}^{z} f(x, y)dz$, for interior points z_0, z, is a differentiable function of z, having for its differential coefficient the function $f(x, y)$, which is therefore also a differentiable function of z at interior points.

(3) Hence if the series $u_0(z) + u_1(z) + \dots$ to ∞ be uniformly convergent over a region, its terms being differentiable functions of z, then its sum $S(z)$ is a differentiable function of z, whose differential coefficient, given by $(1/2\pi i) \int 2\pi i/(t - z)^2$, is obtainable by differentiating the series. This theorem, unlike (1), does not hold for functions of a real variable.

(4) If the region of definition of a differentiable function $f(z)$ include the region bounded by two concentric circles of radii r, R, with centre at the origin, and z_0 be an interior point of this region,

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(t)dt}{R^t - z_0} - \frac{1}{2\pi i} \int \frac{f(t)dt}{r^t - z_0},$$

where the integrals are both counter-clockwise round the two circumferences respectively; putting in the first $(t - z_0)^{-1} = \sum_{n=0} z_0^n/t^{n+1}$, and in the second $(t - z_0)^{-1} = -\sum_{n=0} t^n/z_0^{n+1}$, we find $f(z_0) = \sum_{-\infty}^{\infty} A_n z_0^n$, wherein $A_n = (1/2\pi i) \int [f(t)/t^{n+1}] dt$, taken round any circle, centre the origin, of radius intermediate between r and R. Particular cases are: (α) when the region of definition of the function includes the whole interior of the outer circle; then we may take $r = 0$, the coefficients A_n for which $n < 0$ all vanish, and the function $f(z_0)$ is expressed for the whole interior $|z_0| < R$ by a power series $\sum_{0}^{\infty} A_n z_0^n$. In other words, *about every interior point c of the region of definition a differentiable function of z is expressible by a power series in z − c*; a very important result.

(β) If the region of definition, though not including the origin, extends to within arbitrary nearness of this on all sides, and at the same time the product $z^m f(z)$ has a finite limit when $|z|$ diminishes to zero, all the coefficients A_n for which $n < -m$ vanish, and we have

$$f(z_0) = A_{-m}z_0^{-m} + A_{-m+1}z_0^{-m+1} + \dots + A_{-1}z_0^{-1} + A_0 + A_1z_0 \dots \text{ to } \infty.$$

Such a case occurs, for instance, when $f(z) = \operatorname{cosec} z$, the number m being unity.

§ 6. *Singular Points.*—The *region of existence* of a differentiable function of z is an unclosed aggregate of points, each of which is an interior point of a neighbourhood consisting wholly of points of the aggregate, at every point of which the function is definite and finite and possesses a unique finite differential coefficient. Every point of the plane, not belonging to the aggregate, which is a limiting point of points of the aggregate, such, that is, that points of the aggregate lie in every neighbourhood of this, is called a *singular point* of the function.

About every interior point z_0 of the region of existence the function may be represented by a power series in $z - z_0$, and the series converges and represents the function over any circle centre at z_0 which contains no singular point in its interior. This has been proved above. And it can be similarly proved, putting $z = 1/\zeta$, that if the region of existence of the function contains all points of the plane for which $|z| > R$, then the function is representable for all such points by a power series in z^{-1} or ζ; in such case we say that the region of existence of the function contains the point $z = \infty$. A series in z^{-1} has a finite limit when $|z| = \infty$; a series in z cannot remain finite for all points z for which $|z| > R$; for if, for $|z| = R$, the sum of a power series $\Sigma a_n z^n$ in z is in absolute value less than M, we have $|a_n| < Mr^{-n}$, and therefore, if M remains finite for all values of r however great, $a_n = 0$. Thus the region of existence of a function if it contains all finite points of the plane cannot contain the point $z = \infty$; such is, for instance, the case of the function $\exp(z) = \Sigma z^n/n!$. This may be regarded as a particular case of a well-known result (§ 7), that the circumference of convergence of any power series representing the function contains at least one singular point. As an extreme case functions exist whose region of existence is circular, there being a singular point in every arc of the circumference, however small; for instance, this is the case for the functions represented for $|z| < 1$ by the series $\sum_{n=0} z^m$, where $m = n^2$, the series $\sum_{n=0} z^m$ where $m = n!$, and the series $\sum_{n=1} z^m/(m + 1)(m + 2)$ where $m = a^n$, a being a positive integer, although in the last case the series actually converges for every point of the circle of convergence $|z| = 1$. If z be a point interior to the circle of convergence of a series representing the function, the series may be rearranged in powers of $z - z_0$; as z_0 approaches to a singular point of the function, lying on the circle of convergence, the radii of convergence of these derived series in $z - z_0$ diminish to zero; when, however, a circle can be put about z_0, not containing any singular point of the function, but containing points outside the circle of convergence of the original series, then the series in $z - z_0$ gives the value of the function for these external points. If the function be supposed to be given only for the interior of the original circle, by the original power series, the series in $z - z_0$ converging beyond the original circle gives what is known as an *analytical continuation* of the function. It appears from what has been proved that the value of the function at 314
all points of its region of existence can be obtained from its value, supposed given by a series in one original circle, by a succession of such processes of analytical continuation.

§ 7. *Monogenic Functions.*—This suggests an entirely different way of formulating the fundamental parts of the theory of functions of a complex variable, which appears to be preferable to that so far followed here.

Starting with a convergent power series, say in powers of z, this series can be arranged in powers of $z - z_0$, about any point z_0 interior to its circle of convergence, and the new series converges certainly for $|z - z_0| < r - |z_0|$, if r be the original radius of convergence. If for every position of z_0 this is the greatest radius of convergence of the derived series, then the original series represents a function existing only within its circle of convergence. If for some position of z_0 the derived series converges for $|z - z_0| < r - |z_0| + D$, then it can be shown that for points z, interior to the original circle, lying in the annulus $r - |z_0| < |z - z_0| < r - |z_0| + D$, the value represented by the derived series agrees with that represented by the original series. If for another point z_1 interior to the original circle the derived series converges for $|z - z_1| < r - |z_1| + E$, and the two circles $|z - z_0| = r - |z_0| + D$, $|z - z_1| = r - |z_1| + E$ have interior points common, lying beyond $|z| = r$, then it can be shown that the values represented by these series at these common points agree. Either series then can be used to furnish an analytical continuation of the function as originally defined. Continuing this process of continuation as far as possible, we arrive at the conception of the function as defined by an aggregate of power series of which every one has points of convergence common with some one or more others; the whole aggregate of points of the plane which can be so reached constitutes the region of existence of the function; the limiting points of this region are the points in whose neighbourhood the derived series have radii of convergence diminishing indefinitely to zero; these are the singular points. The circle of convergence of any of the series has at least one such singular point upon its circumference. So regarded the function is called a *monogenic* function, the epithet having reference to the single origin, by one power series, of the expressions representing the function; it is also sometimes called a *monogenic analytical* function, or simply an *analytical* function; all that is necessary to define it is the value of the function and of all its differential coefficients, at some one point of the plane; in the method previously followed here it was necessary to suppose the function differentiable at every point of its region of existence. The theory of the integration of a monogenic function, and Cauchy's theorem, that $\int f(z)dz = 0$ over a closed path, are at once deducible from the corresponding results applied to a single power series for the interior of its circle of convergence. There is another advantage belonging to the theory of monogenic functions: the theory as originally given here applies in the first instance only to single valued functions; a monogenic function is by no means necessarily single valued—it may quite well happen that starting from a particular power series, converging over a certain circle, and applying the process of analytical continuation over a closed path back to an interior point of this circle, the value obtained does not agree with the initial value. The notion of basing the theory of functions on the theory of power series is, after Newton, largely due to Lagrange, who has some interesting remarks in this regard at the beginning of his *Théorie des fonctions analytiques*. He applies the idea, however, primarily to functions of a real variable for which the expression by power series is only of very limited validity; for functions of a complex variable probably the systematization of the theory owes most to Weierstrass, whose use of the word monogenic is that adopted above. In what follows we generally suppose this point of view to be regarded as fundamental.

§ 8. *Some Elementary Properties of Single Valued Functions.*—A *pole* is a singular point of the function $f(z)$ which is not a singularity of the function $1/f(z)$; this latter function is therefore, by the definition, capable of representation about this point, z_0, by a series $[f(z)]^{-1} = \Sigma a_n(z - z_0)^n$. If herein a_0 is not zero we can hence derive a representation for $f(z)$ as a power series about z_0, contrary to the hypothesis that z_0 is a singular point for this function. Hence $a_0 = 0$; suppose also $a_1 = 0$, $a_2 = 0$, ... $a_{m-1} = 0$, but $a_m \pm 0$. Then $[f(z)]^{-1} = (z - z_0)^m[a_m + a_{m+1} (z - z_0) + ...]$, and hence $(z - z_0)^m f(z) = a_m{}^{-1} + \Sigma b_n (z - z_0)^n$, namely, the expression of $f(z)$ about $z = z_0$ contains a finite number of negative powers of $z - z_0$ and a (finite or) infinite number of positive powers. Thus a pole is always an isolated singularity.

The integral $\int f(z)dz$ taken by a closed circuit about the pole not containing any other singularity is at once seen to be $2\pi i A_1$, where A_1 is the coefficient of $(z - z_0)^{-1}$ in the expansion of $f(z)$ at the pole; this coefficient has therefore a certain uniqueness, and it is called the *residue of f(z) at the pole*. Considering a region in which there are no other singularities than poles, all these being interior points, *the integral* $(1/2\pi i) \int f(z)dz$ *round the boundary of this region is equal to the sum of the residues at the included poles*, a very important result. Any singular point of a function which is not a pole is called an *essential singularity*; if it be isolated the function is capable, in the neighbourhood of this point, of approaching arbitrarily near to any assigned value. For, the point being isolated, the function can be represented, in its neighbourhood, as we have proved, by a series $\Sigma_{-\infty}^{\infty} a_n(z - z_0)^n$; it thus cannot remain finite in the immediate neighbourhood of the point. The point is necessarily an isolated essential singularity also of the function $\{f(z) - A\}^{-1}$ for if this were expressible by a power series about the point, so would also the function $f(z)$ be; as $\{f(z) - A\}^{-1}$ approaches infinity, so does $f(z)$ approach the arbitrary value A. Similar remarks apply to the point $z = \infty$, the function being regarded as a function of $\zeta = z^{-1}$. In the neighbourhood of an essential singularity, which is a limiting point also of poles, the function clearly becomes infinite. For an essential singularity which is not isolated the same result does not necessarily hold.

A single valued function is said to be an *integral* function when it has no singular points except $z = \infty$. Such is, for instance, an integral polynomial, which has $z = \infty$ for a pole, and the functions exp (z) which has $z = \infty$ as an essential singularity. A function which has no singular points for finite values of z other than poles is called a *meromorphic* function. If it also have a pole at $z = \infty$ it is a *rational* function; for then, if a_1, ... a_s be its finite poles, of orders m_1; m_2, ... m_s, the product $(z - a_1)^{m_1} ... (z - a_s)^{m_s} f(z)$ is an integral function with a pole at infinity, capable therefore, for large values of z, of an expression $(z^{-1})^{-m} \Sigma_{r=0} a_r(z^{-1})^r$; thus $(z - a_1)^{m1} ... (z - a_s)^{m_s} f(z)$ is capable of a form $\Sigma_{r=0} b_r z^r$, but $z^{-m} \Sigma_{r=0} b_r z^r$ remains finite for $z = \infty$. Therefore $b_{r+1} = b_{r+2} = ... = 0$, andf(z) is a rational function.

If for a single valued function F(z) every singular point in the finite part of the plane is isolated there can only be a finite number of these in any finite part of the plane, and they can be taken to be a_1, a_2, a_3, ... with $|a_1| \lesseqgtr |a_2| \lesseqgtr |a_3|$ ... and limit $|a_n| = \infty$. About a_s the function is expressible as $\Sigma_{-\infty}^{\infty} A_n(z - a_s)^n$; let $f_s(z) = \Sigma_{-\infty}^{1} A^n(z - a_s)^n$ be the sum of the negative powers in this expansion. Assuming $z = 0$ not to be a singular point, let $f_s(z)$ be expanded in powers of z, in the form $\Sigma_{n=0} C_n z^n$, and μ_s be chosen so that $F_s(z) = f_s(z) - \Sigma_1^{\mu_s - 1} C_n z^n = \Sigma_{\mu_s}^{\infty} C_n z^n$ is, for $|z| < r_s < |a_s|$, less in absolute value than the general term ε_s of a fore-agreed convergent series of real positive terms. Then the series $\varphi(z) = \Sigma_{s=1}^{\infty} F_s(z)$ converges uniformly in any finite region of the plane, other than at the points a_s, and is expressible about any point by a power series, and near a_s, $\varphi(z) - f_s(z)$ is expressible by a power series in $z - a_s$. Thus $F(z) - \varphi(z)$ is an integral function. In particular when all the finite singularities of F(z) are poles, F(z) is hereby expressed as the sum of an integral function and a series of rational functions. The condition $|F_s(z)| < \varepsilon_s$ is imposed only to render the series $\Sigma F_s(z)$ uniformly convergent; this condition may in particular cases be satisfied by a series $\Sigma G_s(z)$ where $G_s(z) = f_s(z) - \Sigma_1^{\nu_s - 1} C_n z^n$ and $\nu_s < \mu_s$. An example of the theorem is the function $\pi \cot \pi z - z^{-1}$ for which, taking at first only half the poles, $f_s(z) = 1/(z - s)$; in this case the series $\Sigma F_s(z)$ where $F_s(z) = (z - s)^{-1} + s^{-1}$ is uniformly convergent; thus $\pi \cot \pi z - z^{-1} - \Sigma_{-\infty}^{\infty} [(z - s)^{-1} + s^{-1}]$, where $s = 0$ is excluded from the summation, is an integral function. It can be proved that this integral function vanishes.

Considering an integral function $f(z)$, if there be no finite positions of z for which this function vanishes, the function $\lambda[f(z)]$ is at once seen to be an integral function, $\varphi(z)$, or $f(z) = \exp[\varphi(z)]$; if however great R may be there be only a finite number of values of z for which $f(z)$ vanishes, say $z = a_1$, ... a_m, then it is at once seen that $f(z) = \exp [\varphi(z)]. (z - a_1)^{h1} ... (z - a_m)^{h_m}$, where $\varphi(z)$ is an integral function, and h_1, ... h_m are positive integers. If, however, $f(z)$ vanish for $z = a_1$, a_2 ... where $|a_1| \lesseqgtr |a2| \lesseqgtr$ ... and limit $|a_n| = \infty$, and if for simplicity we assume that $z - 0$ is not a zero and all the zeros a_1, a_2, ... are of the first order, we find, by applying the preceding theorem to the function $[1 / f(z)] [df(z) / dz]$, that $f(z) = \exp [\varphi(z)] \Pi_{n=1}^{\infty} \{(1 - z/a_n) \exp \varphi_n(z)\}$, where $\varphi(z)$ is an integral function, and $\varphi_n(z)$ is an integral polynomial of the form $\varphi_n(z) = z/a_n + z^2/2a_n{}^2 + ... + z^s/sa_n{}^s$. The number s may be the same for all values of n, or it may increase indefinitely with n; it is sufficient in any case to take $s = n$. In particular for the function $\sin \pi x / \pi x$, we have 315

$$\frac{\sin \pi x}{\pi x} = \Pi_{-\infty}^{\infty} \left\{ \left(1 - \frac{x}{n}\right) \exp\left(\frac{x}{n}\right) \right\},$$

where $n = 0$ is excluded from the product. Or again we have

$$\frac{1}{\Gamma(x)} = x e^{Cx} \Pi_{n=1}^{\infty} \left\{ \left(1 + \frac{x}{n}\right) \exp\left(-\frac{x}{n}\right) \right\},$$

where C is a constant, and $\Gamma(x)$ is a function expressible when x is real and positive by the integral $\int_0^{\infty} e^{-t} t^{x-1} dt$.

There exist interesting investigations as to the connexion of the value of s above, the law of increase of the modulus of the integral function $f(z)$, and the law of increase of the coefficients in the series $f(z) = \Sigma a_n z^n$ as n increases (see the bibliography below under *Integral Functions*). It can be shown, moreover, that an integral function actually assumes every finite complex value, save, in exceptional cases, one value at most. For instance, the function exp (z) assumes every finite value except zero (see below under § 21, *Modular Functions*).

The two theorems given above, the one, known as Mittag-Leffler's theorem, relating to the expression as a sum of simpler functions of a function whose singular points have the point $z = \infty$ as their only limiting point, the other, Weierstrass's factor theorem, giving the expression of an integral function as a product of factors each with only one zero in the finite part of the plane, may be respectively generalized as follows:—

I. If a_1, a_2, a_3, ... be an infinite series of isolated points having the points of the aggregate (c) as their limiting points, so that in any neighbourhood of a point of (c) there exists an infinite number of the points a_1, a_2, ..., and with every point a_i there be associated a polynomial in $(z - a_i)^{-1}$, say g_i; then there exists a single valued function whose region of existence excludes only the points (a) and the points (c), having in a point a_i a pole whereat the expansion consists of the terms g_i, together with a power series in $z - a_i$; the function is expressible as an infinite series of terms $g_i - \gamma_i$, where γ_i is also a rational function.

II. With a similar aggregate (a), with limiting points (c), suppose with every point a_i there is associated a positive integer r_i. Then there exists a single valued function whose region of existence excludes only the points (c), vanishing to order r_i at the point a_i, but not elsewhere, expressible in the form

$$\prod_{n=1}^{\infty} \left(1 - \frac{a_n - c_n}{z - c_n}\right)^{r_n} \exp(g_n),$$

where with every point a_n is associated a proper point c_n of (c), and

$$g_n = r_n \sum_{s=1}^{\mu_n} \frac{1}{s} \left(\frac{a_n - c_n}{z - c_n}\right)^s,$$

μ_n being a properly chosen positive integer.

If it should happen that the points (c) determine a path dividing the plane into separated regions, as, for instance, if $a_n = R(1 - n^{-1}) \exp(i\pi \sqrt{2} \cdot n)$, when (c) consists of the points of the circle $|z| = R$, the product expression above denotes different monogenic functions in the different regions, not continuable into one another.

§ 9. *Construction of a Monogenic Function with a given Region of Existence.*—A series of isolated points interior to a given region can be constructed in infinitely many ways whose limiting points are the boundary points of the region, or are boundary points of the region of such denseness that one of them is found in the neighbourhood of every point of the boundary, however small. Then the application of the last enunciated theorem gives rise to a function having no singularities in the interior of the region, but having a singularity in a boundary point in every small neighbourhood of every boundary point; this function has the given region as region of existence.

§ 10. *Expression of a Monogenic Function by means of Rational Functions in a given Region.*—Suppose that we have a region R_0 of the plane, as previously explained, for all the interior or boundary points of which z is finite, and let its boundary points, consisting of one or more closed polygonal paths, no two of which have a point in common, be called C_0. Further suppose that all the points of this region, including the boundary points, are interior points of another region R, whose boundary is denoted by C. Let z be restricted to be within or upon the boundary of C_0; let a, b, ... be finite points upon C or outside R. Then when b is near enough to a, the fraction $(a - b)/(z - b)$ is arbitrarily small for all positions of z; say

$$\left| \frac{a - b}{z - b} \right| < \varepsilon, \text{ for } |a - b| < \eta;$$

the rational function of the complex variable t,

$$\frac{1}{t - a} \left[1 - \left(\frac{a - b}{t - a} \right)^n \right],$$

in which n is a positive integer, is not infinite at $t = a$, but has a pole at $t = b$. By taking n large enough, the value of this function, for all positions z of t belonging to R_0, differs as little as may be desired from $(t - a)^{-1}$. By taking a sum of terms such as

$$F = \sum A_p \left\{ \frac{1}{t - a} \left[1 - \left(\frac{a - b}{t - b} \right)^n \right] \right\}^p,$$

we can thus build a rational function differing, in value, in R_0, as little as may be desired from a given rational function

$$f = \sum A_p (t - a)^{-p},$$

and differing, outside R or upon the boundary of R, from f, in the fact that while f is infinite at $t = a$, F is infinite only at $t = b$. By a succession of steps of this kind we thus have the theorem that, given a rational function of t whose poles are outside R or upon the boundary of R, and an arbitrary point c outside R or upon the boundary of R, which can be reached by a finite continuous path outside R from all the poles of the rational function, we can build another rational function differing in R_0 arbitrarily little from the former, whose poles are all at the point c.

Now any monogenic function $f(t)$ whose region of definition includes C and the interior of R can be represented at all points z in R_0 by

$$f(z) = \frac{1}{2\pi i} \int \frac{f(t)dt}{t - z},$$

where the path of integration is C. This integral is the limit of a sum

$$S = \frac{1}{2\pi i} \sum \frac{f(t_i)\,(t_{i+1} - t_i)}{t_i - z},$$

where the points t_i are upon C; and the proof we have given of the existence of the limit shows that the sum S converges to $f(z)$ uniformly in regard to z, when z is in R_0, so that we can suppose, when the subdivision of C into intervals $t_{i+1} - t_i$, has been carried sufficiently far, that

$$|S - f(z)| < \varepsilon,$$

for all points z of R_0, where ε is arbitrary and agreed upon beforehand. The function S is, however, a rational function of z with poles upon C, that is external to R_0. We can thus find a rational function differing arbitrarily little from S, and therefore arbitrarily little from $f(z)$, for all points z of R_0, with poles at arbitrary positions outside R_0 which can be reached by finite continuous curves lying outside R from the points of C.

In particular, to take the simplest case, if C_0, C be simple closed polygons, and Γ be a path to which C approximates by taking the number of sides of C continually greater, we can find a rational function differing arbitrarily little from $f(z)$ for all points of R_0 whose poles are at one finite point c external to Γ. By a transformation of the form $t - c = r^{-1}$, with the appropriate change in the rational function, we can suppose this point c to be at infinity, in which case the rational function becomes a polynomial. Suppose $\varepsilon_1, \varepsilon_2, \ldots$ to be an indefinitely continued sequence of real positive numbers, converging to zero, and P_r to be the polynomial such that, within C_0, $|P_r - f(z)| < \varepsilon_r$; then the infinite series of polynomials

$$P_1(z) + \{P_2(z) - P_1(z)\} + \{P_3(z) - P_2(z)\} + \ldots,$$

whose sum to n terms is $P_n(z)$, converges for all finite values of z and represents $f(z)$ within C_0.

When C consists of a series of disconnected polygons, some of which may include others, and, by increasing indefinitely the number of sides of the polygons C, the points C become the boundary points Γ of a region, we can suppose the poles of the rational function, constructed to approximate to $f(z)$ within R_0, to be at points of Γ. A series of rational functions of the form

$$H_1(z) + \{H_2(z) - H_1(z)\} + \{H_3(z) - H_2(z)\} + \ldots$$

then, as before, represents $f(z)$ within R_0. And R_0 may be taken to coincide as nearly as desired with the interior of the region bounded by Γ.

§ 11. *Expression of* $(1 - z)^{-1}$ *by means of Polynomials. Applications.*—We pursue the ideas just cursorily explained in some further detail.

Let c be an arbitrary real positive quantity; putting the complex variable $\zeta = \xi + i\eta$, enclose the points $\zeta = l$, $\zeta = 1 + c$ by means of (i.) the straight lines $\eta = \pm a$, from $\xi = l$ to $\xi = 1 + c$, (ii.) a semicircle convex to $\zeta = 0$ of equation $(\xi - 1)^2 + \eta^2 = a^2$, (iii.) a semicircle concave to $\zeta = 0$ of equation $(\xi - 1 - c)^2 + \eta^2 = a^2$. The quantities c and a are to remain fixed. Take a positive integer r so that $1/r\ (c/a)$ is less than unity, and put $\sigma = 1/r\ (c/a)$. Now take

$$c_1 = 1 + c/r,\ c_2 = 1 + 2c/r,\ \ldots\ c_r = 1 + c;$$

if $n_1, n_2, \ldots n_r$, be positive integers, the rational function 316

$$\frac{1}{1-\zeta}\left\{1 - \left(\frac{c_1 - 1}{c_1 - \zeta}\right)^{n_1}\right\}$$

is finite at $\zeta = 1$, and has a pole of order n_1 at $\zeta = c_1$; the rational function

$$\frac{1}{1-\zeta}\left\{1 - \left(\frac{c_1 - 1}{c_1 - \zeta}\right)^{n_1}\right\}\left\{1 - \left(\frac{c_2 - c_1}{c_2 - \zeta}\right)^{n_2}\right\}^{n_1}$$

is thus finite except for $\zeta = c_2$, where it has a pole of order $n_1 n_2$; finally, writing

$$x_s = \left(\frac{c_s - c_{s-1}}{c_s - \zeta}\right)^{n_s},$$

the rational function

$$U = (1 - \zeta)^{-1}\,(1 - x_1)\,(1 - x_2)^{n_1}\,(1 - x_3)^{n_1 n_2} \ldots (1 - x_r)^{n_1 n_2 \cdots n_{r-1}}$$

has a pole only at $\zeta = 1 + c$, of order $n_1 n_2 \ldots n_r$.

The difference $(1 - \zeta)^{-1} - U$ is of the form $(1 - \zeta)^{-1}P$, where P, of the form

$$1 - (1 - \rho_1)\,(1 - \rho_2)\ldots(1 - \rho_k),$$

in which there are equalities among $\rho_1, \rho_2, \ldots \rho_k$, is of the form

$$\Sigma\rho_1 - \Sigma\rho_1\rho_2 + \Sigma\rho_1\rho_2\rho_3 - \ldots;$$

therefore, if $|r_i| = |\rho_i|$, we have

$$|P| < \Sigma\, r_1 + \Sigma\, r_1 r_2 + \Sigma\, r_1 r_2 r_3 + \ldots < (1 + r_1)\,(1 + r_2)\ldots(1 + r_k) - 1;$$

now, so long as ζ is without the closed curve above described round $\zeta = 1$, $\zeta = 1 + c$, we have

$$\left|\frac{1}{1-\zeta}\right| < \frac{1}{a},\ \left|\frac{c_m - c_{m-1}}{c_m - \zeta}\right| < \frac{c/r}{a} < \sigma,$$

and hence

$$|(1 - \zeta)^{-1} - U| < a^{-1}\,\{(1 + \sigma^{n_1})\,(1 + \sigma^{n_2})^{n_1}\,(1 + \sigma^{n_3})^{n_1 n_2} \ldots (1 + \sigma^{n_r})^{n_1 n_2 \cdots n_{r-1}} - 1\}.$$

Take an arbitrary real positive ε, and μ, a positive number, so that $\varepsilon^{mu} - 1 < \varepsilon a$, then a value of n_1 such that $\sigma^{n_1} < \mu/(1 + \mu)$ and therefore $\sigma^{n_1}/(1 - \sigma^{n_1} < \mu$, and values for $n_2, n_3 \ldots$ such that $\sigma^{n_2} < 1/n_1\ \sigma^{2n_1}$, $\sigma^{n_3} < 1/n_1 n_2\ \sigma^{3n_1}$, … $\sigma^{n}{}_r < 1/(n_1 \ldots n_{r-1})\ \sigma^{n_r\, n_1}$; then, as $1 + x < e^x$, we have

$$|(-\zeta)^{-1} - U| < a^{-1}\,\{\exp\,(\sigma^{n_1} + n_1\sigma^{n_2} + n_1 n_2\sigma^{n_3} + \ldots + n_1 n_2 \ldots n_{r-1}\sigma^{n_r}) - 1\},$$

and therefore less than

$$a^{-1}\{\exp(\sigma^{n_1} + \sigma^{2n_1} + ... + \sigma^{n_r n_1}) - 1\},$$

which is less than

$$\frac{1}{a}\left[\exp\left(\frac{\sigma^{n_1}}{1-\sigma^{n_1}}\right) - 1\right]$$

and therefore less than ε.

The rational function U, with a pole at $\zeta = 1 + c$, differs therefore from $(1 - \zeta)^{-1}$, for all points outside the closed region put about $\zeta = 1$, $\zeta = l + c$, by a quantity numerically less than ε. So long as a remains the same, r and σ will remain the same, and a less value of ε will require at most an increase of the numbers n_1, n_2, ... n_r; but if a be taken smaller it may be necessary to increase r, and with this the complexity of the function U.

Now put

$$z = \frac{c\zeta}{c + 1 - \zeta}, \qquad \zeta = \frac{(c+1)z}{c+z};$$

thereby the points $\zeta = 0, 1, 1 + c$ become the points $z = 0, 1, \infty$, the function $(1 - z)^{-1}$ being given by $(1 - z)^{-1} = c(c + 1)^{-1}(1 - \zeta)^{-1} + (c + 1)^{-1}$; the function U becomes a rational function of z with a pole only at $z = \infty$, that is, it becomes a polynomial in z, say $[(c + 1)/c]\, H - 1/c$, where H is also a polynomial in z, and

$$\frac{1}{1-z} - H = \frac{c}{c+1}\left[\frac{1}{1-\zeta} - U\right];$$

the lines $\eta = \pm a$ become the two circles expressed, if $z = x + iy$, by

$$(x + c)^2 + y^2 = \pm\frac{c(c+1)}{a}\,y,$$

the points $(\eta = 0, \xi = 1 - a)$, $(\eta = 0, \xi = 1 + c + a)$ become respectively the points $(y = 0, x = c(1 - a)/(c + a)$, $(y = 0, x = -c(l + c + a)/a)$, whose limiting positions for $a = 0$ are respectively $(y = 0, x = 1)$, $(y = 0, x = -\infty)$. The circle $(x + c)^2 + y^2 = c(c + 1)y/a$ can be written

$$y = \frac{(x+c)^2}{2\mu} + \frac{(x+c)^4}{2\mu}\{\mu + \sqrt{[\mu^2 - (x+c)^2]}\}^{-2},$$

where $\mu = \frac{1}{2}c(c + 1)/a$; its ordinate y, for a given value of x, can therefore be supposed arbitrarily small by taking a sufficiently small.

We have thus proved the following result; taking in the plane of z any finite region of which every interior and boundary point is at a finite distance, however short, from the points of the real axis for which $1 \lesssim x \lesssim \infty$, we can take a quantity a, and hence, with an arbitrary c, determine a number r; then corresponding to an arbitrary ε_s, we can determine a polynomial P_s, such that, for all points interior to the region, we have

$$|(1 - z^{-1}) - P_s| < \varepsilon_s;$$

thus the series of polynomials

$$P_1 + (P_2 - P_1) + (P_3 - P_2) + ...,$$

constructed with an arbitrary aggregate of real positive numbers ε_1, ε_2, ε_3, ... with zero as their limit, converges uniformly and represents $(1 - z)^{-1}$ for the whole region considered.

§ 12. *Expansion of a Monogenic Function in Polynomials, over a Star Region.*—Now consider any monogenic function ƒ(z) of which the origin is not a singular point; joining the origin to any singular point by a straight line, let the part of this straight line, produced beyond the singular point, lying between the singular point and $z = \infty$, be regarded as a barrier in the plane, the portion of this straight line from the origin to the singular point being erased. Consider next any finite region of the plane, whose boundary points constitute a path of integration, in a sense previously explained, of which every point is at a finite distance greater than zero from each of the barriers before explained; we suppose this region to be such that any line joining the origin to a boundary point, when produced, does not meet the boundary again. For every point x in this region R we can then write

$$2\pi i f(x) = \int \frac{f(t)}{t}\,\frac{f(t)}{1 - xt^{-1}},$$

where ƒ(x) represents a monogenic branch of the function, in case it be not everywhere single valued, and t is on the boundary of the region. Describe now another region R_0 lying entirely within R, and let x be restricted to be within R_0 or upon its boundary; then for any point t on the boundary of R, the points z of the plane for which zt^{-1} is real and positive and equal to or greater than 1, being points for which $|z| = |t|$ or $|z| > |t|$, are without the region R_0, and not infinitely near to its boundary points. Taking then an arbitrary real positive ε we can determine a polynomial in xt^{-1}, say $P(xt^{-1})$, such that for all points x in R_0 we have

$$|(1 - xt^{-1})^{-1} - P(xt^{-1})| < \varepsilon;$$

the form of this polynomial may be taken the same for all points t on the boundary of R, and hence, if E be a proper variable quantity of modulus not greater than ε,

$$\left| 2\pi i f(x) - \int \frac{dt}{t} f(t)P(xt^{-1}) \right| = \left| \int \frac{dt}{t} f(t)E \right| \lesssim \varepsilon LM,$$

where L is the length of the path of integration, the boundary of R, and M is a real positive quantity such that upon this boundary $|t^{-1} f(t)| < M$. If now

$$P(xt^{-1}) = c_0 + c_1 xt^{-1} + ... + c_m x^m t^{-m},$$

and

$$\frac{1}{2\pi i} \int t^{-r-1} f(t)dt = \mu_r,$$

this gives

$$|f(x) - \{c_0\mu_0 + c_1\mu_1 x + \dots + c_m\mu_m x^m\}| \leqq \varepsilon LM/2\pi,$$

where the quantities μ_0, μ_1, μ_2, ... are the coefficients in the expansion of $f(x)$ about the origin.

If then an arbitrary finite region be constructed of the kind explained, excluding the barriers joining the singular points of $f(x)$ to $x = \infty$, it is possible, corresponding to an arbitrary real positive number σ, to determine a number m, and a polynomial $Q(x)$, of order m, such that for all interior points of this region

$$|f(x) - Q(x)| < \sigma.$$

Hence as before, within this region $f(x)$ can be represented by a series of polynomials, converging uniformly; when $f(x)$ is not a single valued function the series represents one branch of the function.

The same result can be obtained without the use of Cauchy’s integral. We explain briefly the character of the proof. If a monogenic function of t, $\varphi(t)$ be capable of expression as a power series in $t - x$ about a point x, for $|t - x| \leqq \rho$, and for all points of this circle $|\varphi(t)| < g$, we know that $|\varphi^{(n)}(x)| < g\rho^{-n}(n!)$. Hence, taking $|z| < \frac{1}{3}\rho$, and, for any assigned positive integer μ, taking m so that for $n > m$ we have $(\mu + n)^\mu < (\frac{3}{2})^n$, we have

$$\left| \frac{\varphi^{(\mu + n)}(x) \cdot z^n}{n!} \right| < \frac{\varphi^{(\mu + n)}(x)}{(\mu + n)!} (\mu + n)^\mu |z|^n < \frac{g}{\rho^{\mu + n}} \left(\frac{3}{2}\right)^n \left(\frac{\rho}{3}\right)^n < \frac{g}{\rho^\mu 2^n},$$

and therefore

$$\varphi^\mu (x + z) = \sum_{n=0}^{m} \frac{\varphi^{(\mu + n)}(x)}{n!} z^n + \varepsilon_\mu,$$

where

$$\left|\varepsilon_\mu\right| < \frac{g}{\rho^\mu} \sum_{n=m+1}^{\infty} \frac{1}{2^n} < \frac{g}{\rho^\mu 2^m}.$$

Now draw barriers as before, directed from the origin, joining the singular point of $\varphi(z)$ to $z = \infty$, take a finite region excluding all these barriers, let ρ be a quantity less than the radii of convergence of all the power series developments of $\varphi(z)$ about interior points of this region, so chosen moreover that no circle of radius ρ with centre at an interior point of the region includes any singular point of $\varphi(z)$, let g be such that $|\varphi(z)| < g$ for all circles of radius ρ whose centres are interior points of the region, and, x being any interior point of the region, choose the positive integer n so that $1/n\ |x| < \frac{1}{3}\rho$; then take the points $a_1 = x/n$, $a_2 = 2x/n$, $a_3 = 3x/n$, ... $a_n = x$; it is supposed that the region is so taken that, whatever x may be, all these are interior points of the region. Then by what has been said, replacing x, z respectively by 0 and x/n, we have

$$\varphi^{(\mu)} (a_1) = \sum_{\lambda 1=0}^{m1} \frac{\varphi^{(\mu + \lambda 1)}(0)}{\lambda_1!} \left(\frac{x}{n}\right)^{\lambda 1} + \alpha_\mu$$ 317

with

$$\alpha_\mu < g/\rho^\mu 2^{m1},$$

provided $(\mu + m_1 + 1)_\mu < (\frac{2}{3})^{m1 + 1}$; in fact for $\mu \leqq 2n^{2n-2}$ it is sufficient to take $m_1 = n^{2n}$; by another application of the same inequality, replacing x, z respectively by a_1 and x/n, we have

$$\varphi^{(\mu)} (a_2) = \sum_{\lambda_2=0}^{m_2} \frac{\varphi^{(\mu + \lambda_2)}(a_1)}{\lambda_2!} \left(\frac{x}{n}\right)^{\lambda_2} + \beta'_\mu,$$

where

$$|\beta'_\mu| < g / \rho^\mu 2^{m_2}$$

provided $(\mu + m_2 + 1)^\mu < (\frac{3}{2})^{m_2} + 1$; we take $m_2 = n^{2n - 2}$, supposing $\mu < 2n^{2n-4}$. So long as $\lambda_2 \leqq m_2 \leqq n^{2n-2}$ and $\mu < 2n^{2n-4}$ we have $\mu + \lambda_2 < 2n^{2n-2}$, and we can use the previous inequality to substitute here for $\varphi^{(\mu + \lambda_2)}(a_1)$. When this is done we find

$$\varphi^{(\mu)} (a_2) = \sum_{\lambda_2=0}^{m_2} \sum_{\lambda_1=0}^{m_1} \frac{\varphi^{(\mu + \lambda_1 + \lambda_2)}(0)}{\lambda_1!\ \lambda_2!} \left(\frac{x}{n}\right)^{\lambda_1 + \lambda_2} + \beta_\mu,$$

where $|\beta\mu| < 2g/\rho^\mu 2^{m_2}$, the numbers m_1, m_2 being respectively n^{2n} and n^{2n-2}.

Applying then the original inequality to $\varphi^{(\mu)}(a_3) = \varphi^{(\mu)}(a_2 + x/n)$, and then using the series just obtained, we find a series for $\varphi^{(\mu)}(a_3)$. This process being continued, we finally obtain

$$\varphi(x) = \sum_{\lambda_1=0}^{m_1} \sum_{\lambda_2=0}^{m_2} \dots \sum_{\lambda_n=0}^{m_n} \frac{\varphi^h(0)}{K} \left(\frac{x}{n}\right)^h + \varepsilon,$$

where $h = \lambda_1 + \lambda_2 + \dots + \lambda_n$, $K = \lambda_1!\ \lambda_2! \dots \lambda_n!$, $m_1 = n^{2n}$, $m_2 = n^{2n-2}$, ..., $m_n = n^2$, $|\varepsilon| < 2g/2^{m_n}$.

By this formula $\varphi(x)$ is represented, with any required degree of accuracy, by a polynomial, within the region in question; and thence can be expressed as before by a series of polynomials converging uniformly (and absolutely) within this region.

§ 13. *Application of Cauchy’s Theorem to the Determination of Definite Integrals.*—Some reference must be made to a method whereby real definite integrals may frequently be evaluated by use of the theorem of the vanishing of the integral of a function of a complex variable round a contour within which the function is single valued and non singular.

We are to evaluate an integral $\int_a^b f(x)dx$; we form a closed contour of which the portion of the real axis from $x = a$ to $x = b$ forms a part, and consider the integral $\int f(z)dz$ round this contour, supposing that the value of this integral can be determined along the curve forming the completion of the contour. The contour being supposed such that, within it, $f(z)$ is a single valued and finite function of the complex variable z save at a finite number of isolated interior points, the contour integral is equal to the sum of the values of $\int f(z)dz$ taken round these points. Two instances will suffice to explain the method. (1) The integral $\int_0^\infty [(\tan x)/x]\,dx$ is convergent if it be understood to mean the limit when ε, ζ, σ, ... all vanish of the sum of the integrals

$$\int_0^{1/2\pi-\varepsilon} \frac{\tan x}{x}\,dx,\quad \int_{1/2\pi+\varepsilon}^{3/2\pi-\zeta} \frac{\tan x}{x}\,dx,\quad \int_{3/2\pi+\zeta}^{5/2\pi-\sigma} \frac{\tan x}{x}\,dx,\ \ldots$$

Now draw a contour consisting in part of the whole of the positive and negative real axis from $x = -n\pi$ to $x = +n\pi$, where n is a positive integer, broken by semicircles of small radius whose centres are the points $x = \pm\frac{1}{2}\pi$, $x = \pm\frac{3}{4}\pi$, ... , the contour containing also the lines $x = n\pi$ and $x = -n\pi$ for values of y between 0 and $n\pi \tan\alpha$, where α is a small fixed angle, the contour being completed by the portion of a semicircle of radius $n\pi \sec\alpha$ which lies in the upper half of the plane and is terminated at the points $x = \pm n\pi$, $y = n\pi \tan\alpha$. Round this contour the integral $\int [(\tan z / z)]\,dz$ has the value zero. The contributions to this contour integral arising from the semicircles of centres $-\frac{1}{2}(2s - 1)\pi$, $+\frac{1}{2}(2s - 1)\pi$, supposed of the same radius, are at once seen to have a sum which ultimately vanishes when the radius of the semicircles diminishes to zero. The part of the contour lying on the real axis gives what is meant by $2\int_0^{n\pi} [(\tan x / x)]\,dx$. The contribution to the contour integral from the two straight portions at $x = \pm n\pi$ is

$$\int_0^{n\pi\tan\alpha} i\,dy \left(\frac{\tan iy}{n\pi + iy} - \frac{\tan iy}{-n\pi + iy} \right)$$

where $i \tan iy$, $= -[\exp(y) - \exp(-y)]/[\exp(y) + \exp(-y)]$, is a real quantity which is numerically less than unity, so that the contribution in question is numerically less than

$$\int_0^{n\pi\tan\alpha} dy\,\frac{2n\pi}{n^2\pi^2 + y^2},\ \text{that is than } 2\alpha.$$

Finally, for the remaining part of the contour, for which, with $R = n\pi \sec\alpha$, we have $z = R(\cos\theta + i\sin\theta) = RE(i\theta)$, we have

$$\frac{dz}{z} = i\,d\theta,\ i\tan z = \frac{\exp(-R\sin\theta)\,E(iR\cos\theta) - \exp(R\sin\theta)\,E(-iR\cos\theta)}{\exp(-R\sin\theta)\,E(iR\cos\theta) + \exp(R\sin\theta)\,E(-iR\cos\theta)};$$

when n and therefore R is very large, the limit of this contribution to the contour integral is thus

$$-\int_\alpha^{\pi-\alpha} d\theta = -(\pi - 2\alpha).$$

Making n very large the result obtained for the whole contour is

$$2\int_0^\infty \frac{\tan x}{x}\,dx - (\pi - 2\alpha) - 2\alpha\varepsilon = 0,$$

where ε is numerically less than unity. Now supposing α to diminish to zero we finally obtain

$$\int_0^\infty \frac{\tan x}{x}\,dx = \frac{\pi}{2}.$$

(2) For another case, to illustrate a different point, we may take the integral

$$\int \frac{z^{a-1}}{1 + z}\,dz,$$

wherein a is real quantity such that $0 < a < 1$, and the contour consists of a small circle, $z = rE(i\theta)$, terminated at the points $x = r\cos\alpha$, $y = \pm r\sin\alpha$, where α is small, of the two lines $y = \pm r\sin\alpha$ for $r\cos\alpha \lesseqgtr x \lesseqgtr R\cos\beta$, where $R\sin\beta = r\sin\alpha$, and finally of a large circle $z = RE(i\varphi)$, terminated at the points $x = R\cos\beta$, $y = \pm R\sin\beta$. We suppose α and β both zero, and that the phase of z is zero for $r\cos a \lesseqgtr x \lesseqgtr R\cos\beta$, $y = r\sin\alpha = R\sin\beta$. Then on $r\cos\alpha \lesseqgtr x \lesseqgtr R\cos\beta$, $y = -r\sin\alpha$, the phase of z will be 2π, and $z^{\alpha-1}$ will be equal to $x^{\alpha-1}\exp[2\pi i(a - 1)]$, where x is real and positive. The two straight portions of the contour will thus together give a contribution

$$[1 - \exp(2\pi i\alpha)] \int_{r\cos\alpha}^{R\cos\beta} \frac{x^{a-1}}{1 + x}\,dx.$$

It can easily be shown that if the limit of $zf(z)$ for $z = 0$ is zero, the integral $\int f(z)dz$ taken round an arc, of given angle, of a small circle enclosing the origin is ultimately zero when the radius of the circle diminishes to zero, and if the limit of $zf(z)$ for $z = \infty$ is zero, the same integral taken round an arc, of given angle, of a large circle whose centre is the origin is ultimately zero when the radius of the circle increases indefinitely; in our case with $f(z) = z^{\alpha-1}/(1 + z)$, we have $zf(z) = z^a/(1 + z)$, which, for $0 < a < 1$, diminishes to zero both for $z = 0$ and for $z = \infty$. Thus, finally the limit of the contour integral when $r = 0$, $R = \infty$ is

$$[1 - \exp(2\pi i\alpha)] \int_0^\infty \frac{x^{\alpha-1}}{1 + x}\,dx.$$

Within the contour $f(z)$ is single valued, and has a pole at $z = 1$; at this point the phase of z is π and z^{a-1} is $\exp[i\pi(a - 1)]$ or $-\exp(i\pi a)$; this is then the residue of $f(z)$ at $z = -1$; we thus have

$$[1 - \exp(2\pi ia)] \int_0^\infty \frac{x^{a-1}}{1 + x}\,dx = -2\pi i\exp(i\pi a),$$

that is

$$\int_0^\infty \frac{x^{a-1}}{1 + x}\,dx = \pi\,\mathrm{cosec}\,(a\pi).$$

§ 14. *Doubly Periodic Functions.*—An excellent illustration of the preceding principles is furnished by the theory of single valued functions having in the finite part of the plane no singularities but poles, which have two periods.

Before passing to this it may be convenient to make here a few remarks as to the periodicity of (single valued) monogenic functions. To say that $f(z)$ is periodic is to say that there exists a constant ω such that for every point z of the interior of the region of existence of $f(z)$ we have $f(z + \omega) = f(z)$. This involves, considering all existing periods $\omega = \rho + i\sigma$, that there exists a lower limit of $\rho^2 + \sigma^2$ other than zero; for otherwise all the differential coefficients of $f(z)$ would be zero, and $f(z)$ a constant; we can then suppose that not both ρ and σ are numerically less than ε, where $\varepsilon > \sigma$. Hence, if g be any real quantity, since the range $(-g, \dots g)$ contains only a finite number of intervals of length ε, and there cannot be two periods $\omega = \rho + i\sigma$ such that $\mu\varepsilon \leqq \rho < (\mu + 1)\varepsilon$, $\nu\varepsilon \leqq \sigma < (\nu + 1)\varepsilon$, where μ, ν are integers, it follows that there is only a finite number of periods for which both ρ and σ are in the interval $(-g \dots g)$. Considering then all the periods of the function which are real multiples of one period ω, and in particular those periods $\lambda\omega$ wherein $0 < \lambda \leqq 1$, there is a lower limit for λ, greater than zero, and therefore, since there is only a finite number of such periods for which the real and imaginary parts both lie between $-g$ and g, a least value of λ, say λ_0. If $\Omega = \lambda_0\omega$ and $\lambda = M\lambda_0 + \lambda'$, where M is an integer and $0 \leqq \lambda' < \lambda_0$, any period $\lambda\omega$ is of the form $M\Omega + \lambda'\omega$; since, however, Ω, $M\Omega$ and $\lambda\omega$ are periods, so also is $\lambda'\omega$, and hence, by the construction of λ_0, we have $\lambda' = 0$; thus all periods which are real multiples of ω are expressible in the form $M\Omega$ where M is an integer, and Ω a period.

If beside ω the functions have a period ω' which is not a real multiple of ω, consider all existing periods of the form $\mu\omega + \nu\omega'$ wherein μ, ν are real, and of these those for which $0 \leqq \mu \leqq 1$, $0 < \nu \leqq 1$; as before there is 318
a least value for ν, actually occurring in one or more periods, say in the period $\Omega' = \mu_0\omega + \nu_0\omega'$; now take, if $\mu\omega + \nu\omega'$ be a period, $\nu = N'\nu_0 + \nu'$, where N' is an integer, and $0 \leqq \nu' < \nu_0$; thence $\mu\omega + \nu\omega' = \mu\omega + N'(\Omega' - \mu_0\omega) + \nu'\omega'$; take then $\mu - N\mu_0 = N\lambda_0 + \lambda'$, where N is an integer and λ_0 is as above, and $0 \leqq \lambda' < \lambda_0$; we thus have a period $N\Omega + N'\Omega' + \lambda'\omega + \nu'\omega'$, and hence a period $\lambda'\omega + \nu'\omega'$, wherein $\lambda' < \lambda_0$, $\nu' < \nu_0$; hence $\nu' = 0$ and $\lambda' = 0$. All periods of the form $\mu\omega + \nu\omega'$ are thus expressible in the form $N\Omega + N'\Omega'$, where Ω, Ω' are periods and N, N' are integers. But in fact any complex quantity, $P + iQ$, and in particular any other possible period of the function, is expressible, with μ, ν real, in the form $\mu\omega + \nu\omega'$; for if $\omega = \rho + i\sigma$, $\omega' = \rho' + i\sigma'$, this requires only $P = \mu\rho + \nu\rho'$, $Q = \mu\sigma + \nu\sigma'$, equations which, since ω'/ω is not real, always give finite values for μ and ν.

It thus appears that if a single valued monogenic function of z be periodic, either all its periods are real multiples of one of them, and then all are of the form $M\Omega$, where Ω is a period and M is an integer, or else, if the function have two periods whose ratio is not real, then all its periods are expressible in the form $N\Omega + N'\Omega'$, where Ω, Ω' are periods, and N, N' are integers. In the former case, putting $\zeta = 2\pi iz/\Omega$, and the function $f(z) = \varphi(\zeta)$, the function $\varphi(\zeta)$ has, like $\exp(\zeta)$, the period $2\pi i$, and if we take $t = \exp(\zeta)$ or $\zeta = \lambda(t)$ the function is a single valued function of t. If then in particular $f(z)$ is an integral function, regarded as a function of t, it has singularities only for $t = 0$ and $t = \infty$, and may be expanded in the form $\sum_{-\infty}^{\infty} a_n t^n$.

Taking the case when the single valued monogenic function has two periods ω, ω' whose ratio is not real, we can form a network of parallelograms covering the plane of z whose angular points are the points $c + m\omega + m'\omega'$, wherein c is some constant and m, m' are all possible positive and negative integers; choosing arbitrarily one of these parallelograms, and calling it the primary parallelogram, all the values of which the function is at all capable occur for points of this primary parallelogram, any point, z', of the plane being, as it is called, *congruent* to a definite point, z, of the primary parallelogram, $z' - z$ being of the form $m\omega + m'\omega'$, where m, m' are integers. Such a function cannot be an integral function, since then, if, in the primary parallelogram $|f(z)| < M$, it would also be the case, on a circle of centre the origin and radius R, that $|f(z)| < M$, and therefore, if $\Sigma a_n z^n$ be the expansion of the function, which is valid for an integral function for all finite values of z, we should have $|a_n| < MR^{-n}$, which can be made arbitrarily small by taking R large enough. The function must then have singularities for finite values of z.

We consider only functions for which these are poles. Of these there cannot be an infinite number in the primary parallelogram, since then those of these poles which are sufficiently near to one of the necessarily existing limiting points of the poles would be arbitrarily near to one another, contrary to the character of a pole. Supposing the constant c used in naming the corners of the parallelograms so chosen that no pole falls on the perimeter of a parallelogram, it is clear that the integral $1/(2\pi i) \int f(z)\, dz$ round the perimeter of the primary parallelogram vanishes; for the elements of the integral corresponding to two such opposite perimeter points as z, $z + \omega$ (or as z, $z + \omega'$) are mutually destructive. This integral is, however, equal to the sum of the residues of $f(z)$ at the poles interior to the parallelogram. Which sum is therefore zero. There cannot therefore be such a function having only one pole of the first order in any parallelogram; we shall see that there can be such a function with two poles only in any parallelogram, each of the first order, with residues whose sum is zero, and that there can be such a function with one pole of the second order, having an expansion near this pole of the form $(z-a)^{-2}$ + (power series in $z - a$).

Considering next the function $\varphi(z) = [f(z)]^{-1}\, df(z)/dz$, it is easily seen that an ordinary point of $f(z)$ is an ordinary point of $\varphi(z)$, that a zero of order m for $f(z)$ in the neighbourhood of which $f(z)$ has a form, $(z - a)^m$ multiplied by a power series, is a pole of $\varphi(z)$ of residue m, and that a pole of $f(z)$ of order n is a pole of $\varphi(z)$ of residue $-n$; manifestly $\varphi(z)$ has the two periods of $f(z)$. We thus infer, since the sum of the residues of $\varphi(z)$ is zero, that for the function $f(z)$, the sum of the orders of its vanishing at points belonging to one parallelogram, Σm, is equal to the sum of the orders of its poles, Σn; which is briefly expressed by saying that the number of its zeros is equal to the number of its poles. Applying this theorem to the function $f(z) - A$, where A is an arbitrary constant, we have the result, that the function $f(z)$ assumes the value A in one of the parallelograms as many times as it becomes infinite. Thus, by what is proved above, every conceivable complex value does arise as a value for the doubly periodic function $f(z)$ in any one of its parallelograms, and in fact at least twice. The number of times it arises is called the *order* of the function; the result suggests a property of rational functions.

Consider further the integral $\int z\, [f'(z)/f(z)]\, dz$, where $f'(z) = df(z)/dz$ taken round the perimeter of the primary parallelogram; the contribution to this arising from two opposite perimeter points such as z and $z + \omega$ is of the form $-\omega \int z\, [f'(z)/f(z)]\, dz$, which, as z increases from z_0 to $z_0 + \omega'$, gives, if λ denote the generalized logarithm, $-\omega\, \{\lambda\, [f(z_0 + \omega')] - \lambda[f(z_0)]\}$, that is, since $f(z_0 + \omega') = f(z_0)$, gives $2\pi iN\omega$, where N is an integer; similarly the result of the integration along the other two opposite sides is of the form $2\pi iN'\omega'$, where N' is an integer. The integral, however, is equal to $2\pi i$ times the sum of the residues of $zf'(z) / f(z)$ at the poles interior to the parallelogram. For a zero, of order m, of $f(z)$ at $z = a$, the contribution to this sum is $2\pi ima$, for a pole of order n at $z = b$ the contribution is $-2\pi inb$; we thus infer that $\Sigma ma - \Sigma nb = N\omega + N'\omega'$; this we express in words by saying that the sum of the values of z where $f(z) = 0$ within any parallelogram is equal to the sum of the values of z where $f(z) = \infty$ save for integral multiples of the periods. By considering similarly the function $f(z) - A$ where A is an arbitrary constant, we prove that each of these sums is equal to the sum of the values of z where the function takes the value A in the parallelogram.

We pass now to the construction of a function having two arbitrary periods ω, ω' of unreal ratio, which has a single pole of the second order in any one of its parallelograms.

For this consider first the network of parallelograms whose corners are the points $\Omega = m\omega + m'\omega'$, where m, m′ take all positive and negative integer values; putting a small circle about each corner of this network, let P be a point outside all these circles; this will be interior to a parallelogram whose corners in order may be denoted by z_0, $z_0 + \omega$, $z_0 + \omega + \omega'$, $z_0 + \omega'$; we shall denote z_0, $z_0 + \omega$ by A_0, B_0; this parallelogram Π_0 is surrounded by eight other parallelograms, forming with Π_0 a larger parallelogram Π_1, of which one side, for instance, contains the points $z_0 - \omega - \omega'$, $z_0 - \omega'$, $z_0 - \omega' + \omega$, $z_0 - \omega' + 2\omega$, which we shall denote by A_1, B_1, C_1, D_1. This parallelogram Π_1 is surrounded by sixteen of the original parallelograms, forming with Π_1 a still larger parallelogram Π_2 of which one side, for instance, contains the points $z_0 - 2\omega - 2\omega'$, $z_0 - \omega - 2\omega'$, $z_0 - 2\omega'$, $z_0 + \omega - 2\omega'$, $z_0 + 2\omega - 2\omega'$, $z_0 + 3\omega - 2\omega'$, which we shall denote by A_2, B_2, C_2, D_2, E_2, F_2. And so on. Now consider the sum of the inverse cubes of the distances of the point P from the corners of all the original parallelograms. The sum will contain the terms

$$S_0 = \frac{1}{PA_0^3} + \left(\frac{1}{PA_1^3} + \frac{1}{PB_1^3} + \frac{1}{PC_1^3} \right) + \left(\frac{1}{PA_2^3} + \frac{1}{PB_2^3} + \dots + \frac{1}{PE_2^3} \right) + \dots$$

and three other sets of terms, each infinite in number, formed in a similar way. If the perpendiculars from P to the sides A_0B_0, $A_1B_1C_1$, $A_2B_2C_2D_2E_2$, and so on, be p, p + q, p + 2q and so on, the sum S_0 is at most equal to

$$\frac{1}{p^3} + \frac{3}{(p+q)^3} + \frac{5}{(p+2q)^3} + \dots + \frac{2n+1}{(p+nq)^3} + \dots$$

of which the general term is ultimately, when n is large, in a ratio of equality with $2q^{-3}n^{-2}$, so that the series S_0 is convergent, as we know the sum Σn^{-2} to be; this assumes that $p \neq 0$; if P be on A_0B_0 the proof for the convergence of $S_0 - 1/PA_0^3$, is the same. Taking the three other sums analogous to S_0 we thus reach the result that the series

$$\varphi(z) = -2\Sigma\,(z - \Omega)^{-3},$$

where Ω is $m\omega + m'\omega'$, and m, m′ are to take all positive and negative integer values, and z is any point outside small circles described with the points Ω as centres, is *absolutely convergent*. Its sum is therefore independent of the order of its terms. By the nature of the proof, which holds for all positions of z outside the small circles spoken of, the series is also clearly *uniformly convergent* outside these circles. Each term of the series being a monogenic function of z, the series may therefore be differentiated and integrated outside these circles, and represents a monogenic function. It is clearly periodic with the periods ω, ω'; for $\varphi(z + \omega)$ is the same sum as $\varphi(z)$ with the terms in a slightly different order. Thus $\varphi(z + \omega) = \varphi(z)$ and $\varphi(z + \omega') = \varphi(z)$.

Consider now the function

$$f(z) = \frac{1}{z^2} + \int_0^z \left\{ \varphi(z) + \frac{2}{z^3} \right\} dz,$$

where, for the subject of integration, the area of uniform convergence clearly includes the point $z = 0$; this gives

$$\frac{df(z)}{dz} = \varphi(z)$$

and

$$f(z) = \frac{1}{z^2} + \Sigma' \left\{ \frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2} \right\},$$

wherein Σ' is a sum excluding the term for which $m = 0$ and $m' = 0$. Hence $f(z + \omega) - f(z)$ and $f(z + \omega') - f(z)$ are both independent of z. Noticing, however, that, by its form, $f(z)$ is an even function of z, and putting $z = -\tfrac{1}{2}\omega$, $z = -\tfrac{1}{2}\omega'$ respectively, we infer that also $f(z)$ has the two periods ω and ω'. In the primary parallelogram Π_0, however, $f(z)$ is only infinite at $z = 0$ in the neighbourhood of which its expansion is of the form z^{-2} + (power series in z). Thus $f(z)$ is such a doubly periodic function as was to be constructed, having in any parallelogram of periods only one pole, of the second order.

It can be shown that any single valued meromorphic function of z with ω and ω' as periods can be expressed rationally in terms of $f(z)$ and $\varphi(z)$, and that $[\varphi(z)]^2$ is of the form $4[f(z)]^3 + Af(z) + B$, where A, B are constants.

To prove the last of these results, we write, for $|z| < |\Omega|$, 319

$$\frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2} = \frac{2z}{\Omega^3} + \frac{3z^2}{\Omega^4} + \dots,$$

and hence, if $\Sigma'\Omega^{-2n} = \sigma_n$, since $\Sigma'\Omega^{-(2n-1)} = 0$, we have, for sufficiently small z greater than zero,

$$f(z) = z^{-2} + 3\sigma_2 \cdot z^2 + 5\sigma_3 \cdot z^4 + \dots$$

and

$$\varphi(z) = -2z^{-3} + 6\sigma_2 \cdot z + 20\sigma_3 \cdot z^3 + \dots;$$

using these series we find that the function

$$F(z) = [\varphi(z)]^2 - 4[f(z)]^3 + 60\sigma_2 f(z) + 140\sigma_3$$

contains no negative powers of z, being equal to a power series in z^2 beginning with a term in z^2. The function F(z) is, however, doubly periodic, with periods ω, ω', and can only be infinite when either $f(z)$ or $\varphi(z)$ is infinite; this follows from its form in $f(z)$ and $\varphi(z)$; thus in one parallelogram of periods it can be infinite only when $z = 0$; we have proved, however, that it is not infinite, but, on the contrary, vanishes, when $z = 0$. Being, therefore, never infinite for finite values of z it is a constant, and therefore necessarily always zero. Putting therefore $f(z) = \zeta$ and $\varphi(z) = d\zeta/dz$ we see that

$$\frac{dz}{d\zeta} = (4\zeta^3 - 60\sigma_2\zeta - 140\sigma_3)^{-1/2}.$$

Historically it was in the discussion of integrals such as

$$\int d\zeta\, (4\zeta^3 - 60\sigma_2\cdot\zeta - 140\sigma_3)^{-1/2},$$

regarded as a branch of Integral Calculus, that the doubly periodic functions arose. As in the familiar case

$$z = \int_0^\zeta (1 - \zeta^2)^{-1/2}\, d\zeta,$$

where $\zeta = \sin z$, it has proved finally to be simpler to regard ζ as a function of z. We shall come to the other point of view below, under § 20, *Elliptic Integrals*.

To prove that any doubly periodic function $F(z)$ with periods ω, ω', having poles at the points $z = a_1, \dots z = a_m$ of a parallelogram, these being, for simplicity of explanation, supposed to be all of the first order, is rationally expressible in terms of $\varphi(z)$ and $f(z)$, and we proceed as follows:—

Consider the expression

$$\Phi(z) = \frac{(\zeta, 1)_m + \eta(\zeta, 1)_{m-2}}{(\zeta - A_1)(\zeta - A_2)\dots(\zeta - A_m)}$$

where $A_s = f(a_s)$, ζ is an abbreviation for $f(z)$ and η for $\varphi(z)$, and $(\zeta, 1)_m$, $(\zeta, 1)_{m-2}$, denote integral polynomials in ζ, of respective orders m and $m - 2$, so that there are 2m unspecified, homogeneously entering, constants in the numerator. It is supposed that no one of the points $a_1, \dots a_m$ is one of the points $m\omega + m'\omega'$ where $f(z) = \infty$. The function $\Phi(z)$ is a monogenic function of z with the periods ω, ω', becoming infinite (and having singularities) only when (1) $\zeta = \infty$ or (2) one of the factors $\zeta - A_s$ is zero. In a period parallelogram including $z = 0$ the first arises only for $z = 0$; since for $\zeta = \infty$, η is in a finite ratio to $\zeta^{3/2}$; the function $\Phi(z)$ for $\zeta = \infty$ is not infinite provided the coefficient of ζ^m in $(\zeta, 1)_m$ is not zero; thus $\Phi(z)$ is regular about $z = 0$. When $\zeta - A_s = 0$, that is $f(z) = f(a_s)$, we have $z = \pm a_s + m\omega + m'\omega'$, and no other values of z, m and m′ being integers; suppose the unspecified coefficients in the numerator so taken that the numerator vanished to the first order in each of the m points $-a_1, -a_2, \dots -a_m$; that is, if $\varphi(a_s) = B_s$, and therefore $\varphi(-a_s) = -B_s$, so that we have the m relations

$$(A_s, 1)_m - B_s(A_s, 1)_{m-2} = 0;$$

then the function $\Phi(z)$ will only have the m poles $a_1, \dots a_m$. Denoting further the m zeros of $F(z)$ by $a_1', \dots a_m'$, putting $f(a_s') = A_s'$, $\varphi(a_s') = B_s'$, suppose the coefficients of the numerator of $\Phi(z)$ to satisfy the further $m - 1$ conditions

$$(A_s', 1)_m + B_s'\,(A_s', 1)_{m-2} = 0$$

for $s = 1, 2, \dots (m - 1)$. The ratios of the 2m coefficients in the numerator of $\Phi(z)$ can always be chosen so that the $m + (m - 1)$ linear conditions are all satisfied. Consider then the ratio

$$F(z) / \Phi(z);$$

it is a doubly periodic function with no singularity other than the one pole a_m'. It is therefore a constant, the numerator of $\Phi(z)$ vanishing spontaneously in a_m'. We have

$$F(z) = A\Phi(z),$$

where A is a constant; by which $F(z)$ is expressed rationally in terms of $f(z)$ and $\varphi(z)$, as was desired.

When $z = 0$ is a pole of $F(z)$, say of order r, the other poles, each of the first order, being $a_1, \dots a_m$, similar reasoning can be applied to a function

$$\frac{(\zeta, 1)_h + \eta(\zeta, 1)_k}{(\zeta - A_1)\dots(\zeta - A_m)},$$

where h, k are such that the greater of $2h - 2m$, $2k + 3 - 2m$ is equal to r; the case where some of the poles $a_1, \dots a_m$ are multiple is to be met by introducing corresponding multiple factors in the denominator and taking a corresponding numerator. We give a solution of the general problem below, of a different form.

One important application of the result is the theorem that the functions $f(z + t)$, $\varphi(z + t)$, which are such doubly periodic function of z as have been discussed, can each be expressed, so far as they depend on z, rationally in terms of $f(z)$ and $\varphi(z)$, and therefore, so far as they depend on z and t, rationally in terms of $f(z)$, $f(t)$, $\varphi(z)$ and $\varphi(t)$. It can in fact be shown, by reasoning analogous to that given above, that

$$f(z + t) + f(z) + f(t) = \tfrac{1}{4}\left[\frac{\varphi(z) - \varphi(t)}{f(z) - f(t)}\right]^2.$$

This shows that if $F(z)$ be any single valued monogenic function which is doubly periodic and of meromorphic character, then $F(z + t)$ is an algebraic function of $F(z)$ and $F(t)$. Conversely any single valued monogenic function of meromorphic character, $F(z)$, which is such that $F(z + t)$ is an algebraic function of $F(z)$ and $F(t)$, can be shown to be a doubly periodic function, or a function obtained from such by degeneration (in virtue of special relations connecting the fundamental constants).

The functions $f(z)$, $\varphi(z)$ above are usually denoted by $\wp(z)$, $\wp'(z)$; further the fundamental differential equation is usually written

$$(\wp'z)^2 = 4(\wp z)^3 - g_2\wp z - g_3,$$

and the roots of the cubic on the right are denoted by e_1, e_2, e_3; for the odd function, $\wp'z$, we have, for the congruent arguments $-\tfrac{1}{2}\omega$ and $\tfrac{1}{2}\omega$, $\wp'(\tfrac{1}{2}\omega) = -\wp'(-\tfrac{1}{2}\omega) = -\wp'(\tfrac{1}{2}\omega)$, and hence $\wp'(\tfrac{1}{2}\omega) = 0$; hence we can take $e_1 = \Re(\tfrac{1}{2}\omega)$, $e_2 = \Re(\tfrac{1}{2}\omega + \tfrac{1}{2}\omega')$, $e_3 = \Re(\tfrac{1}{2}\omega)$. It can then be proved that $[\Re(z) - e_1][\Re(z + \tfrac{1}{2}\omega) - e_1] = (e_1 - e_2)(e_1 - e_3)$, with similar equations for the other half periods. Consider more particularly the function $\Re(z) - e_1$; like $\Re(z)$ it has a pole of the second order at $z = 0$, its expansion in its neighbourhood being of the form $z^{-2}(1 - e_1z^2 + Az^4 + \dots)$; having no other pole, it has therefore either two zeros, or a double zero in a period parallelogram (ω, ω'). In fact near its zero $\tfrac{1}{2}\omega$ its expansion is $(x - \tfrac{1}{2}\omega)\Re'(\tfrac{1}{2}\omega) + \tfrac{1}{2}(z - \tfrac{1}{2}\omega)^2 \Re''(\tfrac{1}{2}\omega) + \dots$; we have seen that $\Re'(\tfrac{1}{2}\omega) = 0$; thus it has a zero of the second order wherever it vanishes. Thus it appears that the square root $[\Re(z) - e_1]^{1/2}$, if we attach a definite sign to it for some particular value of z, is a single valued function of z; for it can at most have two values, and the only small circuits in the plane which could lead to an interchange of these values are those about either a pole or a zero, neither of which, as we have seen, has this effect; the function is therefore single valued for any circuit. Denoting the function, for a moment, by $f_1(z)$, we have $f_1(z + \omega) = \pm f_1(z)$, $f_1(z + \omega') = \pm f_1(z)$; it can be seen by considerations of continuity that the right sign in either of these equations does not vary with z; not both these signs can be positive, since the function has only one pole, of the first order, in a parallelogram (ω, ω'); from the expansion of $f_1(z)$ about $z = 0$, namely $z^{-1}(1 - \tfrac{1}{2}e_1z^2 + \dots)$, it follows that $f_1(z)$ is an odd function, and hence $f_1(-\tfrac{1}{2}\omega') = -f_1(\tfrac{1}{2}\omega')$, which is not zero since $[f_1(\tfrac{1}{2}\omega')]^2 = e_3 - e_1$, so that we have $f_1(z + \omega') = -f_1(z)$; an equation $f_1(z + \omega) = -f_1(z)$ would then give $f_1(z + \omega + \omega') = f_1(z)$, and hence $f_1(\tfrac{1}{2}\omega + \tfrac{1}{2}\omega') = f_1(-\tfrac{1}{2}\omega - \tfrac{1}{2}\omega')$, of which the latter is $-f_1(\tfrac{1}{2}\omega + \tfrac{1}{2}\omega')$; this would give $f_1(\tfrac{1}{2}\omega + \tfrac{1}{2}\omega') = 0$, while $[f_1(\tfrac{1}{2}\omega + \tfrac{1}{2}\omega')]^2 = e_2 - e_1$. We thus infer that $f_1(z + \omega) = f_1(z)$, $f_1(z + \omega') = -f_1(z)$, $f_1(z + \omega + \omega') = -f_1(z)$. The function $f_1(z)$ is thus doubly periodic with the periods ω and $2\omega'$; in a parallelogram of which two sides are ω and $2\omega'$ it has poles at $z = 0$, $z = \omega'$ each of the first order, and zeros of the first order at $z = \tfrac{1}{2}\omega$, $z = \tfrac{1}{2}\omega + \omega'$; it is thus a doubly periodic function of the second order with two different poles of the first order in its parallelogram $(\omega, 2\omega')$. We may similarly consider the functions $f_2(z) = [\Re(z) - e_2]^{1/2}$, $f_3(z) = [\Re(z) - e_3]^{1/2}$; they give

$$f_2(z + \omega + \omega') = f_2(z),\ f_2(z + \omega) = -f_2(z),\ f_2(z + \omega') = -f_2(z),\ \ f_3(z + \omega') = f_3 z,\ f_3(z + \omega) = -f_3(z),\ f_3(z + \omega + \omega') = -f_3(z).$$

Taking $u = z(e_1 - e_3)^{1/2}$, with a definite determination of the constant $(e_1 - e_3)^{1/2}$, it is usual, taking the preliminary signs so that for $z = 0$ each of $zf_1(z)$, $zf_2(z)$, $zf_3(z)$ is equal to $+1$, to put

$$\mathrm{sn}(u) = \frac{(e_1 - e_3)^{1/2}}{f_3(z)},\quad \mathrm{cn}(u) = \frac{f_1(z)}{f_3(z)},\quad \mathrm{dn}(u) = \frac{f_2(z)}{f_3(z)},$$

$$k^2 = (e_2 - e_3)/(e_1 - e_3),\quad K = \tfrac{1}{2}\omega(e_1 - e_3)^{1/2},\quad iK' = \tfrac{1}{2}\omega'(e_1 - e_3)^{1/2};$$

thus $\mathrm{sn}(u)$ is an odd doubly periodic function of the second order with the periods $4K$, $2iK$, having poles of the first order at $u = iK'$, $u = 2K + iK'$, and zeros of the first order at $u = 0$, $u = 2K$; similarly $\mathrm{cn}(u)$, $\mathrm{dn}(u)$ are even doubly periodic functions whose periods can be written down, and $\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1$, $k^2\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1$; if $x = \mathrm{sn}(u)$ we at once find, from the relations given here, that

$$\frac{du}{dx} = [(1 - x^2)(1 - k^2x^2)]^{-1/2};$$

if we put $x = \sin\varphi$ we have

$$\frac{du}{d\varphi} = [1 - k^2\sin^2\varphi]^{-1/2},$$

and if we call φ the amplitude of u, we may write $\varphi = \mathrm{am}(u)$, $x = \sin\cdot\mathrm{am}(u)$, which explains the origin of the notation $\mathrm{sn}(u)$. Similarly $\mathrm{cn}(u)$ is an abbreviation of $\cos\cdot\mathrm{am}(u)$, and $\mathrm{dn}(u)$ of $\Delta\mathrm{am}(u)$, where $\Delta(\varphi)$ meant $(1 - k^2\sin^2\varphi)^{1/2}$. The addition equation for each of the functions $f_1(z)$, $f_2(z)$, $f_3(z)$ is very simple, being

$$f(z + t) = \tfrac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial i}\right)\log\frac{f(z) + f(t)}{f(z) - f(t)} = \frac{f(z)f'(t) - f(t)f'(z)}{f^2(z) - f^2(t)},$$

where $f_1'(z)$ means $df_1(z)/dz$, which is equal to $-f_2(z)\cdot f_3(z)$, and $f^2(z)$ means $[f(z)]^2$. This may be verified directly 320
by showing, if R denote the right side of the equation, that $\partial R/\partial z = \partial R/\partial t$; this will require the use of the differential equation

$$[f_1'^{(z)}]^2 = [f_1^2(z) + e_1 - e_2][f_1^2(z) + e_1 - e_3],$$

and in fact we find

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{dt^2}\right)\log[f(z) + f(t)] = f^2(z) - f^2(t) = \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{dt^2}\right)\log[f(z) - f(t)];$$

hence it will follow that R is a function of $z + t$, and R is at once seen to reduce to $f(z)$ when $t = 0$. From this the addition equation for each of the functions $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, $\mathrm{dn}(u)$ can be deduced at once; if s_1, c_1, d_1, s_2, c_2, d_2 denote respectively $\mathrm{sn}(u_1)$, $\mathrm{cn}(u_1)$, $\mathrm{dn}(u_1)$, $\mathrm{sn}(u_2)$, $\mathrm{cn}(u_2)$, $\mathrm{dn}(u_2)$, they can be put into the forms

$$\mathrm{sn}(u_1 + u_2) = (s_1c_2d_2 + s_2c_1d_1)/D,\ \ \mathrm{cn}(u_1 + u_2) = (c_1c_2 - s_1s_2d_1d_2)/D,\ \ \mathrm{dn}(u_1 + u_2) = (d_1d_2 - k^2s_1s_2c_1c_2)/D,$$

where

$$D = 1 - k^2s_1^2s_2^2.$$

The introduction of the function $f_1(z)$ is equivalent to the introduction of the function $\Re(z; \omega, 2\omega')$ constructed from the periods ω, $2\omega'$ as was $\Re(z)$ from ω and ω'; denoting this function by $\Re_1(z)$ and its differential coefficient by $\Re'_1(z)$, we have in fact

$$f_1(z) = \tfrac{1}{2}\frac{\Re'_1(z)}{\Re_1(\omega') - \Re_1(z)}$$

as we see at once by considering the zeros and poles and the limit of $zf_1(z)$ when $z = 0$. In terms of the function $\Re_1(z)$ the original function $\Re(z)$ is expressed by

$$\Re(z) = \Re_1(z) + \Re_1(z + \omega') - \Re_1(\omega'),$$

as a consideration of the poles and expansion near $z = 0$ will show.

A function having ω, ω' for periods, with poles at two arbitrary points a, b and zeros at a', b', where $a' + b' = a + b$ save for an expression $m\omega + m'\omega'$, in which m, m′ are integers, is a constant multiple of

$$\{\wp [z - ½(a' + b')] - \wp [a' - ½(a' + b')]\} / \{\wp [z - ½(a + b)] - \wp [a - ½(a + b)]\};$$

if the expansion of this function near z = a be

$$\lambda(z - a)^{-1} + \mu + \Sigma_{n=1} \mu_n (z - a)^n,$$

the expansion near z = b is

$$-\lambda (z - b)^{-1} + \mu + \Sigma_{n=1} (-1)^n \mu_n (z - b)^n,$$

as we see by remarking that if $z' - b = -(z - a)$ the function has the same value at z and z′; hence the differential equation satisfied by the function is easily calculated in terms of the coefficients in the expansions.

From the function $\wp(z)$ we can obtain another function, termed the Zeta-function; it is usually denoted by $\zeta(z)$, and defined by

$$\zeta(z) - \frac{1}{z} = \int_0^{\pi} \left[\frac{1}{z^2} - \wp(z) \right] dz = \Sigma' \left(\frac{1}{z - \Omega} + \frac{1}{\Omega} + \frac{z}{\Omega^2} \right),$$

for which as before we have equations

$$\zeta(z + \omega) = \zeta(z) + 2\pi i\eta, \quad \zeta(z + \omega') = \zeta(z) + 2\pi i\eta',$$

where 2η, $2\eta'$ are certain constants, which in this case do not both vanish, since else $\zeta(z)$ would be a doubly periodic function with only one pole of the first order. By considering the integral

$$\int \zeta(z)dz$$

round the perimeter of a parallelogram of sides ω, ω′ containing z = 0 in its interior, we find $\eta\omega' - \eta'\omega = 1$, so that neither of η, η′ is zero. We have $\zeta'(z) = -\wp(z)$. From $\zeta(z)$ by means of the equation

$$\frac{\sigma(z)}{z} = \exp \left\{ \int_0^z \left[\zeta(x) - \frac{1}{z} \right] dz \right\} = \Pi' \left[\left(1 - \frac{z}{\Omega}\right) \exp \left(\frac{z}{\Omega} + \frac{z^2}{2\Omega^2} \right) \right],$$

we determine an integral function $\sigma(z)$, termed the Sigma-function, having a zero of the first order at each of the points $z = \Omega$; it can be seen to satisfy the equations

$$\frac{\sigma(z + \omega)}{\sigma(z)} = -\exp [2\pi i\eta(z + ½\omega)], \quad \frac{\sigma(z + \omega')}{\sigma(z)} = -\exp [2\pi i\eta' (z + ½\omega')].$$

By means of these equations, if $a_1 + a_2 + \ldots + a_m = a'_1 + a'_2 + \ldots + a'_m$, it is readily shown that

$$\frac{\sigma(z - a'_1)\, \sigma(z - a'_2) \ldots \sigma(z - a'_m)}{\sigma(z - a_1)\, \sigma(z - a_2) \ldots \sigma(z - a_m)}$$

is a doubly periodic function having $a_1, \ldots a_m$ as its simple poles, and $a'_1, \ldots a'_m$ as its simple zeros. Thus the function $\sigma(z)$ has the important property of enabling us to write any meromorphic doubly periodic function as a product of factors each having one zero in the parallelogram of periods; these form a generalization of the simple factors, z − a, which have the same utility for rational functions of z. We have $\zeta(z) = \sigma'(z)/\sigma(z)$.

The functions $\zeta(z)$, $\wp(z)$ may be used to write any meromorphic doubly periodic function F(z) as a sum of terms having each only one pole; for if in the expansion of F(z) near a pole z = a the terms with negative powers of z − a be

$$A_1(z - a)^{-1} + A_2(z - a)_{-2} + \ldots + A_{m+1}(z - a)^{-(m+1)},$$

then the difference

$$F(z) - A_1\zeta (z - a) - A_2\wp (z - a) - \ldots + \frac{A_{m+1}}{m!} (-1)^m \wp^{m-1} (z - a)$$

will not be infinite at z = a. Adding to this a sum of further terms of the same form, one for each of the poles in a parallelogram of periods, we obtain, since the sum of the residues A is zero, a doubly periodic function without poles, that is, a constant; this gives the expression of F(z) referred to. The indefinite integral $\int F(z)dz$ can then be expressed in terms of z, functions $\wp(z - a)$ and their differential coefficients, functions $\zeta(z - a)$ and functions $\log\sigma(z - a)$.

§ 15. *Potential Functions. Conformal Representation in General.*—Consider a circle of radius a lying within the region of existence of a single valued monogenic function, u + iv, of the complex variable z, = x + iy, the origin z = 0 being the centre of this circle. If $z = rE(i\varphi) = r(\cos\varphi + i \sin\varphi)$ be an internal point of this circle we have

$$u + iv = \frac{1}{2\pi i} \int \frac{(U + iV)}{t - z} dt,$$

where U + iV is the value of the function at a point of the circumference and $t = aE(i\theta)$; this is the same as

$$u + iv = \frac{1}{2\pi} \int \frac{(U + iV) [1 - (r/a) E (i\theta - i\varphi)]}{1 + (r/a)^2 - 2(r/a) \cos (\theta - \varphi)} d\theta.$$

If in the above formula we replace z by the external point $(a^2/r) E(i\varphi)$ the corresponding contour integral will vanish, so that also

$$0 = \frac{1}{2\pi} \int \frac{(U + iV) [(r/a)^2 - (r/a) E (i\theta - i\varphi)]}{1 + (r/a)^2 - 2(r/a) \cos (\theta - \varphi)} d\theta;$$

hence by subtraction we have

$$u = \frac{1}{2\pi} \int \frac{U(a^2 - r^2)}{a^2 + r^2 - 2ar \cos (\theta - \varphi)} d\theta,$$

and a corresponding formula for v in terms of V. If O be the centre of the circle, Q be the interior point z, P the point $aE(i\theta)$ of the circumference, and ω the angle which QP makes with OQ produced, this integral is at once found to be the same as

$$u = \frac{1}{\pi} \int U d\omega - \frac{1}{2\pi} \int U d\theta$$

of which the second part does not depend upon the position of z, and the equivalence of the integrals holds for every arc of integration.

Conversely, let U be any continuous real function on the circumference, U_0 being the value of it at a point P_0 of the circumference, and describe a small circle with centre at P_0 cutting the given circle in A and B, so that for all points P of the arc AP_0B we have $|U - U_0| < \varepsilon$, where ε is a given small real quantity. Describe a further circle, centre P_0 within the former, cutting the given circle in A′ and B′, and let Q be restricted to lie in the small space bounded by the arc $A'P_0B'$ and this second circle; then for all positions of P upon the greater arc AB of the original circle QP^2 is greater than a definite finite quantity which is not zero, say $QP^2 > D^2$. Consider now the integral

$$u' = \frac{1}{2\pi} \int U \frac{(a^2 - r^2)}{a^2 + r^2 - 2ar\cos(\theta - \varphi)} d\theta = \frac{1}{\pi} \int U d\omega - \frac{1}{2\pi} \int U d\theta,$$

which we evaluate as the sum of two, respectively along the small arc AP_0B and the greater arc AB. It is easy to verify that, for the whole circumference,

$$U_0 = \frac{1}{2\pi} \int U_0 \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \varphi)} d\theta = \frac{1}{\pi} \int U_0\, d\omega - \frac{1}{2\pi} \int U_0\, d\theta.$$

Hence we can write

$$u' - U_0 = \frac{1}{2\pi} \int_{AP_0B} (U - U_0) d\omega - \frac{1}{2\pi} \int_{AP_0B} (U - U_0) d\theta + \frac{1}{2\pi} \int_{AB} (U - U_0) \frac{(a^2 - r^2)}{QP^2} d\theta.$$

If the finite angle between QA and QB be called Φ and the finite angle AOB be called Θ, the sum of the first two components is numerically less than

$$\frac{\varepsilon}{2\pi} (\Phi + \Theta).$$

If the greatest value of $|(U - U_0)|$ on the greater arc AB be called H, the last component is numerically less than

$$\frac{H}{D^2} (a^2 - r^2)$$

of which, when the circle, of centre P_0, passing through A′B′ is sufficiently small, the factor $a^2 - r^2$ is arbitrarily small. Thus it appears that u′ is a function of the position of Q whose limit, when Q, interior to the original circle, approaches indefinitely near to P_0, is U_0. From the form

$$u' = \frac{1}{\pi} \int U d\omega - \frac{1}{2\pi} \int U d\theta,$$

since the inclination of QP to a fixed direction is, when Q varies, P remaining fixed, a solution of the differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0,$$

where z, = x + iy, is the point Q, we infer that u′ is a differentiable function satisfying this equation; indeed, 321
when r < a, we can write

$$\frac{1}{2\pi} \int U \frac{(a^2 - r^2)}{a^2 + r^2 - 2ar\cos(\theta - \varphi)} d\theta = \frac{1}{2\pi} \int U \left[1 + 2\frac{r}{a} \cos(\theta - \varphi) + 2\frac{r^2}{a^2} \cos 2(\theta - \varphi) + \ldots \right] d\theta$$

$$= a_0 + a_1 x + b_1 y + a_2 (x^2 - y^2) + 2b_2 xy + \ldots,$$

where

$$a_0 = \frac{1}{2\pi} \int U d\theta, \quad a_1 = \frac{1}{\pi} \int \frac{U\cos\theta}{a} d\theta, \quad b_1 = \frac{1}{\pi} \int \frac{U\sin\theta}{a} d\theta,$$

$$a_2 = \frac{1}{\pi} \int \frac{U\cos 2\theta}{a^2} d\theta, \quad b_2 = \frac{1}{\pi} \int \frac{U\sin 2\theta}{a^2} d\theta.$$

In this series the terms of order n are sums, with real coefficients, of the various integral polynomials of dimension n which satisfy the equation $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2$; the series is thus the real part of a power series in z, and is capable of differentiation and integration within its region of convergence.

Conversely we may suppose a function, P, defined for the interior of a finite region R of the plane of the real variables x, y, capable of expression about any interior point x_0, y_0 of this region by a power series in $x - x_0$, $y - y_0$, with real coefficients, these various series being obtainable from one of them by continuation. For any region R_0 interior to the region specified, the radii of convergence of these power series will then have a lower limit greater than zero, and hence a finite number of these power series suffice to specify the function for all points interior to R_0. Each of these series, and therefore the function, will be differentiable; suppose that at all points of R_0 the function satisfies the equation

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial P^2}{\partial y^2} = 0,$$

we then call it a monogenic potential function. From this, save for an additive constant, there is defined another potential function by means of the equation

$$Q = \int^{(x, y)} \left(\frac{\partial P}{\partial x} dy - \frac{\partial P}{\partial y} dx \right).$$

The functions P, Q, being given by a finite number of power series, will be single valued in R_0, and $P + iQ$ will be a monogenic function of z within R_0. In drawing this inference it is supposed that the region R_0 is such that every closed path drawn in it is capable of being deformed continuously to a point lying within R_0, that is, is *simply connected*.

Suppose in particular, c being any point interior to R_0, that P approaches continuously, as z approaches to the boundary of R, to the value log r, where r is the distance of c to the points of the perimeter of R. Then the function of z expressed by

$$\zeta = (z - c) \exp(-P - iQ)$$

will be developable by a power series in $(z - z_0)$ about every point z_0 interior to R_0, and will vanish at $z = c$; while on the boundary of R it will be of constant modulus unity. Thus if it be plotted upon a plane of ζ the boundary of R will become a circle of radius unity with centre at $\zeta=0$, this latter point corresponding to $z=c$. A closed path within R_0, passing once round $z=c$, will lead to a closed path passing once about $\zeta = 0$. Thus every point of the interior of R will give rise to one point of the interior of the circle. The converse is also true, but is more difficult to prove; in fact, the differential coefficient $d\zeta/dz$ does not vanish for any point interior to R. This being assumed, we obtain a conformal representation of the interior of the region R upon the interior of a circle, in which the arbitrary interior point c of R corresponds to the centre of the circle, and, by utilizing the arbitrary constant arising in determining the function Q, an arbitrary point of the boundary of R corresponds to an arbitrary point of the circumference of the circle.

There thus arises the problem of the determination of a real monogenic potential function, single valued and finite within a given arbitrary region, with an assigned continuous value at all points of the boundary of the region. When the region is circular this problem is solved by the integral $1/\pi \int U d\omega - 1/\pi \int U d\theta$ previously given. When the region is bounded by the outermost portions of the circumferences of two overlapping circles, it can hence be proved that the problem also has a solution; more generally, consider a finite simply connected region, whose boundary we suppose to consist of a single closed path in the sense previously explained, ABCD; joining A to C by two non-intersecting paths AEC, AFC lying within the region, so that the original region may be supposed to be generated by the overlapping regions AECD, CFAB, of which the common part is AECF; suppose now the problem of determining a single valued finite monogenic potential function for the region AECD with a given continuous boundary value can be solved, and also the same problem for the region CFAB; then it can be shown that the same problem can be solved for the original area. Taking indeed the values assigned for the original perimeter ABCD, assume arbitrarily values for the path AEC, continuous with one another and with the values at A and C; then determine the potential function for the interior of AECD; this will prescribe values for the path CFA which will be continuous at A and C with the values originally proposed for ABC; we can then determine a function for the interior of CFAB with the boundary values so prescribed. This in its turn will give values for the path AEC, so that we can determine a new function for the interior of AECD. With the values which this assumes along CFA we can then again determine a new function for the interior of CFAB. And so on. It can be shown that these functions, so alternately determined, have a limit representing such a potential function as is desired for the interior of the original region ABCD. There cannot be two functions with the given perimeter values, since their difference would be a monogenic potential function with boundary value zero, which can easily be shown to be everywhere zero. At least two other methods have been proposed for the solution of the same problem.

A particular case of the problem is that of the conformal representation of the interior of a closed polygon upon the upper half of the plane of a complex variable t. It can be shown without much difficulty that if a, b, c, ... be real values of t, and α, β, γ, ... be n real numbers, whose sum is $n - 2$, the integral

$$z = \int (t - a)^{\alpha-1} (t - b)^{\beta-1} \dots dt,$$

as t describes the real axis, describes in the plane of z a polygon of n sides with internal angles equal to $\alpha\pi$, $\beta\pi$, ..., and, a proper sign being given to the integral, points of the upper half of the plane of t give rise to interior points of the polygon. Herein the points a, b, ... of the real axis give rise to the corners of the polygon; the condition $\Sigma\alpha = n - 2$ ensures merely that the point $t = \infty$ does not correspond to a corner; if this condition be not regarded, an additional corner and side is introduced in the polygon. Conversely it can be shown that the conformal representation of a polygon upon the half plane can be effected in this way; for a polygon of given position of more than three sides it is necessary for this to determine the positions of all but three of a, b, c, ...; three of them may always be supposed to be at arbitrary positions, such as $t = 0$, $t = 1$, $t = \infty$.

As an illustration consider in the plane of $z = x + iy$, the portion of the imaginary axis from the origin to $z = ih$, where h is positive and less than unity; let C be this point $z = ih$; let BA be of length unity along the positive real axis, B being the origin and A the point $z = 1$; let DE be of length unity along the negative real axis, D being also the origin and E the point $z = -1$; let EFA be a semicircle of radius unity, F being the point $z = i$. If we put $\zeta = [(z^2 + h^2)/(1 + h^2z^2)]^{1/2}$, with $\zeta = 1$ when $z = 1$, the function is single valued within the semicircle, in the plane of z, which is slit along the imaginary axis from the origin to $z = ih$; if we plot the value of ζ upon another plane, as z describes the continuous curve ABCDE, ζ will describe the real axis from $\zeta = 1$ to $\zeta = -1$, the point C giving $\zeta = 0$, and the points B, D giving the points $\zeta = \pm h$. Near $z = 0$ the expansion of ζ is $\zeta - h = z^2 (1 - h^4 / 2h) + \dots$, or $\zeta + h = -z^2 (1 - h^4 / 2h) + \dots$; in either case an increase of $\frac{1}{2}\pi$ in the phase of z gives an increase of π in the phase of $\zeta - h$ or $\zeta + h$. Near $z = ih$ the expansion of ζ is $\zeta = (z - ih)^{1/2} [2ih/(1 - h^4)]^{1/2} + \dots$, and an increase of 2π in the phase of $z - ih$ also leads to an increase of π in the phase of ζ. Then as z describes the semicircle EFA, ζ also describes a semicircle of radius unity, the point $z = i$ becoming $\zeta = i$. There is thus a conformal representation of the interior of the slit semicircle in the z-plane, upon the interior of the whole semicircle in the ζ-plane, the function

$$z = [(\zeta^2 - h^2) / (1 - h^2\zeta^2)]^{1/2}$$

being single valued in the latter semicircle. By means of a transformation $t = (\zeta + 1)^2 / (\zeta - 1)^2$, the semicircle in the plane of ζ can further be conformably represented upon the upper half of the whole plane of t.

As another illustration we may take the conformal representation of an equilateral triangle upon a half plane. Taking the elliptic function $\wp(u)$ for which $\wp'^2(u) = 4\wp^3(u) - 4$, so that, with $\varepsilon = \exp(\frac{2}{3}\pi i)$, we have $e_1 = 1$, $e_2 = \varepsilon^2$, $e_3 = \varepsilon$, the half periods may be taken to be

$$\tfrac{1}{2}\omega = \int_1^\infty dt \quad , \quad \tfrac{1}{2}\omega' = \int_{e_3}^\infty dt$$

$$\frac{}{2(t^3-1)^{1/2}} \qquad \frac{}{2(t^3-1)^{1/2}} = \tfrac{1}{2}\varepsilon\omega;$$

drawing the equilateral triangle whose vertices are O, of argument O, A of argument ω, and B of argument $\omega + \omega' = -\varepsilon^2\omega$, and the equilateral triangle whose angular points are O, B and C, of argument ω′, let E, of argument $\tfrac{1}{3}(2\omega + \omega')$, and D, of argument $\tfrac{1}{3}(\omega + 2\omega')$, be the centroids of these triangles respectively, and let BE, OE, AE cut OA, AB, BO in K, L, H respectively, and BD, OD, CD cut OC, BC, OB in F, G, H respectively; then if $u = \xi + i\eta$ be any point of the interior of the triangle OEH and $v = \varepsilon u_0 = \varepsilon(\xi - i\eta)$ be any point of the interior of the triangle OHD, the points respectively of the ten triangles OEK, EKA, EAL, ELB, EBH, DHB, DBG, DGC, DCF, DFO are at once seen to be given by $-\varepsilon v$, $\omega + \varepsilon u$, $\omega - \eta^2 v$, $\omega + \omega' + \varepsilon^2 u$, $\omega + \omega' - v$, $\omega + \omega' - u$, $\omega + \omega' + \varepsilon v$, $\omega' - \varepsilon u$, $\omega' + \varepsilon^2 v$, $-\varepsilon^2 u$. Further, when u is real, since the term $-2(u + m\omega + m'\varepsilon^2\omega)^{-3}$, which is the conjugate complex of $-2(u + m\omega + m'\varepsilon^2\omega)^3$, arises in the infinite sum which expresses $\mathfrak{R}'(u)$, namely as $-2(u + \mu\omega + \mu'\varepsilon\omega)^{-3}$, where $\mu = m - m'$, $\mu' = -m'$, it follows that $\mathfrak{R}'(u)$ is real; in a similar way we prove that $\mathfrak{R}'(u)$ is pure imaginary when u is pure imaginary, and that $\mathfrak{R}'(u) = \mathfrak{R}'(\varepsilon u) = \mathfrak{R}'(\varepsilon^2 u)$, as also that for $v = \varepsilon u_0$, $\mathfrak{R}'(v)$ is the conjugate complex of $\mathfrak{R}'(u)$. Hence it follows that the variable

$$t = \tfrac{1}{2}\, i\mathfrak{R}'(u)$$

takes each real value once as u passes along the perimeter of the triangle ODE, being as can be shown respectively ∞, 1, 0, − 1 at O, D, H, E, and takes every complex value of imaginary part positive once in the interior of this triangle. This leads to 322

$$u = \tfrac{1}{3}\, i \int_t^{\infty} (t^2 - 1)^{-2/3} dt$$

in accordance with the general theory.

It can be deduced that $\tau = t^2$ represents the triangle ODH on the upper half plane of τ, and $\zeta = (i - \tau^{-1})^{1/2}$ represents similarly the triangle OBD.

§ 16. *Multiple valued Functions. Algebraic Functions.*—The explanations and definitions of a monogenic function hitherto given have been framed for the most part with a view to single valued functions. But starting from a power series, say in $z - c$, which represents a single value at all points of its circle of convergence, suppose that, by means of a derived series in $z - c'$, where c′ is interior to the circle of convergence, we can continue the function beyond this, and then by means of a series derived from the first derived series we can make a further continuation, and so on; it may well be that when, after a closed circuit, we again consider points in the first circle of convergence, the value represented may not agree with the original value. One example is the case $z^{1/2}$, for which two values exist for any value of z; another is the generalized logarithm λ(z), for which there is an infinite number of values. In such cases, as before, the region of existence of the function consists of all points which can be reached by such continuations with power series, and the singular points, which are the limiting points of the point-aggregate constituting the region of existence, are those points in whose neighbourhood the radii of convergence of derived series have zero for limit. In this description the point $z = \infty$ does not occupy an exceptional position, a power series in $z - c$ being transformed to a series in 1/z when z is near enough to c by means of $z - c = c(1 - cz^{-1})\,[1 - (1 - cz^{-1})]^{-1}$, and a series in 1/z to a series in $z - c$, when z is near enough to c, by means of $1/z = 1/c\,[1 + (z - c / c)]^{-1}$.

The commonest case of the occurrence of multiple valued functions is that in which the function s satisfies an algebraic equation $f(s, z) = p_0 s^n + p_1 s^{n-1} + \dots + p_n = 0$, wherein $p_0, p_1, \dots p_n$ are integral polynomials in z. Assuming $f(s, z)$ incapable of being written as a product of polynomials rational in s and z, and excepting values of z for which the polynomial coefficient of s^n vanishes, as also the values of z for which beside $f(s, z) = 0$ we have also $\partial f(s, z)/\partial s = 0$, and also in general the point $z = \infty$, the roots of this equation about any point $z=c$ are given by n power series in $z - c$. About a finite point $z = c$ for which the equation $\partial f(s, z)/\partial s = 0$ is satisfied by one or more of the roots s of $f(s, z) = 0$, the n roots break up into a certain number of cycles, the r roots of a cycle being given by a set of power series in a radical $(z - c)^{1/r}$, these series of the cycle being obtainable from one another by replacing $(z - c)^{1/r}$ by $\omega(z - r)^{1/r}$, where ω, equal to $\exp(2\pi ih/r)$, is one of the rth roots of unity. Putting then $z - c = t^r$ we may say that the r roots of a cycle are given by a single power series in t, an increase of 2π in the phase of t giving an increase of 2πr in the phase of $z - c$. This single series in t, giving the values of s belonging to one cycle in the neighbourhood of $z = c$ when the phase of $z - c$ varies through 2πr, is to be looked upon as defining a single *place* among the aggregate of values of z and s which satisfy $f(s, z) = 0$; two such places may be at the same *point* ($z = c$, $s = d$) without coinciding, the corresponding power series for the neighbouring points being different. Thus for an ordinary value of z, $z = c$, there are n places for which the neighbouring values of s are given by n power series in $z - c$; for a value of z for which $\partial f(s, z)/\partial s = 0$ there are less than n places. Similar remarks hold for the neighbourhood of $z = \infty$; there may be n places whose neighbourhood is given by n power series in z^{-1} or fewer, one of these being associated with a series in t, where $t = (z^{-1})^{1/r}$; the sum of the values of r which thus arise is always n. In general, then, we may say, with t of one of the forms $(z - c)$, $(z - c)^{1/r}$, z^{-1}, $(z^{-1})^{1/r}$. that the neighbourhood of any place (c, d) for which $f(c, d) = 0$ is given by a pair of expressions $z = c + P(t)$, $s = d + Q(t)$, where P(t) is a (particular case of a) power series vanishing for $t = 0$, and Q(t) is a power series vanishing for $t = 0$, and t vanishes at (c, d), the expression $z - c$ being replaced by z^{-1} when c is infinite, and similarly the expression $s - d$ by s^{-1} when d is infinite. The last case arises when we consider the finite values of z for which the polynomial coefficient of s^n vanishes. Of such a pair of expressions we may obtain a continuation by writing $t = t_0 + \lambda_1\tau + \lambda_2\tau^2 + \dots$, where τ is a new variable and λ_1 is not zero; in particular for an ordinary finite place this equation simply becomes $t = t_0 + \tau$. It can be shown that all the pairs of power series $z = c + P(t)$, $s = d + Q(t)$ which are necessary to represent all pairs of values of z, s satisfying the equation $f(s, z) = 0$ can be obtained from one of them by this process of continuation, a fact which we express by saying that the equation $f(s, z) = 0$ defines a *monogenic algebraic construct.* With less accuracy we may say that an irreducible algebraic equation $f(s, z) = 0$ determines a single monogenic function s of z.

Any rational function of z and s, where $f(s, z) = 0$, may be considered in the neighbourhood of any place (c, d) by substituting therein $z = c + P(t)$, $s = d + Q(t)$; the result is necessarily of the form $t^m H(t)$, where H(t) is a power series in t not vanishing for t=0 and m is an integer. If this integer is positive, the function is said to vanish to order m at the place; if this integer is negative, $= -\mu$, the function is infinite to order μ at the place. More generally, if A be an arbitrary constant, and, near (c, d), $R(s, z) - A$ is of the form $t^m H(t)$, where m is positive, we say that R(s, z) becomes m times equal to A at the place; if R(s, z) is infinite of order μ at the place, so also is R(s, z) − A. It can be shown that the sum of the values of m at all the places, including the places $z = \infty$, where R(s, z) vanishes, which we call the number of zeros of R(s, z) on the algebraic construct, is finite, and equal to the sum of the values of μ where R(s, z) is infinite, and more generally equal to the sum of the values of m where R(s, z) = A; this we express by saying that a rational function R(s, z) takes any value (including ∞) the same number of times on the algebraic construct; this number is called the *order* of the rational function.

That the total number of zeros of R(s, z) is finite is at once obvious, these values being obtainable by rational elimination of s between $f(s, z) = 0$, $R(s, z) = 0$. That the number is equal to the total number of infinities is best deduced by means of a theorem which is also of more general utility. Let R(s, z) be any rational function of s, z, which are connected by $f(s, z) = 0$; about any place (c, d) for which $z = c + P(t)$, $s = d + Q(t)$, expand the product

$$R(s, z) \frac{dz}{dt}$$

in powers of t and pick out the coefficient of t^{-1}. There is only a finite number of places of this kind. The theorem is that the sum of these coefficients of t^{-1} is zero. This we express by

$$\left[R(s, z) \frac{dz}{dt} \right]_{t^{-1}} = 0.$$

The theorem holds for the case n=1, that is, for rational functions of one variable z; in that case, about any finite point we have $z - c = t$, and about $z = \infty$ we have $z^{-1} = t$, and therefore $dz/dt = -t^{-2}$; in that case, then, the theorem is that in any rational function of z,

$$\Sigma \left(\frac{A_1}{z - a} + \frac{A_2}{(z - a)^2} + \dots + \frac{A_m}{(z - a)^m} \right) + Pz^h + Qz^{h-1} + \dots + R,$$

the sum ΣA_1 of the sum of the residues at the finite poles is equal to the coefficient of 1/z in the expansion, in ascending powers of 1/z, about $z = \infty$; an obvious result. In general, if for a finite place of the algebraic construct associated with $f(s, z) = 0$, whose neighbourhood is given by $z = c + t^r$, $s = d + Q(t)$, there be a coefficient of t^{-1} in R(s, z) dz/dt, this will be r times the coefficient of t^{-r} in R(s, z) or $R[d + Q(t), c + t^r]$, namely will be the coefficient of t^{-r} in the sum of the r series obtainable from $R[d + Q(t), c + t^r]$ by replacing t by ωt, where ω is an rth root of unity; thus the sum of the coefficients of t^{-1} in R(s, z) dz/dt for all the places which arise for z = c, and the corresponding values of s, is equal to the coefficient of $(z - c)^{-1}$ in $R(s_1, z) + R(s_2 z) + \dots + R(s_n, z)$, where $s_1, \dots s_n$ are the n values of s for a value of z near to z = c; this latter sum $\Sigma R(s_i, z)$ is, however, a rational function of z only. Similarly, near $z = \infty$, for a place given by $z^{-1} = t^r$, $s = d + Q(t)$, or $s^{-1} = Q(t)$, the coefficient of t^{-1} in R(s, z) dz/dt is equal to −r times the coefficient of t^r in $R[d + Q(t), t^{-r}]$, that is equal to the negative coefficient of z^{-1} in the sum of the r series $R[d + Q(\omega t), t^{-r}]$, so that, as before, the sum of the coefficients of t^{-1} in R(s, z) dz/dt at the various places which arise for $z = \infty$ is equal to the negative coefficient of z^{-1} in the same rational function of z, $\Sigma R(s_i, z)$. Thus, from the corresponding theorem for rational functions of one variable, the general theorem now being proved is seen to follow.

Apply this theorem now to the rational function of s and z,

$$\frac{1}{R(s, z)} \frac{dR(s, z)}{dz};$$

at a zero of R(s, z) near which $R(s, z) = t^m H(t)$, we have

$$\frac{1}{R(s, z)} \frac{dR(s, z)}{dz} \frac{dz}{dt} = \frac{d}{dt} \{\lambda [R(s, z)] \},$$

where λ denotes the generalized logarithmic function, that is equal to

$$mt^{-1} + \text{power series in } t;$$

similarly at a place for which $R(s, z) = t^{-\mu}K(t)$; the theorem

$$\left[\frac{1}{R(s, z)} \frac{dR(s, z)}{dz} \frac{dz}{dt} \right]_{t^{-1}} = 0$$

thus gives $\Sigma m = \Sigma \mu$, or, in words, the total number of zeros of R(s, z) on the algebraic construct is equal to the total number of its poles. The same is therefore true of the function R(s, z) − A, where A is an arbitrary constant; thus the number in question, being equal to the number of poles of R(s, z) − A, is equal also to the number of times that R(s, z) = A on the algebraic construct.

We have seen above that all single valued doubly periodic meromorphic functions, with the same periods, 323 are rational functions of two variables s, z connected by an equation of the form $s^2 = 4z^3 + Az + B$. Taking account of the relation connecting these variables s, z with the argument of the doubly periodic functions (which was above denoted by z), it can then easily be seen that the theorem now proved is a generalization of the theorem proved previously establishing for a doubly periodic function a definite *order*. There exists a generalization of another theorem also proved above for doubly periodic functions, namely, that the sum of the values of the argument in one parallelogram of periods for which a doubly periodic function takes a given value is independent of that value; this generalization, known as Abel’s Theorem, is given § 17 below.

§ 17. *Integrals of Algebraic Functions.*—In treatises on Integral Calculus it is proved that if R(z) denote any rational function, an indefinite integral $\int R(z)dz$ can be evaluated in terms of rational and logarithmic functions, including the inverse trigonometrical functions. In generalization of this it was long ago discovered that if $s^2 = az^2 + bz + c$ and R(s, z) be any rational function of s, z any integral $\int R(s, z)dz$ can be evaluated in terms of rational functions of s, z and logarithms of such functions; the simplest case is $\int s^{-1}dz$ or $\int (az^2 + bz + c)^{-1/2}dz$. More generally if f(s, z) = 0 be such a relation connecting s, z that when θ is an appropriate rational function of s and z both s and z are rationally expressible, in virtue of $f(s, z) = 0$ in terms of θ, the integral $\int R(s, z)dz$ is reducible to a form $\int H(\theta)d\theta$, where $H(\theta)$ is rational in θ, and can therefore also be evaluated by rational functions and logarithms of rational functions of s and z. It was natural to inquire whether a similar theorem holds for integrals $\int R(s, z)dz$ wherein s^2 is a cubic polynomial in z. The answer is in the negative. For instance, no one of the three integrals

$$\int \frac{dz}{s}, \int \frac{zdz}{s}, \int \frac{dz}{(z-c)s}$$

can be expressed by rational and logarithms of rational functions of s and z; but it can be shown that every integral $\int R(s, z)dz$ can be expressed by means of integrals of these three types together with rational and logarithms of rational functions of s and z (see below under § 20, *Elliptic Integrals*). A similar theorem is true when s^2 = quartic polynomial in z; in fact when $s^2 = A(z-a)(z-b)(z-c)(z-d)$, putting $y = s(z-a)^{-2}$, $x = (z-a)^{-1}$, we obtain y^2 = cubic polynomial in x. Much less is the theorem true when the fundamental relation $f(s, z) = 0$ is of more general type. There exists then, however, a very general theorem, known as *Abel's Theorem*, which may be enunciated as follows: Beside the rational function R(s, z) occurring in the integral $\int R(s, z)dz$, consider another rational function H(s, z); let (a_1), ... (a_m) denote the places of the construct associated with the fundamental equation $f(s, z) = 0$, for which H(s, z) is equal to one value A, each taken with its proper multiplicity, and let (b_1), ... (b_m) denote the places for which H(s, z) = B, where B is another value; then the sum of the m integrals $\int_{(a_i)}^{(b_i)} R(s, z)dz$ is equal to the sum of the coefficients of t^{-1} in the expansions of the function

$$R(s, z)\frac{dz}{dt}\lambda\left(\frac{H(s, z)-B}{H(s, z)-A}\right),$$

where λ denotes the generalized logarithmic function, at the various places where the expansion of R(s, z)dz/dt contains negative powers of t. This fact may be obtained at once from the equation

$$\left[\frac{1}{H(s, z)-\mu} R(s, z)\frac{dz}{dt}\right]_{t^{-1}} = 0,$$

wherein μ is a constant. (For illustrations see below, under § 20, *Elliptic Integrals*.)

§ 18. *Indeterminateness of Algebraic Integrals.*—The theorem that the integral $\int_a^x f(z)dz$ is independent of the path from a to z, holds only on the hypothesis that any two such paths are equivalent, that is, taken together from the complete boundary of a region of the plane within which f(z) is finite and single valued, besides being differentiable. Suppose that these conditions fail only at a finite number of isolated points in the finite part of the plane. Then any path from a to z is equivalent, in the sense explained, to any other path together with closed paths beginning and ending at the arbitrary point a each enclosing one or more of the exceptional points, these closed paths being chosen, when f(z) is not a single valued function, so that the final value of f(z) at a is equal to its initial value. It is necessary for the statement that this condition may be capable of being satisfied.

For instance, the integral $\int_1^z z^{-1}dz$ is liable to an additive indeterminateness equal to the value obtained by a closed path about z = 0, which is equal to 2πi; if we put $u = \int_1^z z^{-1}dz$ and consider z as a function of u, then we must regard this function as unaffected by the addition of 2πi to its argument u; we know in fact that z = exp (u) and is a single valued function of u, with the period 2πi. Or again the integral $\int_0^z (1+z^2)^{-1}dz$ is liable to an additive indeterminateness equal to the value obtained by a closed path about either of the points z = ±i; thus if we put $u = \int_0^z (1+z^2)^{-1}dz$, the function z of u is periodic with period π, this being the function tan (u). Next we take the integral $u = \int_{(0)}^{(z)} (1-z^2)^{-1/2}dz$, agreeing that the upper and lower limits refer not only to definite values of z, but to definite values of z each associated with a definite determination of the sign of the associated radical $(1-z^2)^{-1/2}$. We suppose 1 + z, 1 − z each to have phase zero for z = 0; then a single closed circuit of z = −1 will lead back to z = 0 with $(1-z^2)^{1/2} = -1$; the additive indeterminateness of the integral, obtained by a closed path which restores the initial value of the subject of integration, may be obtained by a closed circuit containing both the points ±1 in its interior; this gives, since the integral taken about a vanishing circle whose centre is either of the points z = ±1 has ultimately the value zero, the sum

$$\int_0^{-1}\frac{dz}{(1-z^2)^{1/2}} + \int_{-1}^{0}\frac{dz}{-(1-z^2)^{1/2}} + \int_0^{1}\frac{dz}{-(1-z^2)^{1/2}} + \int_1^{0}\frac{dz}{(1-z^2)^{1/2}},$$

where, in each case, $(1-z^2)^{1/2}$ is real and positive; that is, it gives

$$-4\int_0^1\frac{dz}{(1-z^2)^{1/2}}$$

or 2π. Thus the additive indeterminateness of the integral is of the form 2kπ, where k is an integer, and the function z of u, which is sin (u), has 2π for period. Take now the case

$$u = \int_{(z_0)}^{(z)}\frac{dz}{\sqrt{\{(z-a)(z-b)(z-c)(z-d)\}}},$$

adopting a definite determination for the phase of each of the factors z − a, z − b, z − c, z − d at the arbitrary point z_0, and supposing the upper limit to refer, not only to a definite value of z, but also to a definite determination of the radical under the sign of integration. From z_0 describe a closed loop about the point z = a, consisting, suppose, of a straight path from z_0 to a, followed by a vanishing circle whose centre is at a, completed by the straight path from a to z_0. Let similar loops be imagined for each of the points b, c, d, no two of these having a point in common. Let A denote the value obtained by the positive circuit of the first loop; this will be in fact equal to twice the integral taken from z_0 along the straight path to a; for the contribution due to the vanishing circle is ultimately zero, and the effect of the circuit of this circle is to change the sign of the subject of integration. After the circuit about a, we arrive back at z_0 with the subject of integration changed in sign; let B, C, D denote the values of the integral taken by the loops enclosing respectively b, c and d when in each case the initial determination of the subject of integration is that adopted in calculating A. If then we take a circuit from z_0 enclosing both a and b but not either c or d, the value obtained will be A − B, and on returning to z_0 the subject of integration will have its initial value. It appears thus that the integral is subject to an additive indeterminateness equal to any one of the six differences such as $A - B$. Of these there are only two linearly independent; for clearly only $A - B$, $A - C$, $A - D$ are linearly independent, and in fact, as we see by taking a closed circuit enclosing all of a, b, c, d, we have $A - B + C - D = 0$; for there is no other point in the plane beside a, b, c, d about which the subject of integration suffers a change of sign, and a circuit enclosing all of a, b, c, d may by putting $z = 1/\zeta$ be reduced to a circuit about $\zeta = 0$ about which the value of the integral is zero. The general value of the integral for any position of z and the associated sign of the radical, when we start with a definite determination of the subject of integration, is thus seen to be of the form $u_0 + m(A - B) + n(A - C)$, where m and n are integers. The value of $A - B$ is independent of the position of z_0, being obtainable by a single closed positive circuit about a and b only; it is thus equal to twice the integral taken once from a to b, with a proper initial determination of the radical under the sign of integration. Similar remarks to the above apply to any integral $\int H(z)dz$, in which $H(z)$ is an algebraic function of z; in any such case $H(z)$ is a rational function of z and a quantity s connected therewith by an irreducible rational algebraic equation $f(s, z) = 0$. Such an integral $\int K(z, s)dz$ is called an Abelian Integral. 324

§ 19. *Reversion of an Algebraic Integral.*—In a limited number of cases the equation $u = \int [z_0 \text{ to } z]\, H(z)dz$, in which $H(z)$ is an algebraic function of z, defines z as a single valued function of u. Several cases of this have been mentioned in the previous section; from what was previously proved under § 14, *Doubly Periodic Functions*, it appears that it is necessary for this that the integral should have at most two linearly independent additive constants of indeterminateness; for instance, for an integral

$$u = \int_{z_0}^{z} [(z - a)(z - b)(z - c)(z - d)(z - e)(z - f)]^{-1/2} dz,$$

there are three such constants, of the form $A - B$, $A - C$, $A - D$, which are not connected by any linear equation with integral coefficients, and z is not a single valued function of u.

§ 20. *Elliptic Integrals.*—An integral of the form $\int R(z, s)dz$, where s denotes the square root of a quartic polynomial in z, which may reduce to a cubic polynomial, and R denotes a rational function of z and s, is called an *elliptic integral*.

To each value of z belong two values of s, of opposite sign; starting, for some particular value of z, with a definite one of these two values, the sign to be attached to s for any other value of z will be determined by the path of integration for z. When z is in the neighbourhood of any finite value z_0 for which the radical s is not zero, if we put $z - z_0 = t$, we can find $s - s_0 =$ a power series in t, say $s = s_0 + Q(t)$; when z is in the neighbourhood of a value, a, for which s vanishes, if we put $z = a + t^2$, we shall obtain $s = tQ(t)$, where $Q(t)$ is a power series in t; when z is very large and s^2 is a quartic polynomial in z, if we put $z^{-1} = t$, we shall find $s^{-1} = t^2Q(t)$; when z is very large and s^2 is a cubic polynomial in z, if we put $z^{-1} = t^2$, we shall find $s^{-1} = t^3Q(t)$. By means of substitutions of these forms the character of the integral $\int R(z, s)dz$ may be investigated for any position of z; in any case it takes a form $\int [Ht^{-m} + Kt^{-m+1} + \dots + Pt^{-1} + R + St + \dots]dt$ involving only a finite number of negative powers of t in the subject of integration. Consider first the particular case $\int s^{-1}dz$; it is easily seen that neither for any finite nor for infinite values of z can negative powers of t enter; the integral is *everywhere finite*, and is said to be of *the first kind*; it can, moreover, be shown without difficulty that no integral $\int R(z, s)dz$, save a constant multiple of $\int s^{-1}dz$, has this property. Consider next, s^2 being of the form $a_0z^4 + 4a_1z^3 + \dots$, wherein a_0 may be zero, the integral $\int (a_0z^2 + 2a_1z)\, s^{-1}dz$; for any finite value of z this integral is easily proved to be everywhere finite; but for infinite values of z its value is of the form $At^{-1} + Q(t)$, where $Q(t)$ is a power series; denoting by $\sqrt{a_0}$ a particular square root of a_0 when a_0 is not zero, the integral becomes infinite for $z = \infty$ for both signs of s, the value of A being $+\sqrt{a_0}$ or $-\sqrt{a_0}$ according as s is $\sqrt{a_0}\cdot z^2 (1 + [2a_1/a_0]\, z^{-1} + \dots)$ or is the negative of this; hence the integral $J_1 = \int ([a_0z^2 + 2a_1z]/s + \sqrt{a_0})\, dz$ becomes infinite when z is infinite, for the former sign of s, its infinite term being $2\sqrt{a_0}\cdot t^{-1}$ or $2a_0\cdot z$, but does not become infinite for z infinite for the other sign of s. When $a_0 = 0$ the signs of s for $z = \infty$ are not separated, being obtained one from the other by a circuit of z about an infinitely large circle, and the form obtained represents an integral becoming infinite as before for $z = \infty$, its infinite part being $2\sqrt{a_1}\cdot t^{-1}$ or $2\sqrt{a_1}\cdot\sqrt{z}$. Similarly if z_0 be any finite value of z which is not a root of the polynomial $f(z)$ to which s^2 is equal, and s_0 denotes a particular one of the determinations of s for $z=z_0$, the integral

$$J_2 = \int \left\{ \frac{s^2_0 + \tfrac{1}{2}(z - z_0)\, f'(z_0)}{(z - z_0)^2\, s} + \frac{s_0}{(z - z_0)^2} \right\} dz,$$

wherein $f'(z) = df(z)/dz$, becomes infinite for $z = z_0$, $s = s_0$, but not for $z = z_0$, $s = -s_0$. its infinite term in the former case being the negative of $2s_0(z - z_0)$. For no other finite or infinite value of z is the integral infinite. If $z = \theta$ be a root of $f(z)$, in which case the corresponding value of s is zero, the integral

$$J_3 = \tfrac{1}{2} f'(\theta) \int \frac{dz}{(z - \theta)\, s}$$

becomes infinite for $z=0$, its infinite part being, if $z - \theta = t^2$, equal to $-[f'(\theta)]^{\frac{1}{2}}\, t^{-1}$: and this integral is not elsewhere infinite. In each of these cases, of the integrals J_1, J_2, J_3, the subject of integration has been chosen so that when the integral is written near its point of infinity in the form $\int [At^{-2} + Bt^{-1} + Q(t)]\, dt$, the coefficient B is zero, so that the infinity is of algebraic kind, and so that, when there are two signs distinguishable for the critical value of z, the integral becomes infinite for only one of these. An integral having only algebraic infinities, for finite or infinite values of z, is called an integral of the *second kind*, and it appears that such an integral can be formed with only one such infinity, that is, for an infinity arising only for one particular, and arbitrary, pair of values (s, z) satisfying the equation $s^2 = f(z)$, this infinity being of the first order. A function having an algebraic infinity of the mth order ($m > 1$), only for one sign of s when these signs are separable, at (1) $z = \infty$, (2) $z = z_0$, (3) $z = a$, is given respectively by $(s\, d/dz)^{m-1} J_1$, $(s\, d/dz)^{m-1} J_2$, $(s\, d/dz)^{m-1} J_3$, as we easily see. If then we have any elliptic integral having algebraic infinities we can, by subtraction from it of an appropriate sum of constant multiples of J_1, J_2, J_3 and their differential coefficients just written down, obtain, as the result, an integral without algebraic infinities. But, in fact, if J, J^1 denote any two of the three integrals J_1, J_2, J_3, there exists an equation $AJ + BJ' + C\int s^{-1}dz =$ rational function of s, z, where A, B, C are properly chosen constants. For the rational function

$$\frac{s + s_0}{z - z_0} + z\sqrt{a_0}$$

is at once found to become infinite for (z_0, s_0), not for $(z_0, -s_0)$, its infinite part for the first point being $2s/(z - z_0)$, and to become infinite for z infinitely large, and one sign of s only when these are separable, its infinite part there being $2z\sqrt{a_0}$ or $2\sqrt{a_1}\sqrt{z}$ when $a_0 = 0$. It does not become infinite for any other pair (z, s) satisfying the relation $s^2 = f(z)$; this is in accordance with the easily verified equation

$$\frac{s + s_0}{z - z^0} + z\sqrt{a_0} - J_1 + J_2 + (a_0 z_0^2 + 2a_1 z_0) \int \frac{dz}{s} = 0;$$

and there exists the analogous equation

$$\frac{s}{z - \theta} + z\sqrt{a_0} - J_1 + J_3 + (a_0\theta^2 + 2a_1\theta) \int \frac{dz}{s}.$$

Consider now the integral

$$P = \int \left(\frac{s + s_0}{z - z_0} + z\sqrt{a_0} \right) \frac{dz}{2s};$$

this is at once found to be infinite, for finite values of z, only for (z_0, s_0), its infinite part being $\log(z - z_0)$, and for $z = \infty$, for one sign of s only when these are separable, its infinite part being $-\log t$, that is $-\log z$ when $a_0 \neq 0$, and $-\log(z^{1/2})$ when $a_0 = 0$. And, if $f(\theta) = 0$, the integral

$$P_1 = \int \left(\frac{s}{z - \theta} + z\sqrt{a_0} \right) \frac{dz}{2s}$$

is infinite at $z = \theta$, $s = 0$ with an infinite part $\log t$, that is $\log (z - \theta)^{1/2}$, is not infinite for any other finite value of z, and is infinite like P for $z = \infty$. An integral possessing such logarithmic infinities is said to be of the third kind.

Hence it appears that any elliptic integral, by subtraction from it of an appropriate sum formed with constant multiples of the integral J_3 and the rational functions of the form $(s\ d/dz)^{m-1} J_1$ with constant multiples of integrals such as P or P_1, with constant multiples of the integral $u = \int s^{-1}dz$, and with rational functions, can be reduced to an integral H becoming infinite only for $z = \infty$, for one sign of s only when these are separable, its infinite part being of the form $A \log t$, that is, $A \log z$ or $A \log(z^{1/2})$. Such an integral $H = \int R(z, s)dz$ does not exist, however, as we at once find by writing $R(z, s) = P(z) + sQ(z)$, where $P(z)$, $Q(z)$ are rational functions of z, and examining the forms possible for these in order that the integral may have only the specified infinity. An analogous theorem holds for rational functions of z and s; there exists no rational function which is finite for finite values of z and is infinite only for $z = \infty$ for one sign of s and to the first order only; but there exists a rational function infinite in all to the first order for each of two or more pairs (z, s), however they may be situated, or infinite to the second order for an arbitrary pair (z, s); and any rational function may be formed by a sum of constant multiples of functions such as

$$\frac{s + s_0}{z - z_0} + z\sqrt{a_0} \text{ or } \frac{s}{z - \theta} + z\sqrt{a_0}$$

and their differential coefficients.

The consideration of elliptic integrals is therefore reducible to that of the three

$$u = \int \frac{dz}{s}, \quad J = \int \left(\frac{a_0 z^2 + 2a_1 z}{s} + z\sqrt{a_0} \right) dz, \quad P = \int \left(\frac{s + s_0}{z - z_0} + z\sqrt{a_0} \right) \frac{dz}{2s}$$

respectively of the first, second and third kind. Now the equation $s^2 = a_0 z^4 + \dots = a_0 (z - \theta)(z - \varphi)(z - \psi)(z - \chi)$, by putting

$$y = 2s (z - \theta)^{-2} [a_0 (\theta - \varphi)(\theta - \psi)(\theta - \chi)]^{-1/2}$$

$$x = \frac{1}{z - \theta} + \frac{1}{3} \left(\frac{1}{\theta - \varphi} + \frac{1}{\theta - \psi} + \frac{1}{\theta - \chi} \right)$$

is at once reduced to the form $y^2 = 4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$, say; and these equations enable us to express s and z rationally in terms of x and y. It is therefore sufficient to consider three elliptic integrals

$$u = \int \frac{dx}{y}, \quad J = \int \frac{xdx}{y}, \quad P = \int \frac{y + y_0}{x - x_0} \frac{dx}{2y}.$$

Of these consider the first, putting

$$u = \int_{(x)}^{(\infty)} \frac{dx}{y},$$

where the limits involve not only a value for x, but a definite sign for the radical y. When x is very large, if we put $x^{-1} = t^2$, $y^{-1} = 2t^3 (1 - \tfrac{1}{4} g_2 t^4 - \tfrac{1}{4} g_3 t^6)^{-1/2}$, we have

$$u = \int_0^t (1 + \tfrac{1}{8} g_2 t^4 + \dots) dt = t + \tfrac{1}{40} g_2 t^5 + \dots,$$

whereby a definite power series in u, valid for sufficiently small value of u, is found for t, and hence a definite power series for x, of the form 325

$$x = u^{-2} + \tfrac{1}{20} g_2 u^2 + \dots$$

Let this expression be valid for $0 < |u| < R$, and the function defined thereby, which has a pole of the second order for $u=0$, be denoted by $\varphi(u)$. In the range in question it is single valued and satisfies the differential equation

$$[\varphi'(u)]^2 = 4[\varphi(u)]^3 - g_2\varphi(u) - g_3;$$

in terms of it we can write $x = \varphi(u)$, $y = -\varphi'(u)$, and, $\varphi'(u)$ being an odd function, the sign attached to y in the original integral for $x = \infty$ is immaterial. Now for any two values u, v in the range in question consider the function

$$F(u, v) = \tfrac{1}{4} \left[\frac{\varphi'(u) - \varphi'(v)}{\varphi(u) - \varphi(v)} \right]^2 - \varphi(u) - \varphi(v);$$

it is at once seen, from the differential equation, to be such that $\partial F/\partial u = \partial F/\partial v$; it is therefore a function of $u + v$; supposing $|u + v| < R$ we infer therefore, by putting $v = 0$, that

$$\varphi(u + v) = \tfrac{1}{4} \left[\frac{\varphi'(u) - \varphi'(v)}{\varphi(u) - \varphi(v)} \right]^2 - \varphi(u) - \varphi(v).$$

By repetition of this equation we infer that if $u_1, \dots u_n$ be any arguments each of which is in absolute value less than R, whose sum is also in absolute value less than R, then $\varphi(u_1 + \dots + u_n)$ is a rational function of the 2n functions $\varphi(u_s)$, $\varphi'(u_s)$; and hence, if $|u| < R$, that

$$\varphi(u) = H \left[\varphi\left(\frac{u}{n}\right), \quad \varphi'\left(\frac{u}{n}\right) \right],$$

where H is some rational function of the arguments $\varphi(u/n)$, $\varphi'(u/n)$. In fact, however, so long as $|u/n| < R$, each of the functions $\varphi(u/n)$, $\varphi'(u/n)$ is single valued and without singularity save for the pole at $u=0$; and a rational function of single valued functions, each of which has no singularities other than poles in a certain region, is also a single valued function without singularities other than poles in this region. We infer, therefore, that the function of u expressed by $H[\varphi(u/n), \varphi'(u/n)]$ is single valued and without singularities other than poles so long as $|u| < nR$; it agrees with $\varphi(u)$ when $|u| < R$, and hence furnishes a continuation of this function over the extended range $|u| < nR$. Moreover, from the method of its derivation, it satisfies the differential equation $[\varphi'(u)]^2 = 4[\varphi(u)]^3 - g_2\varphi(u) - g_3$. This equation has therefore one solution which is a single valued monogenic function with no singularities other than poles for any finite part of the plane, having in particular for $u = 0$, a pole of the second order; and the method adopted for obtaining this near $u=0$ shows that the differential equation has no other such solution. This, however, is not the only solution which is a single valued meromorphic function, a the functions $\varphi(u + \alpha)$, wherein α is arbitrary, being such. Taking now any range of values of u, from $u = 0$, and putting for any value of u, $x = \varphi(u)$, $y = -\varphi'(u)$, so that $y^2=4x^3-g_2x-g_3$, we clearly have

$$u = \int_{(x, y)}^{(\infty)} \frac{dx}{y};$$

conversely if $x_0 = \varphi(u_0)$, $y_0 = -\varphi'(u_0)$ and ξ, η be any values satisfying $\eta_2 = 4\xi^2 - g_2\xi - g_3$, which are sufficiently near respectively to x_0, y_0, while v is defined by

$$v - u_0 = - \int_{(x_0, y_0)}^{(\xi, \eta)} \frac{d\xi}{\eta},$$

then ξ, η are respectively $\varphi(v)$ and $-\varphi'(v)$; for this equation leads to an expansion for $\xi - x_0$ in terms of $v = u_0$ and only one such expansion, and this is obtained by the same work as would be necessary to expand $\varphi(v)$ when v is near to u_0; the function $\varphi(u)$ can therefore be continued by the help of this equation, from $v = u_0$, provided the lower limit of $|\xi - x_0|$ necessary for the expansions is not zero in the neighbourhood of any value (x_0, y_0). In fact the function $\varphi(u)$ can have only a finite number of poles in any finite part of the plane of u; each of these can be surrounded by a small circle, and in the portion of the finite part of the plane of u which is outside these circles, the lower limit of the radii of convergence of the expansions of $\varphi(u)$ is greater than zero; the same will therefore be the case for the lower limit of the radii $|\xi - x_0|$ necessary for the continuations spoken of above provided that the values of (ξ, η) considered do not lead to infinitely increasing values of v; there does not exist, however, any definite point (ξ_0, η_0) in the neighbourhood of which the integral $\int_{(x_0, y_0)}^{(\xi, \eta)} d\xi/\eta$ increases indefinitely, it is only by a path of infinite length that the integral can so increase. We infer therefore that if (ξ, η) be any point, where $\eta_2 = 4\xi^3 - g_2\xi - g_3$, and v be defined by

$$v = \int_{(\xi, \eta)}^{(\infty)} \frac{dx}{y},$$

then $\xi = \varphi(v)$ and $\eta = -\varphi'(v)$. Thus this equation determines (ξ, η) without ambiguity. In particular the additive indeterminatenesses of the integral obtained by closed circuits of the point of integration are periods of the function $\varphi(u)$; by considerations advanced above it appears that these periods are sums of integral multiples of two which may be taken to be

$$\omega = 2 \int_{e_1}^{\infty} \frac{dx}{y}, \quad \omega' = 2 \int_{e_3}^{\infty} \frac{dx}{y};$$

these quantities cannot therefore have a real ratio, for else, being periods of a monogenic function, they would, as we have previously seen, be each integral multiples of another period; there would then be a closed path for (x, y), starting from an arbitrary point (x_0, y_0), other than one enclosing two of the points $(e_1, 0)$, $(e_2, 0)$, $(e_3, 0)$, (∞, ∞), which leads back to the initial point (x_0, y_0), which is impossible. On the whole, therefore, it appears that the function $\varphi(u)$ agrees with the function $\wp(u)$ previously discussed, and the discussion of the elliptic integrals can be continued in the manner given under § 14, *Doubly Periodic Functions*.

§ 21. *Modular Functions.*—One result of the previous theory is the remarkable fact that if

$$\omega = 2 \int_{e_1}^{\infty} \frac{dx}{y}, \quad \omega' = 2 \int_{e_3}^{\infty} \frac{dx}{y};$$

where $y^2 = 4(x - e_1)(x - e_2)(x - e_3)$, then we have

$$e_1 = (\tfrac{1}{2}\omega)^{-2} + \Sigma' \{[(m + \tfrac{1}{2})\omega + m'\omega']^{-2} - [m\omega + m'\omega']^{-2}\},$$

and a similar equation for e_3, where the summation refers to all integer values of m and m′ other than the one pair $m = 0$, $m' = 0$. This, with similar results, has led to the consideration of functions of the complex ratio ω'/ω.

It is easy to see that the series for $\wp(u)$, $u^{-2} + \Sigma'[(u + m\omega + m'\omega')^2 - (m\omega + m'\omega')^2]$, is unaffected by replacing ω, ω′ by two quantities Ω, Ω′ equal respectively to $p\omega + q\omega'$, $p'\omega' + q'\omega'$, where p, q, p′, q′ are any integers for which $pq' - p'q = \pm 1$; further it can be proved that all substitutions with integer coefficients $\Omega = p\omega + q\omega'$, $\Omega' = p'\omega + q'\omega'$, wherein $pq' - p'q = 1$, can be built up by repetitions of the two particular substitutions ($\Omega = -\omega'$, $\Omega' = \omega$), ($\Omega = \omega$, $\Omega' = \omega + \omega'$). Consider the function of the ratio ω'/ω expressed by

$$h = -\wp(\tfrac{1}{2}\omega') / \wp(\tfrac{1}{2}\omega);$$

it is at once seen from the properties of the function $\wp(u)$ that by the two particular substitutions referred to we obtain the corresponding substitutions for h expressed by

$$h' = 1/h, \qquad h' = 1 - h;$$

thus, by all the integer substitutions $\Omega = p\omega + q\omega'$, $\Omega' = p'\omega + q'\omega'$, in which $pq' - p'q = 1$, the function h can only take one of the six values h, 1/h, 1 − h, 1/(1 − h), h/(h − 1), (h − 1)/h, which are the roots of an equation in θ,

$$\frac{(1-\theta+\theta^2)^3}{\theta^2(1-\theta)^2} = \frac{(1-h+h^2)^3}{h^2(1-h)^2};$$

the function of τ, = ω′/ω, expressed by the right side, is thus unaltered by every one of the substitutions $\tau' = (p' + q'\tau / p + q\tau)$, wherein p, q, p′, q′ are integers having $pq' - p'q = 1$. If the imaginary part σ, of τ, which we may write $\tau = \rho + i\sigma$, is positive, the imaginary part of τ′, which is equal to $\sigma(pq' - p'q)/[(p + q\rho)^2 + q^2\sigma^2]$, is also positive; suppose σ to be positive; it can be shown that the upper half of the infinite plane of the complex variable τ can be divided into regions, all bounded by arcs of circles (or straight lines), no two of these regions overlapping, such that any substitution of the kind under consideration, $\tau' = (p' + q'\tau)/(p + q\tau)$ leads from an arbitrary point τ, of one of these regions, to a point τ′ of another; taking $\tau = \rho + i\sigma$, one of these regions may be taken to be that for which $-\tfrac{1}{2} < \rho < \tfrac{1}{2}$, $\rho^2 + \sigma^2 > 1$, together with the points for which ρ is negative on the curves limiting this region; then every other region is obtained from this so-called fundamental region by one and only one of the substitutions $\tau = (p' + q'\tau)/(p + q\tau)$, and hence by a definite combination of the substitutions $\tau' = -1/\tau$, $\tau' = 1 + \tau$. Upon the infinite half plane of τ, the function considered above,

$$z(\tau) = \tfrac{4}{27}\,\frac{[\wp^2(\tfrac{1}{2}\omega) + \wp(z(\tfrac{1}{2}\omega)\,\wp(\tfrac{1}{2}\omega') + \wp^2(\tfrac{1}{2}\omega')]^3}{\wp^2(\tfrac{1}{2}\omega)\,\wp^2(\tfrac{1}{2}\omega')\,[\wp(\tfrac{1}{2}\omega) + \wp(\tfrac{1}{2}\omega']^2}$$

is a single valued monogenic function, whose only essential singularities are the points $\tau' = (p' + q'\tau)/(p + q\tau)$ for which $\tau = \infty$, namely those for which τ′ is any real rational value; the real axis is thus a line over which the function z(τ) cannot be continued, having an essential singularity in every arc of it, however short; in the fundamental region, z(τ) has thus only the single essential singularity, $r = \rho + i\sigma$, where $\sigma = \infty$; in this fundamental region z(τ) takes any assigned complex value just once, the relation $z(\tau') = z(\tau)$ requiring, as can be shown, that τ′ is of the form $(p' + q'\tau)/(p + q\tau)$, in which p, q, p′, q′ are integers with $pq' - p'q = 1$; the function z(τ) has thus a similar behaviour in every other of the regions. The division of the plane into regions is analogous to the division of the plane, in the case of doubly periodic functions, into parallelograms; in that case we considered only functions without essential singularities, and in each of the regions the function assumed every complex value twice, at least. Putting, as another function of τ, $J(\tau) = z(\tau)\,[z(\tau) - 1]$, it can be shown that $J(\tau) = 0$ for $\tau = \exp(\tfrac{2}{3}\pi i)$, that $J(\tau) = 1$ for $\tau = i$, these being values of τ on the boundary of the fundamental region; like z(τ) it has an essential singularity for $\tau = \rho + i\sigma$, $\sigma = +\infty$. In the theory of linear 326
differential equations it is important to consider the inverse function τ(J); this is infinitely many valued, having a cycle of three values for circulation of J about J = 0 (the circuit of this point leading to a linear substitution for τ of period 3, such as $\tau' = -(1 + \tau)^{-1}$), having a cycle of two values about J = 1 (the circuit leading to a linear substitution for τ of period 2, such as $\tau' = -\tau^{-1}$), and having a cycle of infinitely many values about J = ∞ (the circuit leading to a linear substitution for τ which is not periodic, such as $\tau' = 1 + \tau$). These are the only singularities for the function τ(J). Each of the functions

$$[J(\tau)]^{1/3}, \qquad [J(\tau) - 1]^{1/2}, \qquad \left[-\frac{\wp(\tfrac{1}{2}\omega) + 2\wp(\tfrac{1}{2}\omega')}{\wp(\tfrac{1}{2}\omega) - \wp(\tfrac{1}{2})\omega')}\right]^{1/8},$$

beside many others (see below), is a single valued function of τ, and is expressible without ambiguity in terms of the single valued function of τ,

$$\eta(\tau) = \exp\left(\frac{i\pi\tau}{12}\right)\prod_{n=1}^{\infty}[1 - \exp(2i\pi n\tau)] = \exp\left(\frac{i\pi\tau}{12}\right)\sum_{m=-\infty}^{\infty}(-1)^m \exp[(3m^2 + m)\,i\pi\tau].$$

It should be remarked, however, that η(τ) is not unaltered by all the substitutions we have considered; in fact

$$\eta(-\tau^{-1}) = (-i\tau)^{\frac{1}{2}}\eta(\tau), \qquad \eta(1 + \tau) = \exp(\tfrac{1}{12}i\pi)\,\eta(\tau).$$

The aggregate of the substitutions $\tau' = (p' + q'\tau)/(p + q\tau)$, wherein p, q, p′, q′ are integers with $pq' - p'q = 1$, represents a *Group*; the function J(τ), unaltered by all these substitutions, is called a *Modular Function*. More generally any function unaltered by all the substitutions of a group of linear substitutions of its variable is called an *Automorphic Function*. A rational function, of its variable h, of this character, is the function $(1 - h + h^2)^3 h^{-2}(1 - h)^{-2}$ presenting itself incidentally above; and there are other rational functions with a similar property, the group of substitutions belonging to any one of these being, what is a very curious fact, associable with that of the rotations of one of the regular solids, about an axis through its centre, which bring the solid into coincidence with itself. Other automorphic functions are the double periodic functions already discussed; these, as we have seen, enable us to solve the algebraic equation $y^2 = 4x^3 - g_2x - g_3$ (and in fact many other algebraic equations, see below, under § 23, *Geometrical Applications of Elliptic Functions*) in terms of single valued functions $x = \wp(u)$, $y = -\wp'(u)$. A similar utility, of a more extended kind, belongs to automorphic functions in general; but it can be shown that such functions necessarily have an infinite number of essential singularities except for the simplest cases.

The modular function J(τ) considered above, unaltered by the group of linear substitutions $\tau' = (p' + q'\tau) / (p + q\tau)$, where p, q, p′, q′ are integers with $pq' - p'q = 1$, may be taken as the independent variable x of a differential equation of the third order, of the form

$$s''' \quad 3 \quad s'' \qquad 1 - \alpha^2 \qquad 1 - \beta^2 \qquad \alpha^2 + \beta^2 - \gamma^2 - 1$$
$$- \qquad\qquad + \qquad + \qquad ,$$

$$\overline{\quad s' \quad} \quad \overline{\ 2\ } \left(\overline{\ s'\ } \right)^2 = \overline{2(x-1)^2} \quad \overline{\ 2x^2\ } \quad \overline{\ 2x\,(x-1)\ }$$

where $s' = ds/dx$, &c., of which the dependent variable s is equal to τ. A differential equation of this form is satisfied by the quotient of two independent integrals of the linear differential equation of the second order satisfied by the hypergeometric functions. If the solution of the differential equation for s be written $s(\alpha,\beta,\gamma, x)$, we have in fact $\tau = s(\tfrac{1}{2}, \tfrac{1}{3}, 0, J)$. If we introduce also the function of τ given by

$$\lambda = \frac{2\mathfrak{P}\,(\tfrac{1}{2}\omega') + \mathfrak{P}\,(\tfrac{1}{2}\omega)}{\mathfrak{P}\,(\tfrac{1}{2}\omega') - \mathfrak{P}\,(\tfrac{1}{2}\omega)},$$

we similarly have $\tau = s(0, 0, 0, \lambda)$; this function λ is a single valued function of τ, which is also a modular function, being unaltered by a group of integral substitutions also of the form $\tau' = (p' + q'\tau)/(p + q\tau)$, with $pq' - p'q = 1$, but with the restriction that p' and q are even integers, and therefore p and q' are odd integers. This group is thus a subgroup of the general modular group, and is in fact of the kind called a self-conjugate subgroup. As in the general case this subgroup is associated with a subdivision of the plane into regions of which any one is obtained from a particular region, called the fundamental region, by a particular one of the substitutions of the subgroup. This fundamental region, putting $\tau = \rho + i\sigma$, may be taken to be that given by $-1 < \rho < 1$, $(\rho + \tfrac{1}{2})^2 + \sigma^2 > \tfrac{1}{4}$, $(\rho - \tfrac{1}{2})^2 + \sigma^2 > \tfrac{1}{4}$, and is built up of six of the regions which arose for the general modular group associated with $J(\tau)$. Within this fundamental region, λ takes every complex value just once, except the values $\lambda = 0, 1, \infty$, which arise only at the angular points $\tau = 0$, $\tau = \infty$, $\tau = -1$ and the equivalent point $\tau = 1$; these angular points are essential singularities for the function $\lambda(\tau)$. For $\lambda(\tau)$ as for $J(\tau)$, the region of existence is the upper half plane of τ, there being an essential singularity in every length of the real axis, however short.

If, beside the plane of τ, we take a plane to represent the values of λ, the function $\tau = s(0, 0, 0, \lambda)$ being considered thereon, the values of τ belonging to the interior of the fundamental region of the τ-plane considered above, will require the consideration of the whole of the λ-plane taken once with the exception of the portions of the real axis lying between $-\infty$ and 0 and between 1 and $+\infty$, the two sides of the first portion corresponding to the circumferences of the τ-plane expressed by $(\rho + \tfrac{1}{2})^2 + \sigma^2 = \tfrac{1}{4}$, $(\rho - \tfrac{1}{2})^2 + \sigma^2 = \tfrac{1}{4}$, while the two sides of the latter portion, for which λ is real and > 1, correspond to the lines of the τ-plane expressed by $\rho = \pm 1$. The line for which λ is real, positive and less than unity corresponds to the imaginary axis of the τ-plane, lying in the interior of the fundamental region. All the values of $\tau = s(0, 0, 0, \lambda)$ may then be derived from those belonging to the fundamental region of the τ-plane by making λ describe a proper succession of circuits about the points $\lambda = 0$, $\lambda = 1$; any such circuit subjects τ to a linear substitution of the subgroup of τ considered, and corresponds to a change of τ from a point of the fundamental region to a corresponding point of one of the other regions.

§ 22. *A Property of Integral Functions deduced from the Theory of Modular Functions.*—Consider now the function $\exp(z)$, for finite values of z; for such values of z, $\exp(z)$ never vanishes, and it is impossible to assign a closed circuit for z in the finite part of the plane of z which will make the function $\lambda = \exp(z)$ pass through a closed succession of values in the plane of λ having $\lambda = 0$ in its interior; the function $s[0, 0, 0, \exp(z)]$, however z vary in the finite part of the plane, will therefore never be subjected to those linear substitutions imposed upon $s(0, 0, 0, \lambda)$ by a circuit of λ about $\lambda = 0$; more generally, if $\varphi(z)$ be an integral function of z, never becoming either zero or unity for finite values of z, the function $\lambda = \varphi(z)$, however z vary in the finite part of the plane, will never make, in the plane of λ, a circuit about either $\lambda = 0$ or $\lambda = 1$, and $s(0, 0, 0, \lambda)$, that is $s[0, 0, 0, \varphi(z)]$, will be single valued for all finite values of z; it will moreover remain finite, and be monogenic. In other words, $s[0, 0, 0, \varphi(z)]$ is also an integral function—whose imaginary part, moreover, by the property of $s(0, 0, 0, \lambda)$, remains positive for all finite values of z. In that case, however, $\exp\{is[0, 0, 0, \varphi(z)]\}$ would also be an integral function of z with modulus less than unity for all finite values of z. If, however, we describe a circle of radius R in the z plane, and consider the greatest value of the modulus of an integral function upon this circle, this certainly increases indefinitely as R increases. We can infer therefore that *an integral function $\varphi(z)$ which does not vanish for any finite value of z, takes the value unity and hence* (by considering the function $A^{-1}\varphi(z)$) *takes every other value for some definite value of z*; or, an integral function for which both the equations $\varphi(z) = A$, $\varphi(z) = B$ are unsatisfied by definite values of z, does not exist, A and B being arbitrary constants.

A similar theorem can be proved in regard to the values assumed by the function $\varphi(z)$ for points z of modulus greater than R, however great R may be, also with the help of modular functions. In general terms it may be stated that it is a very exceptional thing for an integral function not to assume every complex value an infinite number of times.

Another application of modular functions is to prove that the function $s(\alpha, \beta, \gamma, \lambda)$ is a single valued function of $\tau = s(0, 0, 0, \lambda)$; for, putting $\tau' = (\tau - i)/(\tau + i)$, the values of τ' which correspond to the singular points $\lambda = 0, 1, \infty$ of $s(\alpha, \beta, \gamma, \lambda)$, though infinite in number, all lie on the circumference of the circle $|\tau'| = 1$, within which therefore $s(\alpha, \beta, \gamma, x)$ is expressible in a form $\sum_{n=0}^{\infty} a_n \tau'^n$. More generally any monogenic function of λ which is single valued save for circuits of the points $\lambda = 0, 1, \infty$, is a single valued function of $\tau = s(0, 0, 0, \lambda)$. Identifying λ with the square of the modulus in Legendre’s form of the elliptical integral, we have $\tau = iK'/K$, where

$$K = \int_0^1 \frac{dt}{\sqrt{[1 - t^2]\,[1 - \lambda t^2]}}, \qquad K' = \int_0^1 \frac{dt}{\sqrt{[1 - t^2]\,[1 - (1 - \lambda)\,t^2]}};$$

functions such as $\lambda^{1/4}$, $(1 - \lambda)^{1/4}$, $[\lambda(1 - \lambda)]^{1/4}$, which have only $\lambda = 0, 1, \infty$ as singular points, were expressed by Jacobi as power series in $q = e^{i\pi\tau}$, and therefore, at least for a limited range of values of τ, as single valued functions of τ; it follows by the theorem given that any product of a root of λ and a root of $1 - \lambda$ is a single valued function of τ. More generally the differential equation

$$x(1 - x)\frac{d^2y}{dx^2} + [\gamma - (\alpha + \beta + 1)x]\frac{dy}{dx} - \alpha\beta\gamma = 0$$

may be solved by expressing both the independent and dependent variables as single valued functions of a single variable τ, the expression for the independent variable being $x = \lambda(\tau)$.

§ 23. *Geometrical Applications of Elliptic Functions.*—Consider any irreducible algebraic equation rational in x, y, $f(x, y) = 0$, of such a form that the equation represents a plane curve of order n with $\frac{1}{2}n(n - 3)$ double points; taking upon this curve $n - 3$ arbitrary fixed points, draw through these and the double points the most general curve of order $n - 2$; this will intersect f in $n(n - 2) - n(n - 3) - (n - 3) = 3$ other points, and will 327
contain homogeneously at least $\frac{1}{2}(n - 1)n - \frac{1}{2}n(n - 3) - (n - 3) = 3$ arbitrary constants, and so will be of the form $\lambda\phi + \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots = 0$, wherein $\lambda_3, \lambda_4, \ldots$ are in general zero. Put now $\xi = \phi_1/\phi$, $\eta = \phi_2/\phi$ and eliminate x, y between these equations and $f(x, y) = 0$, so obtaining a rational irreducible equation $F(\xi, \eta) = 0$, representing a further plane curve. To any point (x, y) of f will then correspond a definite point (ξ, η) of F.

For a general position of (x, y) upon f the equations $\phi_1(x', x')/\phi(x', x') = \phi_1(x, y)/\phi(x, y)$, $\phi_2(x', x')/\phi(x', x') = \phi_2(x, y)/\phi(x, y)$, subject to $f(x', x') = 0$, will have the same number of solutions (x′, x′); if their only solution is $x' = x$, $x' = y$, then to any position (ξ, η) of F will conversely correspond only one position (x, y) of f. If these equations have another solution beside (x, y), then any curve $\lambda\phi + \lambda_1\phi_1 + \lambda_2\phi_2 = 0$ which passes (through the double points of f and) through the $n - 2$ points of f constituted by the fixed $n - 3$ points and a point (x_0, y_0), will necessarily pass through a further point, say (x_0', y_0'), and will have only one further intersection with f; such a curve, with the $n - 2$ assigned points, beside the double points, of f, will be of the form $\mu\psi + \mu_1\psi_1 + \ldots = 0$, where $\mu_2, \mu_3, \ldots$ are generally zero; considering the curves $\psi + t\psi_1 = 0$, for variable t, one of these passes through a further arbitrary point of f, by choosing t properly, and conversely an arbitrary value of t determines a single further point of f; the co-ordinates of the points of f are thus rational functions of a parameter t, which is itself expressible rationally by the co-ordinates of the point; it can be shown algebraically that such a curve has not $\frac{1}{2}(n - 3)n$ but $\frac{1}{2}(n - 3)n + 1$ double points. We may therefore assume that to every point of F corresponds only one point of f, and there is a birational transformation between these curves; the coefficients in this transformation will involve rationally the co-ordinates of the $n - 3$ fixed points taken upon f, that is, at the least, by taking these to be consecutive points, will involve the co-ordinates of one point of f, and will not be rational in the coefficients of f unless we can specify a point of f whose co-ordinates are rational in these. The curve F is intersected by a straight line $a\xi + b\eta + c = 0$ in as many points as the number of unspecified intersections of f with $a\phi + b\phi_1 + c\phi_2 = 0$, that is, 3; or F will be a cubic curve, without double points.

Such a cubic curve has at least one point of inflection Y, and if a variable line YPQ be drawn through Y to cut the curve again in P and Q, the locus of a point R such that YR is the harmonic mean of YP and YQ, is easily proved to be a straight line. Take now a triangle of reference for homogeneous co-ordinates XYZ, of which this straight line is $Y = 0$, and the inflexional tangent at Y is $Z = 0$; the equation of the cubic curve will then be of the form

$$ZY^2 = aX^3 + bX^2Z + cXZ^2 + dZ^3;$$

by putting X equal to $\lambda X + \mu Z$, that is, choosing a suitable line through Y to be $X = 0$, and choosing λ properly, this is reduced to the form

$$ZY^2 = 4X^3 - g_2XZ^2 - g_3Z^3,$$

of which a representation is given, valid for every point, in terms of the elliptic functions $\wp(u)$, $\wp'(u)$, by taking $X = Z\wp(u)$, $Y = Z\wp'(u)$. The value of u belonging to any point is definite save for sums of integral multiples of the periods of the elliptic functions, being given by

$$u = \int_{(\infty)}^{(x)} \frac{ZdX - XdZ}{ZY},$$

where (∞) denotes the point of inflection.

It thus appears that the co-ordinates of any point of a plane curve, f, of order n with $\frac{1}{2}(n - 3)n$ double points are expressible as elliptic functions, there being, save for periods, a definite value of the argument u belonging to every point of the curve. It can then be shown that if a variable curve, ϕ, of order m be drawn, passing through the double points of the curve, the values of the argument u at the remaining intersections of ϕ with f, have a sum which is unaffected by variation of the coefficients of ϕ, save for additive aggregates of the periods. In virtue of the birational transformation this theorem can be deduced from the theorem that if any straight line cut the cubic $y^2 = 4x^3 - g_2x - g_3$, in points (u_1), (u_2), (u_3), the sum $u_1 + u_2 + u_3$ is zero, or a period; or the general theorem is a corollary from Abel's theorem proved under § 17, *Integrals of Algebraic Functions*. To prove the result directly for the cubic we remark that the variation of one of the intersections (x, y) of the cubic with the straight line $y = mx + n$, due to a variation δm, δn in m and n, is obtained by differentiation of the equation for the three abscissae, namely the equation

$$F(x) = 4x^3 - g_2x - g_3 - (mx + n)^2 = 0,$$

and is thus given by

$$\frac{dx}{y} = \frac{x\delta m + \delta n}{F'(x)},$$

and the sum of three such fractions as that on the right for the three roots of $F(x) = 0$ is zero; hence $u_1 + u_2 + u_3$ is independent of the straight line considered; if in particular this become the inflexional tangent each of u_1, u_2, u_3 vanishes. It may be remarked in passing that $x_1 + x_2 + x_3 = \frac{1}{4}m^2$, and hence is $\frac{1}{4}\{(y_1 - y_2)/(x_1 - x_2)\}^2$; so that we have another proof of the addition equation for the function $\wp(u)$. From this theorem for the cubic curve many of its geometrical properties, as for example those of its inflections, the properties of inscribed polygons, of the three kinds of corresponding points, and the theory of residuation, are at once obvious. And similar results hold for the curve of order n with $\frac{1}{2}(n - 3)n$ double points.

§ 24. *Integrals of Algebraic Functions in Connexion with the Theory of Plane Curves.*—The developments which have been explained in connexion with elliptic functions may enable the reader to appreciate the vastly more extensive theory similarly arising for any algebraical irrationality, $f(x, y) = o$.

The algebraical integrals $\int R(x, y)dx$ associated with this may as before be divided into those of the *first kind*, which have no infinities, those of the *second kind*, possessing only algebraical infinities, and those of the *third kind*, for which logarithmic infinities enter. Here there is a certain number, p, greater than unity, of linearly independent integrals of the first kind; and this number p is unaltered by any birational transformation of the fundamental equation $f(x, y) = 0$; a rational function can be constructed with poles of the first order at $p + 1$ arbitrary positions (x, y), satisfying $f(x, y) = 0$, but not with a fewer number unless their positions are chosen properly, a property we found for the case $p = 1$; and p is the number of linearly independent curves of order $n - 3$ passing through the double points of the curve of order n expressed by $f(x, y) = 0$. Again any integral of the second kind can be expressed as a sum of p integrals of this kind, with poles of the first order at arbitrary positions, together with rational functions and integrals of the first kind; and an integral of the second kind can be found with one pole of the first order of arbitrary position, and an integral of the third kind with two logarithmic infinities, also of arbitrary position; the corresponding properties for $p = 1$ are proved above.

There is, however, a difference of essential kind in regard to the inversion of integrals of the first kind; if $u = \int R(x, y)dx$ be such an integral, it can be shown, in common with all algebraic integrals associated with $f(x, y) = 0$, to have $2p$ linearly independent additive constants of indeterminateness; the upper limit of the integral cannot therefore, as we have shown, be a single valued function of the value of the integral. The corresponding theorem, if $\int R_i(x, y)dx$ denote one of the integrals of the first kind, is that the p equations

$$\int R_i (x_1, y_1)dx_1 + \dots + \int R_i (x_p, y_p)dx_p = u_i,$$

determine the rational symmetric functions of the p positions $(x_1, y_1), \dots (x_p, y_p)$ as single valued functions of the p variables, $u_1, \dots u_p$. It is thus necessary to enter into the theory of functions of several independent variables; and the equation $f(x, y) = 0$ is thus not, in this way, capable of solution by single valued functions of one variable. That solution in fact is to be sought with the help of automorphic functions, which, however, as has been remarked, have, for $p > 1$, an infinite number of essential singularities.

§ 25. *Monogenic Functions of Several Independent Variables.*—A monogenic function of several independent complex variables $u_i, \dots u_p$ is to be regarded as given by an aggregate of power series all obtainable by continuation from any one of them in a manner analogous to that before explained in the case of one independent variable. The singular points, defined as the limiting points of the range over which such continuation is possible, may either be *poles*, or *polar points of indetermination*, or *essential singularities*.

A pole is a point $(u^{(0)}{}_1, \dots u^{(0)}{}_p)$ in the neighbourhood of which the function is expressible as a quotient of converging power series in $u_1 - u^{(0)}{}_1 \dots u_p - u^{(0)}{}_p$; of these the denominator series D must vanish at $(u^{(0)}{}_1, \dots u^{(0)}{}_p)$, since else the fraction is expressible as a power series and the point is not a singular point, but the numerator series N must not also vanish at $(u^{(0)}{}_1, \dots u^{(0)}{}_p)$, or if it does, it must be possible to write $D = MD_0$, $N = MN_0$, where M is a converging power series vanishing at $(u^{(0)}{}_1, \dots u^{(0)}{}_p)$, and N_0 is a converging power series, in $(u_1 - u^{(0)}{}_1 \dots u_p - u^{(0)}{}_p)$, not so vanishing. A polar point of indetermination is a point about which the function can be expressed as a quotient of two converging power series, both of which vanish at the point. As in such a simple case as $(Ax + By)/ (ax + by)$, about $x = 0$, $y = 0$, it can be proved that then the function can be made to approach to any arbitrarily assigned value by making the variables $u_1, \dots u_p$ approach to $u^{(0)}{}_1, \dots u^{(0)}{}_p$ by a proper path. It is the necessary existence of such polar points of indetermination, which in case $p > 2$ are not merely isolated points, which renders the theory essentially more difficult than that of functions of one variable. An essential singularity is any which does not come under one of the two former descriptions and includes very various possibilities. A point at infinity in this theory is one for which any one of the variables $u_1, \dots u_p$ is indefinitely great; such points are brought under the preceding definitions by means of 328
the convention that for $u^{(0)}{}_i = \infty$, the difference $u_i - u^{(0)}{}_i$ is to be understood to stand for $u^{-1}{}_i$. This being so, a single valued function of $u_1, \dots u_p$ without essential singularities for infinite or finite values of the variables can be shown, by induction, to be, as in the case of $p = 1$, necessarily a rational function of the variables. A function having no singularities for finite values of all the variables is as before called an integral function; it is expressible by a power series converging for all finite values of the variables; a single valued function having for finite values of the variables no singularities other than poles or polar points of indetermination is called a meromorphic function; as for $p = 1$ such a function can be expressed as a quotient of two integral functions having no common zero point other than the points of indetermination of the function; but the proof of this theorem is difficult.

The single valued functions which occur, as explained above, in the inversion of algebraic integrals of the first kind, for $p > 1$, are meromorphic. They must also be periodic, unaffected that is when the variables $u_1, \dots u_p$ are *simultaneously* increased each by a proper constant, these being the additive constants of indeterminateness for the p integrals $\int R_i(x, y)dx$ arising when (x, y) makes a closed circuit, the same for each integral. The theory of such single valued meromorphic periodic functions is simpler than that of meromorphic functions of several variables in general, as it is sufficient to consider only finite values of the variables; it is the natural extension of the theory of doubly periodic functions previously discussed. It can be shown to reduce, though the proof of this requires considerable developments of which we cannot speak, to the theory of a single integral function of $u_1, \dots u_p$, called the *Theta Function*. This is expressible as a series of positive and negative integral powers of quantities $\exp(c_1u_1)$, $\exp(c_2u_2)$, ... $\exp(c_pu_p)$, wherein $c_1, \dots c_p$ are proper constants; for $p = 1$ this theta function is essentially the same as that above given under a different form (see § 14, *Doubly Periodic Functions*), the function $\sigma(u)$. In the case of $p = 1$, all meromorphic functions periodic with the same two periods have been shown to be rational functions of two of them connected by a single algebraic equation; in the same way all meromorphic functions of p variables, periodic with the same sets of simultaneous periods, $2p$ sets in all, can be shown to be expressible rationally in terms of $p + 1$ such periodic functions connected by a single algebraic equation. Let $x_1, \dots x_p, y$ denote $p + 1$ such functions; then each of the partial derivatives $dx_i/\partial u_i$ will equally be a meromorphic function of the same periods, and so expressible rationally in terms of $x_1, \dots x_p, y$; thus there will exist p equations of the form

$$dx_i = R_1du_1 + \dots + R_pdu_p,$$

and hence p equations of the form

$$du_i = H_{i,1}dx_1 + \dots + H_{i,p}dx_p,$$

wherein $H_{i,j}$ are rational functions of $x_1, \dots x_p, y$, these being connected by a fundamental algebraic (rational) equation, say $f(x_1, \dots x_p, y) = 0$. This then is the generalized form of the corresponding equation for $p = 1$.

§ 26. *Multiply-Periodic Functions and the Theory of Surfaces.*—The theory of algebraic integrals $\int R(x, y)dx$, wherein x, y are connected by a rational equation $f(x, y) = 0$, has developed concurrently with the theory of algebraic curves; in particular the existence of the number p invariant by all birational transformations is one result of an extensive theory in which curves capable of birational correspondence are regarded as equivalent; this point of view has made possible a general theory of what might otherwise have remained a collection of isolated theorems.

In recent years developments have been made which point to a similar unity of conception as possible for surfaces, or indeed for algebraic constructs of any number of dimensions. These developments have been in two directions, at first followed independently, but now happily brought into the most intimate connexion. On the analytical side, E. Picard has considered the possibility of classifying integrals of the form $\int(Rds + Sdy)$, belonging to a surface $f(x, y, z) = 0$, wherein R and S are rational functions of x, y, z, according as they are (1) everywhere finite, (2) have poles, which then lie along curves upon the surface, or (3) have logarithmic infinities, also then lying along curves, and has brought the theory to a high degree of perfection. On the geometrical side A. Clebsch and M. Noether, and more recently the Italian school, have considered the geometrical characteristics of a surface which are unaltered by birational transformation. It was first remarked that for surfaces of order n there are associated surfaces of order $n - 4$, having properties in relation thereto analogous to those of curves of order $n - 3$ for a plane curve of order n; if such a surface $f(x, y, z) = 0$ have a double curve with triple points triple also for the surface, and $\varphi(x, y, z) = 0$ be a surface of order $n - 4$ passing through the double curve, the double integral

$$\int \int \frac{\varphi \, dx \, dy}{\partial f/\partial z}$$

is everywhere finite; and, the most general everywhere finite integral of this form remains invariant in a birational transformation of the surface f, the theorem being capable of generalization to algebraic constructs of any number of dimensions. The number of linearly independent surfaces of order $n - 4$, possessing the requisite particularity in regard to the singular lines and points of the surface, is thus a number invariant by birational transformation, and the equality of these numbers for two surfaces is a necessary condition of their being capable of such transformation. The number of surfaces of order m having the assigned particularity in regard to the singular points and lines of the fundamental surface can be given by a formula for a surface of given singularity; but the value of this formula for $m = n - 4$ is not in all cases equal to the actual number of surfaces of order $n - 4$ with the assigned particularity, and for a cone (or ruled surface) is in fact negative, being the negative of the deficiency of the plane section of the cone. Nevertheless this number for $m = n - 4$ is also found to be invariant for birational transformation. This number, now denoted by p_a, is then a second invariant of birational transformation. The former number, of actual surfaces of order $n - 4$ with the assigned particularity in regard to the singularities of the surface, is now denoted by p_g. The difference $p_g - p_a$, which is never negative, is a most important characteristic of a surface. When it is zero, as in the case of the general surface of order n, and in a vast number of other ordinary cases, the surface is called regular.

On a plane algebraical curve we may consider linear series of sets of points, obtained by the intersection with it of curves $\lambda\varphi + \lambda_1\varphi_1 + \ldots = 0$, wherein $\lambda, \lambda_1, \ldots$ are variable coefficients; such a series consists of the sets of points where a rational function of given poles, belonging to the construct $f(x, y) = 0$, has constant values. And we may consider series of sets of points determined by variable curves whose coefficients are algebraical functions, not necessarily rational functions, of parameters. Similarly on a surface we may consider linear systems of curves, obtained by the intersection with the given surface of variable surfaces $\lambda\varphi + \lambda_1\varphi_1 + \ldots = 0$, and may consider algebraic systems, of which the individual curve is given by variable surfaces whose coefficients are algebraical, not necessarily rational, functions of parameters. Of a linear series upon a plane curve there are two numbers manifestly invariant in birational transformation, the *order*, which is the number of points forming a set of the series, and the *dimension*, which is the number of parameters $\lambda_1/\lambda, \lambda_2/\lambda, \ldots$ entering linearly in the equation of the series. The series is *complete* when it is not contained in a series of the same order but of higher dimension. So for a linear system of curves upon a surface, we have three invariants for birational transformation; the *order*, being in the number of variable intersections of two curves of the system, the *dimension*, being the number of linear parameters $\lambda_1/\lambda, \lambda_2/\lambda, \ldots$ in the equation for the system, and the *deficiency* of the individual curves of the system. Upon any curve of the linear system the other curves of the system define a linear series, called the *characteristic* series; but even when the linear system is complete, that is, not contained in another linear system of the same order and higher dimension, it does not follow that the characteristic series is complete; it may be contained in a series whose dimension is greater by $p_g - p_a$ than its own dimension. When this is so it can be shown that the linear system of curves is contained in an algebraic system whose dimension is greater by $p_g - p_a$ than the dimension of the linear system. The extra $p = p_g - p_a$ variable parameters so entering may be regarded as the independent co-ordinates of an algebraic construct $f(y, x_1, \ldots x_p) = 0$; this construct has the property that its co-ordinates are single valued meromorphic functions of p variables, which are periodic, possessing 2p systems of periods; the p variables are expressible in the forms

$$u_i = \int R_1(x, y) \, dx_1 + \ldots + R_p(x, y) \, dx_p,$$

wherein $R_i(x, y)$ denotes a rational function of $x_1, \ldots x_p$ and y. The original surface has correspondingly p integrals of the form $\int(R \, dx + S \, dy)$, wherein R, S are rational in x, y, z, which are everywhere finite; and it can be shown that it has no other such integrals. From this point of view, then, the number p, $= p_g - p_a$ is, for a surface, analogous to the deficiency of a plane curve; another analogy arises in the comparison of the theorems: for a plane curve of zero deficiency there exists no algebraic series of sets of points which does not consist of sets belonging to a linear series; for a surface for which $p_g - p_a = 0$ there exists no algebraic system of curves not contained in a linear system.

But whereas for a plane curve of deficiency zero, the co-ordinates of the points of the curve are rational functions of a single parameter, it is not necessarily the case that for a surface having $p_g - p_a = 0$ the co-ordinates of the points are rational functions of two parameters; it is necessary that $p_g - p_a = 0$, but this is not sufficient. For surfaces, beside the p_g linearly independent surfaces of order $n - 4$ having a definite particularity at the singularities of the surface, it is useful to consider surfaces of order $k(n - 4)$, also having each a definite particularity at the singularities, the number of these, not containing the original surface as component, which are linearly independent, is denoted by P_k. It can then be stated that a sufficient condition for a surface to be rational consists of the two conditions $p_a = 0$, $P_2 = 0$. More generally it becomes a problem to classify surfaces according to the values of the various numbers which are invariant under birational transformation, and to determine for each the simplest form of surface to which it is birationally equivalent. Thus, for example, the hyperelliptic surface discussed by Humbert, of which the co-ordinates are 329 meromorphic functions of two variables of the simplest kind, with four sets of periods, is characterized by $p_g = 1$, $p_a = -1$; or again, any surface possessing a linear system of curves of which the order exceeds twice the deficiency of the individual curves diminished by two, is reducible by birational transformation to a ruled surface or is a rational surface. But beyond the general statement that much progress has already been made in this direction, of great interest to the student of the theory of functions, nothing further can be added here.

BIBLIOGRAPHY.—The learner will find a lucid introduction to the theory in E. Goursat, *Cours d'analyse mathématique*, t. ii. (Paris, 1905), or, with much greater detail, in A.R. Forsyth, *Theory of Functions of a Complex Variable* (2nd ed., Cambridge, 1900); for logical rigour in the more difficult theorems, he should consult W.F. Osgood, *Lehrbuch der Functionentheorie*, Bd. i. (Leipzig, 1906-1907); for greater precision in regard to the necessary quasi-geometrical axioms, beside the indications attempted here, he should consult W.H. Young, *The Theory of Sets of Points* (Cambridge, 1906), chs. viii.-xiii., and C. Jordan, *Cours d'analyse*, t. i. (Paris, 1893), chs. i., ii.; a comprehensive account of the *Theory of Functions of Real Variables* is by E.W. Hobson (Cambridge, 1907). Of the theory regarded as based after Weierstrass upon the theory of power series, there is J. Harkness and F. Morley, *Introduction to the Theory of Analytic Functions* (London, 1898), an elementary treatise; for the theory of the convergence of series there is also T.J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series* (London, 1908); but the student should consult the collected works of Weierstrass (Berlin, 1894 ff.), and the writings of Mittag-Leffler in the early volumes of the *Acta mathematica*; earlier expositions of the theory of functions on the basis of power series are in C. Méray, *Leçons nouvelles sur l'analyse infinitésimale* (Paris, 1894), and in Lagrange's books on the Theory of Functions. An account of the theory of potential in its applications to the present theory is found in most treatises; in particular consult E. Picard, *Traité d'analyse*, t. ii. (Paris, 1893). For elliptic functions there is an introductory book, P. Appell and E. Lacour, *Principes de la théorie des fonctions elliptiques et applications* (Paris, 1897), beside the treatises of G.H. Halphen, *Traité des fonctions elliptiques et de leurs applications* (three parts, Paris, 1886 ff.), and J. Tannery et J. Molk, *Éléments de la théorie des fonctions elliptiques* (Paris, 1893 ff.); a book, A.G. Greenhill, *The Applications of Elliptic Functions* (London, 1892), shows how the functions enter in problems of many kinds. For modular functions there is an extensive treatise, F. Klein and R. Fricke, *Theorie der elliptischen Modulfunctionen* (Leipzig, 1890); see also the most interesting smaller volume, F. Klein, *Über das Ikosaeder* (Leipzig, 1884) (also obtainable in English). For the theory of Riemann's surface, and algebraic integrals, an interesting introduction is P. Appeil and E. Goursat, *Théorie des fonctions algébriques et de leurs intégrales*; for Abelian functions see also H. Stahl, *Theorie der Abel'schen Functionen* (Leipzig, 1896), and H.F. Baker, *An Introduction to the Theory of Multiply Periodic Functions* (Cambridge, 1907), and H.F. Baker, *Abel's Theorem and the Allied Theory, including the Theory of the Theta Functions* (Cambridge, 1897); for theta functions of one variable a standard work is C.G. Jacobi, *Fundamenta nova, &c.* (Königsberg, 1828); for the general theory of theta functions, consult W. Wirtinger, *Untersuchungen über Theta-Functionen* (Leipzig, 1895). For a history of the theory of algebraic functions consult A. Brill and M. Noether, *Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit, Bericht der deutschen Mathematiker-Vereinigung* (1894); and for a special theory of algebraic functions, K. Hensel and G. Landsberg, *Theorie der algebraischen Function u.s.w.* (Leipzig, 1902). The student will, of course, consult also Riemann's and Weierstrass's *Ges. Werke*. For the applications to geometry in general an important contribution, of permanent value, is E. Picard and G. Simart, *Théorie des fonctions algébriques de deux variables indépendantes* (Paris, 1897-1906). This work contains, as Note v. t. ii. p. 485, a valuable summary by MM. Castelnuovo and Enriques, *Sur quelques résultats nouveaux dans la théorie des surfaces algébriques*, containing many references to the numerous memoirs to be found, for the most part, in the transactions of scientific societies and the mathematical journals of Italy.

Beside the books above enumerated there exists an unlimited number of individual memoirs, often of permanent importance and only imperfectly, or too elaborately, reproduced in the pages of the volumes in which the student will find references to them. The German *Encyclopaedia of Mathematics*, and the Royal Society's *Reference Catalogue of Current Scientific Literature, Pure Mathematics*, published yearly, should also be consulted.

(H. F. BA.)

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1 The word "function" (from Lat. *fungi*, to perform) has many uses, with the fundamental sense of an activity special or proper to an office, business or profession, or to an organ of an animal or plant, the definite work for which the organ is an apparatus. From the use of the word, as in the Italian *funzione*, for a ceremony of the Roman Church, "function" is often employed for a public ceremony of any kind, and loosely of a social entertainment or gathering.

FUNDY, BAY OF, an inlet of the North Atlantic, separating New Brunswick from Nova Scotia. It is 145 m. long and 48 m. wide at the mouth, but gradually narrows towards the head, where it divides into Chignecto Bay to the north, which subdivides into Shepody Bay and Cumberland Basin (the French Beaubassin), and Minas Channel, leading into Minas Basin, to the east and south. Off its western shore opens Passamaquoddy Bay, a magnificent sheet of deep water with good anchorage, receiving the waters of the St Croix river and forming part of the boundary between New Brunswick and the state of Maine, The Bay of Fundy is remarkable for the great rise and fall of the tide, which at the head of the bay has been known to reach 62 ft. In Passamaquoddy Bay the rise and fall is about 25 ft., which gradually increases toward the narrow upper reaches. At spring tides the water in the Bay of Fundy is 19 ft. higher than it is in Bay Verte, in Northumberland Strait, only 15 m. distant. Though the bay is deep, navigation is rendered dangerous by the violence and rapidity of the tide, and in summer by frequent fogs. At low tide, at such points as Moncton or Amherst, only an expanse of red mud can be seen, and the tide rushes in a bore or crest from 3 to 6 ft. in height. Large areas of fertile marshes are situated at the head of the bay, and the remains of a submerged forest show that the land has subsided in the latest geological period at least 40 ft. The bay receives the waters of the St Croix and St John rivers, and has numerous harbours, of which the chief are St Andrews (on Passamaquoddy Bay) and St John in New Brunswick, and Digby and Annapolis (on an inlet known as Annapolis Basin) in Nova Scotia. It was first explored by the Sieur de Monts (d. *c.* 1628) in 1604 and named by him La Baye Française.

FUNERAL RITES, the ceremonies associated with different methods of disposing of the dead. (See also Burial and Burial Acts; Cemetery; and Cremation.) In general we have little record, except in their tombs, of races which, in a past measured not merely by hundreds but by thousands of years, occupied the earth; and exploration of these often furnishes our only clue to the religions, opinions, customs, institutions and arts of long vanished societies. In the case of the great culture folks of antiquity, the Babylonians, Egyptians, Hindus, Persians, Greeks and Romans, we have, besides their monuments, the evidence of their literatures, and so can know nearly as much of their rites as we do of our own. The rites of modern savages not only help us to interpret prehistoric monuments, but explain peculiarities in our own rituals and in those of the culture folks of the past of which the significance was lost or buried under etiological myths. We must not then confine ourselves to the rites of a few leading races, neglecting their less fortunate brethren who have never achieved civilization. It is better to try to classify the rites of all races alike according as they embody certain leading conceptions of death, certain fears, hopes, beliefs entertained about the dead, about their future, and their relations with the living.

The main ideas, then, underlying funeral rites may roughly be enumerated as follows:

1. The pollution or taboo attaching to a corpse.

2. Mourning.

3. The continued life of the dead as evinced in the housing and equipment of the dead, in the furnishing of food for them, and in the orientation and posture assigned to the body.

4. Communion with the dead in a funeral feast and otherwise.

5. Sacrifice for the dead and expiation of their sins.

6. Death witchery.

7. Protection of the dead from ghouls.

8. Fear of ghosts.

1. A dead body is unclean, and the uncleanness extends to things and persons which touch it. Hence the Jewish law (Num. v. 2) enacted that “whoever is unclean by the dead shall be put outside the camp, that they defile not the camp in the midst whereof the Lord dwells.” Such persons were unclean until the even, and might not eat of the holy things unless they bathed their flesh in water. A high priest might on no account “go in to any dead body” (Lev. xxi. 11). Why a corpse is so widely tabooed is not certain; but it is natural to see one reason in the corruption which in warm climates soon sets in. The common experience that where one has died another is likely to do so may also have contributed, though, of course, there was no scientific idea of infection. The old Persian scriptures are full of this taboo. He who has touched a corpse is “powerless in mind, tongue and hand” (*Zend Avesta* in *Sacred Books of the East*, pt. i. p. 120), and the paralysis is inflicted by the innumerable *drugs* or evil spirits which invest a corpse. Fire and earth, being alike creations of the good and pure god Ahuramazda, a body must not be burned or buried; and so the ancient Persians and their 330 descendants the Parsees build Dakmas or “towers of silence” on hill-tops far from human habitations. Inside these the corpses are laid on a flagged terrace which drains into a central pit. Twice a year the bones, picked clean by dogs and birds of prey, are collected in the pit, and when it is full another tower is built. In ancient times perhaps the bodies of the magi or priests alone were exposed at such expense; the common folk were covered with wax and laid in the earth, the wax saving the earth from pollution. In Rome and Greece the corpse was buried by night, lest it should pollute the sunlight; and a trough of water was set at the door of the house of death that men might purify themselves when they came out, before mixing in general society. Priests and magistrates in Rome might not meet or look on a corpse, for they were thereby rendered unclean and incapable of fulfilling their official duties without undergoing troublesome rites of purification. At a Roman funeral, when the remains had been laid in the tomb, all present were sprinkled with lustral water from a branch of olive or laurel called *aspergillum*; and when they had gone home they were asperged afresh and stepped over a fire. The house was also swept out with a broom, probably lest the ghost of the dead should be lying about the floor. Many races, to avoid pollution, destroy the house and property of the deceased. Thus the Navahos pull down the hut in which he died, leaving its ruins on the ground; but if it be an expensive hut, a shanty is extemporized alongside, into which the dying man is transferred before death. No one will use the timbers of a hut so ruined. A burial custom of the Solomon Islands, noted by R.H. Codrington (*The Melanesians*, p. 255), may be dictated by the same scruple. There “the mourners having hung up a dead man’s arms on his house make great lamentations; all remains afterwards untouched, the house goes to ruin, mantled, as time goes on, with the vines of the growing yams, a picturesque and indeed, perhaps, a touching sight; for these things are not set up that they may in a ghostly manner accompany their former owner.” H. Oldenberg (*Religion des Veda*, p. 426) describes how Hindus shave themselves and cut off their nails after a death, at the same time that they wash, renew the hearth fire, and furnish themselves with new vessels. For the hair and nails may harbour pollution, just as the medieval Greeks believed that evil spirits could lurk in a man’s beard (Leo Allatius, *De opinionibus quorundam Graecorum*). The dead man’s body is shorn and the nails cut for a kindred reason; for it must be purified as much as can be before it is burned as an offering on the pyre and before he enters on a new sphere of existence.

2. We are accustomed to regard mourning costume as primarily an outward sign of our grief. Originally, however, the special garb seems to have been intended to warn the general public that persons so attired were unclean. In ancient Rome mourners stayed at home and avoided all feasts and amusements; laying aside gold, purple and jewels, they wore black dresses called *lugubria* or even skins. They cut neither hair nor beard, nor lighted fire. Under the emperors women began to wear white. On the west coast of Africa negroes wear white, on the Gold Coast red. The Chinese wear hemp, which is cheap, for mourning dress must as a rule be destroyed when the season of grief is past to get rid of the taboo. Among the Aruntas of Australia the wives of a dead man smear themselves with white pipe-clay until the last ceremonies are finished, sometimes adding ashes—this not to conceal themselves from the ghost (which may partly be the aim of some mourning costumes), but to show the ghost that they are duly sorrowing for their loss. These widows must not talk except on their hands for a whole year. “Among the Maoris,” says Frazer (*Golden Bough*, i. 323), “anyone who had handled a corpse, helped to convey it to the grave, or touched a dead man’s bones; was cut off from all intercourse and almost all communication with mankind. He could not enter any house, or come into contact with any person or thing, without utterly bedevilling them. He might not even touch food with his hands, which had become so frightfully tabooed or unclean as to be quite useless. Food would be set for him on the ground, and he would then sit or kneel down, and, with his hands carefully held behind his back, would gnaw at it as best he could.” Often a degraded outcast was kept in a village to feed mourners. Such a taboo is strictly similar to those which surround a sacred chief or his property, a menstruous woman or a homicide, rendering them dangerous to themselves and to all who approach them.

3. Primitive folk cannot conceive of a man’s soul surviving apart from his body, nor of another life as differing from this, and the dead must continue to enjoy what they had here. Accordingly the Patagonians kill horses at the grave that the dead may ride to *Alhuemapu*, or country of the dead. After a year they collect a chief’s bones, arrange them, tie them together and dress them in his best garments with beads and feathers. Then they lay him with his weapons in a square pit, round which dead horses are placed set upright on their feet by stakes. As late as 1781 in Poland F. Casimir’s horse was slain and buried with him. In the Caucasus a Christian lady’s jewels are buried with her. The Hindus used to burn a man’s widow on his pyre, because he could not do without her; and St Boniface commends the self-sacrifice of the Wend widows who in his day burned themselves alive on their husbands’ pyres.

The tumuli met with all over the north of Europe (in the Orkneys alone 2000 remain) are regular houses of the dead, models of those they occupied in life. The greater the dignity of the deceased, the loftier was his barrow. Silbury hill is 170 ft. high; the tomb of Alyattes, father of Croesus, was a fourth of a league round; the Pyramids are still the largest buildings in existence; at Oberea in Tahiti is a barrow 267 ft. long, 87 wide and 44 high. Some Eskimo just leave a dead man’s body in his house, and shut it up, often leaving by his side a dog’s head to guide him on his last journey, along with his tools and kayak. The Sea Dyaks set a chief adrift in his war canoe with his weapons. So in Norse story Hake “was laid wounded on a ship with the dead men and arms; the ship was taken out to sea and set on fire.” The Viking was regularly buried in his ship or boat under a great mound. He sailed after death to Valhalla. In the ship was laid a stone as anchor and the tools, clothes, weapons and treasures of the dead. The Egyptians, whose land was the gift of the river Nile, equally believed that the dead crossed over water, and fashioned the hearse in the form of a boat. Hence perhaps was derived the Greek myth of Charon and the Styx, and the custom, which still survives in parts of Europe, of placing a coin in the mouth of the dead with which to pay the ferryman. The Egyptians placed in the tomb books of a kind to guide the dead to the next world. The Copts in a later age did the same, and to this custom we owe the recovery in Egypt of much ancient literature. The Armenians till lately buried with a priest his missal or gospel.

In Egyptian entombments of the XIIth to the XIVth dynasties were added above the sepulchres what Professor Petrie terms soul-houses, viz. small models of houses furnished with couch and table, &c., for the use of the *ka* or double whenever it might wish to come above ground and partake of meats and drinks. They recall, in point of size, the hut-urns of the Etruscans, but the latter had another use, for they contain incinerated remains. Etruscan tombs, like those of Egypt and Asia Minor, were made to resemble the dwelling-houses of the living, and furnished with coffered ceilings, panelled walls, couches, stools, easy chairs with footstools attached, all hewn out of the living rock (Dennis, *Cities and Cemeteries of Etruria*, vol i. p. lxx.).

Of the old Peruvian mummies in the Kircherian Museum at Rome, several are of women with babies in their arms, whence it is evident that a mother had her suckling buried with her; it would console her in the next world and could hardly survive her in this. The practice of burying ornaments, tools and weapons with the dead characterizes the inhumations of the Quaternary epoch, as if in that dim and remote age death was already regarded as the portal of another life closely resembling this. The cups, tools, weapons, ornaments and other articles deposited with the dead are often carefully broken or turned upside down and inside out; for the soul or *manes* of objects is liberated by such fracture or inversion and so passes into the dead man’s 331
use and possession. For the same reason where the dead are burned, their properties are committed to the flames. The ghost of the warrior has a ghostly sword and buckler to fight with and a ghostly cup to drink from, and he is also nourished by the impalpable odour and reek of the animal victims sacrificed over his grave. Instead of valuable objects cheap images and models are often substituted; and why not, if the mere ghosts of the things are all that the wraith can enjoy? Thus Marco Polo (ii. 76) describes how in the land of Kinsay (Hang-chau) “the friends and relations make a great mourning for the deceased, and clothe themselves in hempen garments, and follow the corpse, playing on a variety of instruments and singing hymns to their idols. And when they come to the burning place they take representations of things cut out of parchment, such as caparisoned horses, male and female slaves, camels, armour, suits of cloth of gold (and money), in great quantities, and these things they put on the fire along with the corpse so that they are all burned with it. And they tell you that the dead man shall have all these slaves and animals of which the effigies are burned, alive in flesh and blood, and the money in gold, at his disposal in the next world; and that the instruments which they have caused to be played at his funeral, and the idol hymns that have been chaunted shall also be produced again to welcome him in the next world.” The manufacture of such paper *simulacra* for consumption at funerals is still an important industry in Chinese cities. The ancient Egyptians,

assured that a man’s *ka* or double shall revivify his body, took pains to guard the flesh from corruption, steeping the corpse in natron and stuffing it with spices. A body so prepared is called a mummy (*q.v.*), and the custom was already of a hoary antiquity in 3200 B.C., when the oldest dated mummy we have was made. The bowels, removed in the process, were placed in jars over the corpse in the tomb, together with writing tablets, books, musical instruments, &c., of the dead. Cemeteries also remain full of mummies of crocodiles, cats, fish, cows and other sacred animals. The Greeks settled in Egypt learned to mummify their dead, but the custom was abhorrent to the Jews, although the Christian belief in the resurrection of the flesh must have been formed to a large extent under Egyptian influence. Half the superiority of the Jewish to other ancient religions lay in this, that it prescribed no funeral rites other than the simplest inhumation.

The dead all over the world and from remote antiquity have been laid not anyhow in the earth, but with the feet and face towards the region in which their future will be spent; the Samoans and Fijians towards the far west whither their souls have preceded them; the Guarayos with head turned eastwards because their god Tamoi has in that quarter “his happy hunting grounds where the dead will meet again” (Tylor, *Prim. Cult.* ii. 422). The legend is that Christ was buried with His head to the west, and the church follows the custom, more ancient than itself, of laying the dead looking to the East, because that is the attitude of prayer, and because at the last trump they will hurry eastwards. So in Eusebius (*Hist. Eccl.* 430.19) a martyr explains to his pagan judge that the heavenly Jerusalem, the fatherland of the pious, lay exactly in the east at the rising place of the sun. Where the body is laid out straight it is difficult to discern the presence of any other idea than that it is at rest. In Scandinavian barrows, *e.g.* in the one opened at Goldhavn in 1830, the skeletons have been found seated on a low stone bench round the wall of the grave chamber facing its opening, which always looks south or east, never north. Here the dead were continuing the drinking bouts they enjoyed on earth.

The Peruvians mummified their dead and placed them jointed and huddled up with knees to chin, looking toward the sunset, with the hands held before the face. In the oldest prehistoric tombs along the Nile the bodies are doubled up in the same position. It would seem as if in these and numerous other similar cases the dead were deliberately given in their graves the attitude of a foetus in the womb, and, as Dr Budge remarks (*Egyptian Ideas of the Future Life*, London, 1899, p. 162), “we may perhaps be justified in seeing in this custom the symbol of a hope that, as the child is born from this position into the world, so might the deceased be born into the life beyond the grave.” The late Quaternary skeletons of the Mentone cave were laid in a layer of ferrugineous earth specially laid down for them, and have contracted a red colour therefrom. Many other prehistoric skeletons found in Italy have a reddish colour, perhaps for the same reason, or because, as often to-day, the bones were stripped of flesh and painted. Ambrose relates that the skeletons of the martyrs Gervasius and Protasius, which he found and deposited A.D. 386 under the altar of his new basilica in Milan, were *mirae magnitudinis ut prisca aetas ferebat*, and were also coloured red. He imagined the red to be the remains of the martyrs’ blood! *Hic sanguis clamat coloris indicio.* Salomon Reinach has rightly divined that what Ambrose really hit upon was a prehistoric tomb. Red earth was probably chosen as a medium in which to lay a corpse because demons flee from red. Sacred trees and stones are painted red, and for the most solemn of their rites savages bedaub themselves with red clay. It is a favourite taboo colour.

4. A feast is an essential feature of every primitive funeral, and in the Irish “wake” it still survives. A dead man’s soul or double has to be fed at the tomb itself, perhaps to keep it from prowling about the homes of the survivors in search of victuals; and such food must also be supplied to the dead at stated intervals for months or years. Many races leave a narrow passage or tube open down to the cavity in which the corpse lies, and through it pour down drinks for the dead. Traces of such tubes are visible in the prehistoric tombs of the British Isles. However, such provision of food is not properly a funeral feast unless the survivors participate. In the Eastern churches and in Russia the departed are thus fed on the ninth, twelfth and fortieth days from death. “Ye appease the shades of the dead with wine and meals,” was the charge levelled at the Catholics by the 4th-century Manichaeans, and it has hardly ceased to be true even now after the lapse of sixteen centuries. The funeral feast proper, however, is either a meal of communion with or in the dead, which accompanies interment, or a banquet off the flesh of victims slain in atonement of the dead man’s sins. Some anthropologists see in the common meal held at the grave “the pledge and witness of the unity of the kin, the chief means, if not of making, at least of repairing and renewing it.”[1] The flesh provided at these banquets is occasionally that of the dead man himself; Herodotus and Strabo in antiquity relate this of several half-civilized races in the East and West, and a similar story is told by Marco Polo of certain Tatars. Nor among modern savages are funeral feasts off the flesh of the dead unknown, and they seem to be intended to effect and renew a sacramental union or kinship of the living with the dead. The Uaupes in the Amazons incinerate a corpse a month after death, pound up the ashes, and mix them with their fermented drink. They believe that the virtues of the dead will thus be passed on to his survivors. The life of the tribe is kept inside the tribe and not lost. Such cannibal sacraments, however, are rare, and, except in a very few cases, the evidence for them weak. The slaying and eating of animal victims, however, at the tomb is universal and bears several meanings, separately or all at once. The animals may be slain in order that their ghosts may accompany the deceased in his new life. This significance we have already dwelt upon. Or it is believed that the shade feeds upon them, as the shades came up from Hades and lapped up out of a trench the blood of the animals slain by Ulysses. The survivors by eating the flesh of a victim, whose blood and soul the dead thus consume, sacramentally confirm the mystic tie of blood kinship with the dead. Or lastly, the victim may be offered for the sins of the dead. His sins are even supposed to be transferred into it and eaten by the priest. Such expiatory sacrifices of animals for the dead survive in the Christian churches of Armenia, Syria and of the East generally. Their vicarious character is emphasized in the prayers which accompany them, but the popular understanding of them probably combines all the meanings above enumerated. It has been suggested by Robertson Smith (*Religion of the Semites*, 336) that the world-wide customs of tearing the hair, rending 332
the garments, and cutting and wounding the body were originally intended to establish a life-bond between the dead and the living. The survivors, he argues, in leaving portions of their hair and garments, and yet more by causing their own blood to stream over the corpse from self-inflicted wounds, by cutting off a finger and throwing it into the grave, leave what is eminently their own with the dead, so drawing closer their tie with him. Conversely, many savages daub themselves with the blood and other effluences of their dead kinsmen, and explain their custom by saying that in this way a portion of the dead is incorporated in themselves. Often the survivors, especially the widows, attach the bones or part of them to their persons and wear them, or at

least keep them in their houses. The retention of the locks of the deceased and of parts of his dress is equally common. There is also another side to such customs. Having in their possession bits of the dead, and being so far in communion with him, the survivors are surer of his friendship. They have ensured themselves against ghosts who are apt to be by nature envious and mischievous. But whatever their original significance, the tearing of cheeks and hair and garments and cutting with knives are mostly expressions of real sorrow, and, as Robertson Smith remarks, of deprecation and supplication to an angry god or spirit. It must not be supposed that the savage or ancient man feels less than ourselves the poignancy of loss.

6. Death-witchery has close parallels in the witch and heretic hunts of the Christians, but, happily for us, only flourishes to-day among savages. Sixty % of the deaths which occur in West Africa are, according to Miss Mary Kingsley—a credible witness—believed to be due to witchcraft and sorcery. The blacks regard old age or effusion of blood as the sole legitimate causes of death. All ordinary diseases are in their opinion due to private magic on the part of neighbours, just as a widespread epidemic marks the active hatred "of some great outraged nature spirit, not of a mere human dabbler in devils."[2] Similarly in Christian countries an epidemic is set down to the wrath of a God offended by the presence of Jews, Arians and other heretics. The duty of an African witch-doctor is to find out who bewitched the deceased, just as it was of an inquisitor to discover the heretic. Every African post-mortem accordingly involves the murder of the person or persons who bewitched the dead man and caused him to die. The death-rate by these means is nearly doubled; but, since the use of poison against an obnoxious neighbour is common, the right person is occasionally executed. It is also well for neighbours not to quarrel, for, if they do and one of them dies of smallpox, the other is likely to be slain as a witch, and his lungs, liver and spleen impaled on a pole at the entrance of the village. It is the same case with the Australian blacks: "no such thing as natural death is realized by the native; a man who dies has of necessity been killed by some other man, or perhaps even by a woman, and sooner or later that man or woman will be attacked. In the normal condition of the tribe every death meant the killing of another individual."[3]

7. Lastly, a primitive interment guards against the double risk of the ghost haunting the living and of ghouls or vampires taking possession of the corpse. The latter end is likely to be achieved if the body is cremated, for then there is no nidus to harbour the demon; but whether, in the remote antiquity to which belong many barrows containing incinerated remains, this motive worked, cannot be ascertained. The Indo-European race seems to have cremated at an early epoch, perhaps before the several races of East and West separated. In Christian funeral rites many prayers are for the protection of the body from violation by vampires, and it would seem as if such a motive dictated the architectural solidity of some ancient tombs. Christian graves were for protection regularly sealed with the cross; and the following is a characteristic prayer from the old Armenian rite for the burial of a layman:

> "Preserve, Almighty Lord, this man's spirit with all saints and with all lovers of Thy holy name. And do Thou seal and guard the sepulchre of Thy servant, Thou who shuttest up the depths and sealest them with Thy almighty right hand ... so let the seal of Thy Lordship abide unmoved upon this man's dwelling-place and upon the shrine which guards Thy servant. And *let not any filthy and unclean devil dare to approach him, such as assail the body and souls of the heathen*, who possess not the birth of the holy font, and have not the dread seal laid upon their graves."

A terrible and revolting picture of the superstitious belief in ghouls which violate Christian tombs is given by Leo Allatius (who held it) in his tract *De opinionibus quorundam Graecorum* (Paris, 1646). It was probably the fear of such demonic assaults on the dead that inspired the insanitary custom of burying the dead under the floors of churches, and as near as possible to the altar. In the Greek Church this practice was happily forbidden by the code of Justinian as well as by the older law in the case of churches consecrated with *Encaenia* and deposition of relics. In the Armenian Church the same rule holds, and Ephrem Syrus in his testament particularly forbade his body to be laid within a church. Such prohibitions, however, are a witness to the tendency in question.

The custom of lighting candles round a dead body and watching at its side all night was originally due to the belief that a corpse, like a person asleep, is specially liable to the assaults of demons. The practice of tolling a bell at death must have had a similar origin, for it was a common medieval belief that the sound of a consecrated bell drives off the demons which when a man dies gather near in the air to waylay his fleeting soul. For a like reason the consecrated bread of the Eucharist was often buried with believers, and St Basil is said to have specially consecrated a Host to be placed in his coffin.

8. Some of the rites described under the previous heads may be really inspired by the fear of the dead haunting the living, but it must be kept in mind that the taboo attaching to a dead body is one thing and fear of a ghost another. A corpse is buried or burned, or scaffolded on a tree, a tower or a house-top, in order to get it out of the way and shield society from the dangerous infection of its taboo; but ghosts *quâ* ghosts need not be feared and a kinsman's ghost usually is not. On the contrary, it is fed and consoled with everything it needs, is asked not to go away but to stay, is in a thousand ways assured of the sorrow and sympathy of the survivors. Even if the body be eaten, it is merely to keep the soul of the deceased inside the circle of kinsmen, and Strabo asserts that the ancient Irish and Massagetae regarded it as a high honour to be so consumed by relatives. In Santa Cruz in Melanesia they keep the bones for arrow heads and store a skull in a box and set food before it "saying that this is the man himself" (R.H. Codrington, *The Melanesians*, p. 264), or the skull and jaw bone are kept and "are called *mangite*, which are *saka*, hot with spiritual power, and by means of which the help of the *lio'a*, the powerful ghost of the man whose relics these are, can be obtained" (*ibid.* p. 262). Here we have the savage analogue to Christian relics. So the Australian natives make pointing sticks out of the small bones of the arm, with which to bewitch enemies.

We may conclude then that in the most primitive societies, where blood-kinship is the only social tie and root of social custom it is the shades, not of kinsmen, but of strangers, who as such are enemies, that are dangerous and uncanny. In more developed societies, however, all ghosts alike are held to be so; and if a ghost walks it is because its body has not been properly interred or because its owner was a malefactor. Still, even allowing for this, it remains true that for a friendly ghost the proper place is the grave and not the homes of the living, and accordingly the Aruntas with cries of *Wah! Wah!* with wearing of fantastic head-dresses, wild dancing and beating of the air with hands and weapons “drive the spirit away from the old camp which it is supposed to haunt,” and which has been set fire to, and hunt it at a run into the grave prepared, and there stamp it down into the earth. “The loud shouting of the men and women shows him that they do not wish to be frightened by him in his present state, and that they will be angry with him if he does not rest.” (Spencer and Gillen, *Native Tribes of Central Australia*, p. 508). In Mesopotamia cemeteries have been discovered where the sepulchral jars were set upside down, clearly by way of hindering the ghosts from escaping into the upper world. In the Dublin museum we see specimens of ancient Celtic tombs showing the same peculiarity. For a like reason perhaps the name of the dead must among the Aruntas not be uttered, nor the grave approached, by certain classes of kinsmen. The same repugnance to naming the dead exists all over the world, and leads survivors who share the dead man’s name to adopt another, at least for a time. If the dead man’s name was that of a plant, tree, animal or stream, that too is changed. Here is a potent cause of linguistic change, that also renders any historical tradition impossible. The survivors seem to fear that the ghost will come when he hears his name called; but it also hangs together with the taboo which hedges round the dead as it does kings, chieftains and priests. 333

Authorities.—B. Spencer and F.J. Gillen, *The Native Tribes of Central Australia* (London, 1899); F.B. Jevons, *Introduction to History of Religion* (London, 1896); E.S. Hartland, *The Legend of Perseus*, vol. ii.; J.G. Frazer, *The Golden Bough* (London, 1900); L.W. Faraday, “Custom and Belief in the Icelandic Sagas,” in *Folk-lore*, vol. xvii. No. 4; E.B. Tylor, *Primitive Culture* (London, 1903); E.A. W. Budge, *The Mummy* (Cambridge, 1893); C. Royer, “Les Rites funéraires aux époques préhistoriques,” *Revue d’anthropologie* (1876); Forrer, *Über die Totenbestattung bei den Pfahlbauern* (Ausland, 1885); J. Lubbock, *Origin of Civilization* (London, 1875) and *Prehistoric Times* (London, 1865); L.A. Muratori, “De antiquis Christianorum sepulchris,” *Anecd. Graeca* (Padua, 1709); Onaphr. Panvinius, *De ritu sepeliendi mortuos apua veteres Christianos*, reprinted in Volbeding’s *Thesaurus* (Leipzig, 1841).

(F. C. C.)

1 E.S. Hartland, *Legend of Perseus* (1895), ii. 278.

2 Mary Kingsley, *West African Studies* (1901), p. 178.

3 B. Spencer and F.J. Gillen, *The Native Tribes of Central Australia* (1899), p. 48.

FUNGI (pl. of Lat. *fungus*, a mushroom), the botanical name covering in the broad sense all the lower cellular Cryptogams devoid of chlorophyll, which arise from spores, and the thallus of which is either unicellular or composed of branched or unbranched tubes or cell-filaments (hyphae) with apical growth, or of more or less complex wefted sheets or tissue-like masses of such (mycelium). The latter may in certain cases attain large dimensions, and even undergo cell-divisions in their interior, resulting in the development of true tissues. The spores, which may be uni- or multicellular, are either abstricted free from the ends of hyphae (acrogenous), or formed from segments in their course (*chlamydospores*) or from protoplasm in their interior (endogenous). The want of chlorophyll restricts their mode of life—which is rarely aquatic—since they are therefore unable to decompose the carbon dioxide of the atmosphere, and renders them dependent on other plants or (rarely) animals for their carbonaceous food-materials. These they obtain usually in the form of carbohydrates from the dead remains of other organisms, or in this or other forms from the living cells of their hosts; in the former case they are termed saprophytes, in the latter parasites. While some moulds (*Penicillium*, *Aspergillus*) can utilize almost any organic food-materials, other fungi are more restricted in their choice—*e.g.* insect-parasites, horn- and feather-destroying fungi and parasites generally. It was formerly the custom to include with the Fungi the Schizomycetes or Bacteria, and the Myxomycetes or Mycetozoa; but the peculiar mode of growth and division, the cilia, spores and other peculiarities of the former, and the emission of naked amoeboid masses of protoplasm, which creep and fuse to streaming plasmodia, with special modes of nutrition and spore-formation of the latter, have led to their separation as groups of organisms independent of the true Fungi. On the other hand, lichens, previously regarded as autonomous plants, are now known to be dual organisms—fungi symbiotic with algae.

The number of species in 1889 was estimated by Saccardo at about 32,000, but of these 8500 were so-called *Fungi imperfecti—i.e.* forms of which we only know certain stages, such as conidia, pycnidia, &c., and which there are reasons for regarding as merely the corresponding stages of higher forms. Saccardo also included about 400 species of Myxomycetes and 650 of Schizomycetes. Allowing for these and for the cases, undoubtedly not few, where one and the same fungus has been described under different names, we obtain Schroeter’s estimate (in 1892) of 20,000 species. In illustration of the very different estimates that have been made, however, may be mentioned that of De Bary in 1872 of 150,000 species, and that of Cooke in 1895 of 40,000, and Massee in 1899 of over 50,000 species, the fact being that no sufficient data are as yet to hand for any accurate census. As regards their geographical distribution, fungi, like flowering plants, have no doubt their centres of origin and of dispersal; but we must not forget that every exchange of wood, wheat, fruits, plants, animals, or other commodities involves transmission of fungi from one country to another; while the migrations of birds and other animals, currents of air and water, and so forth, are particularly efficacious in transmitting these minute organisms. Against this, of course, it may be argued that parasitic forms can only go where their hosts grow, as is proved to be the case by records concerning the introduction of *Puccinia malvacearum*, *Peronospora viticola*, *Hemileia vastatrix*, &c. Some fungi—*e.g.* moulds and yeasts—appear to be distributed all over the earth. That the north temperate regions appear richest in fungi may be due only to the fact that North America and Europe have been much more thoroughly investigated than other countries; it is certain that the tropics are the home of very numerous species. Again, the accuracy of the statement that the fleshy Agaricini, Polyporei, *Pezizae*, &c., are relatively rarer in the tropics may depend on the fact that they are more difficult to collect and remit for identification than the abundantly recorded woody and coriaceous forms of these regions. When we remember that many parts of the world are practically unexplored as regards fungi, and that new species are constantly being discovered in the United States, Australia and northern Europe—the best explored of all—it is clear that no very accurate census of fungi can as yet be made, and no generalizations of value as to their geographical distribution are possible.

The existence of fossil fungi is undoubted, though very few of the identifications can be relied on as regards species or genera. They extend back beyond the Carboniferous, where they occur as hyphae, &c., preserved in the fossil woods, but the best specimens are probably those in amber and in siliceous petrifactions of more recent origin.

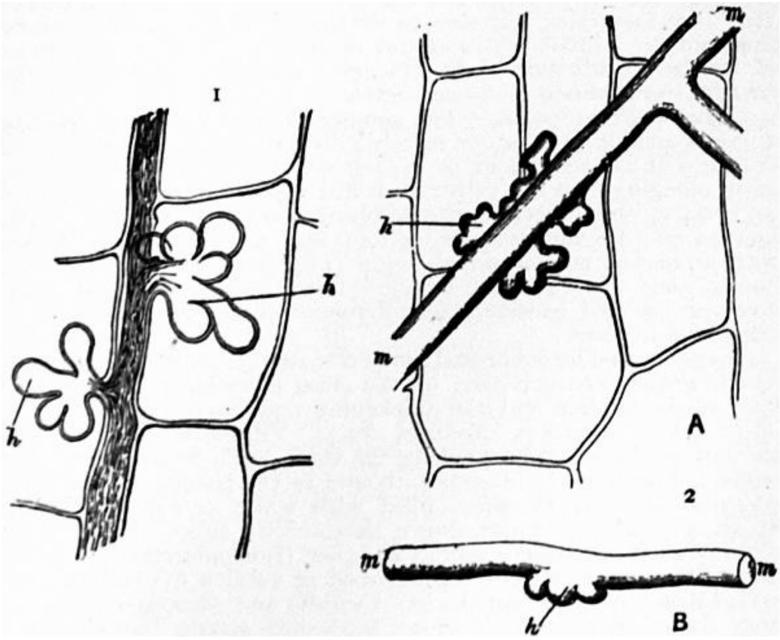

Fig. 1.—1, *Peronospora parasitica* (De Bary). Mycelium with haustoria (*h*); 2, *Erysiphe*; A and B, mycelium (*m*), with haustoria (*h*). (After De Bary.)

Organs.—Individual hyphae or their branches often exhibit specializations of form. In many Basidiomycetes minute branches arise below the septa; their tips curve over the outside of the latter, and fuse with the cell above just beyond it, forming a *clamp-connexion*. Many parasitic hyphae put out minute lateral branches, which pierce the cell-wall of the host and form a peg-like (*Trichosphaeria*), sessile (*Cystopus*), or stalked (*Hemileia*), knot-like, or a more or less branched (*Peronospora*) or coiled (*Protomyces*) haustorium. In *Rhizopus* certain hyphae creep horizontally on the surface of the substratum, and then anchor their tips to it by means of a tuft of short branches (*appressorium*), the walls of which soften and gum themselves to it, then 334
another branch shoots out from the tuft and repeats the process, like a strawberry-runner. Appressoria are also formed by some parasitic fungi, as a minute flattening of the tip of a very short branch (*Erysiphe*), or the swollen end of any hypha which comes in contact with the surface of the host (*Piptocephalis*, *Syncephalis*), haustoria piercing in each case the cell-wall below. In *Botrytis* the appressoria assume the form of dense tassels of short branches. In *Arthrobotrys* side-branches of the mycelium sling themselves around the host (*Tylenchus*) much as tendrils round a support.

Many fungi (*Phallus*, *Agaricus*, *Fumago*, &c.) when strongly growing put out ribbon-like or cylindrical cords, or sheet-like mycelial plates of numerous parallel hyphae, all growing together equally, and fusing by anastomoses, and in this way extend long distances in the soil, or over the surfaces of leaves, branches, &c. These mycelial strands may be white and tender, or the outer hyphae may be hard and black, and very often the resemblance of the subterranean forms to a root is so marked that they are termed rhizomorphs. The outermost hyphae may even put forth thinner hyphae, radiating into the soil like root-hairs, and the convergent tips may be closely appressed and so divided by septa as to resemble the root-apex of a higher plant (*Armillaria mellea*).

Sclerotia.—Fungi, like other plants, are often found to store up large quantities of reserve materials (oil, glycogen, carbohydrates, &c.) in special parts of their vegetative tissues, where they lie accumulated between a period of active assimilation and one of renewed activity, forming reserves to be consumed particularly during the formation of large fructifications. These reserve stores may be packed away in single hyphae or in swollen cells, but the hyphae containing them are often gathered into thick cords or mycelial strands (*Phallus*, mushroom, &c.), or flattened and anastomosing ribbons and plates, often containing several kinds of hyphae (*Merulius lacrymans*). In other cases the strands undergo differentiation into an outer layer with blackened, hardened cell-walls and a core of ordinary hyphae, and are then termed rhizomorphs (*Armillaria mellea*), capable not only of extending the fungus in the soil, like roots, but also of lying dormant, protected by the outer casing. Such aggregations of hyphae frequently become knotted up into dense masses of interwoven and closely packed hyphae, varying in size from that of a pin’s head or a pea (*Peziza*, *Coprinus*) to that of a man’s fist or head, and weighing 10 to 25 ℔ or more (*Polyporus Mylittae*, *P. tumulosus*, *Lentinus Woermanni*, *P. Sapurema*, &c.). The interwoven hyphae fuse and branch copiously, filling up all interstices. They also undergo cutting up by numerous septa into short cells, and these often divide again in all planes, so that a pseudoparenchyma results, the walls of which may be thickened and swollen internally, or hardened and black on the exterior. In many cases the swollen cell-walls serve as reserves, and sometimes the substance is so thickly deposited in strata as to obliterate the lumen, and the hyphae become nodular (*Polyporus sacer*, *P. rhinoceros*, *Lentinus Woermanni*). The various sclerotia, if kept moist, give rise to the fructifications of the fungi concerned, much as a potato tuber does to a potato plant, and in the same way the reserve materials are consumed. They are principally Polyporei, Agaricini, Pezizae; none are known among the Phycomycetes, Uredineae or Ustilagineae. The functions of mycelial strands, rhizomorphs and sclerotia are not only to collect and store materials, but also to extend the fungus, and in many cases similar strands act as organs of attack. The same functions of storage in advance of fructification are also exercised by the stromata so common in Ascomycetes.

Tissue Differentiations.—The simpler mycelia consist of hyphae all alike and thin-walled, or merely differing in the diameter of the branches of various orders, or in their relations to the environment, some plunging into the substratum like roots, others remaining on its surface, and others (aerial hyphae) rising into the air. Such hyphae may be multicellular, or they may consist of simple tubes with numerous nuclei and no septa (*Phycomycetes*), and are then non-cellular. In the more complex tissue-bodies of higher fungi, however, we find considerable differences in the various layers or strands of hyphae.

An epidermis-like or cortical protective outer layer is very common, and is usually characterized by the close septation of the densely interwoven hyphae and the thickening and dark colour of their outer walls (sclerotia, *Xylaria*, &c.). Fibre-like hyphae with the lumen almost obliterated by the thick walls occur in mycelial cords (*Merulius*). Latex-tubes abound in the tissues of *Lactarius*, *Stereum*, *Mycena*, *Fistulina*, filled with white or coloured milky fluids, and Istvanffvi has shown that similar tubes with fluid or oily contents are widely spread in other Hymenomycetes. Sometimes fatty oil or watery sap is found in swollen hyphal ends, or such tubes contain coloured sap. Cystidia and paraphyses may be also classed here. In *Merulius lacrymans* Hartig has observed thin-walled hyphae with large lumina, the septa of which are perforated like those of sieve-tubes.

As regards its composition, the cell-wall of fungi exhibits variations of the same kind as those met with in higher plants. While the fundamental constituent is a cellulose in many Mucorini and other Phycomycetes, in others bodies like pectose, callose, &c., commonly occur, and Wisselingh’s researches show that chitin, a gluco-proteid common in animals, forms the main constituent in many cases, and is probably deposited directly as such, though, like the other substances, it may be mixed with cellulose. As in other cell-walls, so here the older membranes may be altered by deposits of various substances, such as resin, calcium oxalate, colouring matters; or more profoundly altered throughout, or in definite layers, by lignification, suberization (*Trametes*, *Daedalea*), or swelling to a gelatinous mucilage (*Tremella*, *Gymnosporangium*), while cutinization of the outer layers is common. One of the most striking alterations of cell-walls is that termed *carbonization*, in which the substance gradually turns black, hard and brittle, as if charred—*e.g. Xylaria*, *Ustulina*, some sclerotia. At the other extreme the cell-walls of many lichen-fungi are soft and colourless, but turn blue in iodine, as does starch. The young cell-wall is always tenuous and flexible, and may remain so throughout, but in many cases thickenings and structural differentiations, as well as the changes referred to above, alter the primary wall considerably. Such thickening may be localized, and *pits* (*e.g. Uredospores*, septa of Basidiomycetes), *spirals*, *reticulations*, *rings*, &c. (capillitium fibres of *Podaxon*, *Calostoma*, *Battarrea*), occur as in the vessels of higher plants, while sculptured networks, pittings and so forth are as common on fungus-spores as they are on pollen grains.

Cell-Contents.—The cells of fungi, in addition to protoplasm, nuclei and sap-vacuoles, like other vegetable cells, contain formed and amorphous bodies of various kinds. Among those directly visible to the microscope are oil drops, often coloured (*Uredineae*) crystals of calcium oxalate (*Phallus*, *Russula*), proteid crystals (*Mucor*, *Pilobolus*, &c.) and resin (Polyporei). The oidia of Erysipheae contain fibrosin bodies and the hyphae of Saprolegnieae cellulin bodies, but starch apparently never occurs. Invisible to the microscope, but rendered visible by reagents, are glycogen, *Mucor*, Ascomycetes, yeast, &c. In addition to these cell-contents we have good indirect evidence of the existence of large series of other bodies, such as proteids, carbohydrates, organic acids, alkaloids, enzymes, &c. These must not be confounded with the numerous substances obtained by chemical analysis of masses of the fungus, as there is often no proof of the manner of occurrence of such bodies, though we may conclude with a good show of probability that some of them also exist preformed in the living cell. Such are sugars (glucose, mannite, &c.), acids (acetic, citric and a whole series of lichen-acids), ethereal oils and resinous bodies, often combined with the intense colours of fungi and lichens, and a number of powerful alkaloid poisons, such as muscarin (*Amanita*), ergotin (*Claviceps*), &c.

Among the enzymes already extracted from fungi are *invertases* (yeasts, moulds, &c.), which split cane-sugar and other complex sugars with hydrolysis into simpler sugars such as dextrose and levulose; *diastases*, which convert starches into sugars (*Aspergillus*, &c.); *cytases*, which dissolve cellulose similarly (*Botrytis*, &c.); *peptases*, using the term as a general one for all enzymes which convert proteids into peptones and other bodies (*Penicillium*, &c.); lipases, which break up fatty oils (*Empusa*, *Phycomyces*, &c.); oxydases, which bring about the oxidations and changes of colour observed in *Boletus*, and *zymase*, extracted by Buchner from yeast, which brings about the conversion of sugar into alcohol and carbon-dioxide. That such enzymes are formed in the protoplasm is evident from the behaviour of hyphae, which have been observed to pierce cell-membranes, the chitinous coats of insects, artificial collodion films and layers of wax, &c. That a fungus can secrete more than one enzyme, according to the materials its hyphae have to attack, has been shown by the extraction of diastase, inulase, trehalase, invertase, maltase, raffinase, malizitase, emulsin, trypsin and lipase from *Aspergillus* by Bourquelot, and similar events occur in other fungi. The same fact is indicated by the wide range of organic substances which can be utilized by *Penicillium* and other moulds, and by the behaviour of parasitic fungi which destroy various cell-contents and tissues. Many of the coloured pigments of fungi are fixed in the cell-walls or excreted to the outside (*Peziza aeruginosa*). Matruchot has used them for staining the living protoplasm of other fungi by growing the two together. Striking instances of coloured mycella are afforded by *Corticium sanguineum*, blood-red; *Elaphomyces Leveillei*, yellow-green; *Chlorosplenium aeruginosum*, verdigris green; and the *Dematei*, brown or black.

Nuclei.—Although many fungi have been regarded as devoid of nuclei, and all have not as yet been proved to contain them, the numerous investigations of recent years have revealed them in the cells of all forms thoroughly examined, and we are justified in concluding that the nucleus is as essential to the cell of a fungus as to that of other organisms. The hyphae of many contain numerous, even hundreds of nuclei (Phycomycetes); those of others have several (*Aspergillus*) in each segment, or only two (*Exoascus*) or one (*Erysiphe*) in each cell. Even the isolated cells of the yeast plant have each one nucleus. As a rule the nuclei of the mycelium are very minute (1.5-2 μ in *Phycomyces*), but those of many asci and spores are large and easily rendered visible. As with other plants, so in fungi the essential process of fertilization consists in the fusion of two nuclei, but owing to the absence of well-marked sexual organs from many fungi, a peculiar interest attaches to certain nuclear fusions in the vegetative cells or in young spores of many forms. Thus in Ustilagineae the chlamydospores, and in Uredineae the teleutospores, each contain two nuclei when young, which fuse as the spores mature. In young asci a similar fusion of two nuclei occurs, and also in basidia, in each case the nucleus of the ascus or of the basidium resulting from the fusion subsequently giving rise by division to the nuclei of the ascospores and basidiospores respectively. The significance of these fusions will be discussed under the various groups. Nuclear division is usually accompanied by all the essential features of karyokinesis.

Spores.—No agreement has ever been arrived at regarding the consistent use of the term spore. This is apparently owing to the facts that too much has been attempted in the definition, and that differences arise according as we aim at a morphological or a physiological definition. Physiologically, any cell or group of cells separated off from a hypha or unicellular fungus, and capable of itself growing out—germinating—to reproduce the fungus, is a spore; but it is evident that so wide a definition does not exclude the ordinary vegetative cells of sprouting fungi, such as yeasts, or small sclerotium like cell-aggregates of forms like *Coniothecium*. Morphologically considered, spores are marked by peculiarities of form, size, colour, place of origin, definiteness in number, mode of preparation, and so forth, such that they can be distinguished more or less sharply from the hyphae which produce them. The only physiological peculiarity exhibited in common by all spores is that they germinate and initiate the production of a new fungus-plant. Whether a spore results from the sexual union of two similar gametes (zygospore) or from the fertilization of an egg-cell by the protoplasm of a male organ (oospore); or is developed asexually as a motile (zoospore) or a quiescent body cut off from a hypha (conidium) or developed along its course (oidium or chlamydospore), or in its protoplasm (endospore), are matters of importance which have their uses in the classification and terminology of spores, though in many respects they are largely of academic interest.

Klebs has attempted to divide spores into three categories as follows: (1) kinospores, arising by relatively simple cell-divisions and subserving rapid dissemination and propagation, *e.g.* zoospores, conidia, endogonidia, stylospores, &c.; (2) paulospores, due to simple rearrangement of cell-contents, and subserving the persistence of the fungus through periods of exigency, *e.g.* gemmae, chlamydospores, resting-cells, cysts, &c.; (3) carpospores, produced by a more or less complex formative process, often in special fructifications, and subserving either or both multiplication and persistence, *e.g.* zygospores, oospores, brand-spores, aecidiospores, ascospores, basidiospores, &c. Little or nothing is gained by these definitions, however, which are especially physiological. In practice these various kinds of spores of fungi receive further special names in the separate groups, and names, moreover, which will appear, to those unacquainted with the history, to have been given without any consistency or regard to general principles; nevertheless, for ordinary purposes these names are far more useful in most cases, owing to their descriptive character, than the proposed new names, which have been only partially accepted.

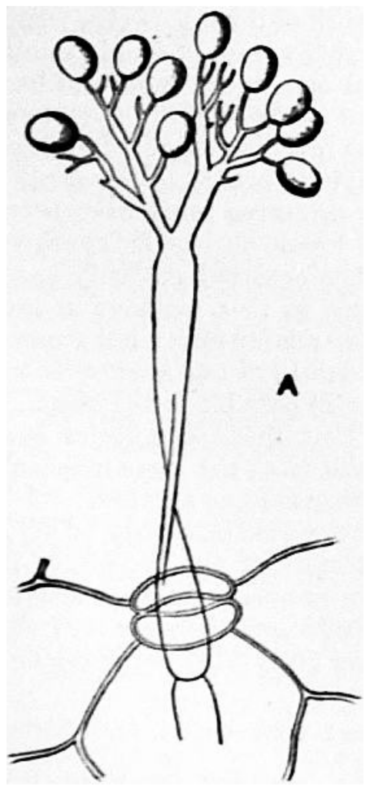

Fig. 2.—*Peronospora parasitica* (De Bary). Conidiophore with conidia.

Sporophores.—In some of the simpler fungi the spores are not borne on or in hyphae which can be distinguished from the vegetative parts or mycelium, but in the vast majority of cases the sporogenous hyphae either ascend free into the air or radiate into the surrounding water as distinct branches, or are grouped into special columns, cushions, layers or complex masses obviously different in colour, consistency, shape and other characters from the parts which gather up and assimilate the food-materials. The term “receptacle” sometimes applied to these spore-bearing hyphae is better replaced by sporophore. The sporophore is obsolete when the spore-bearing hyphae are not sharply distinct from the mycelium, simple when the constituent hyphae are isolated, and compound when the latter are conjoined. The chief distinctive characters of the sporogenous hyphae are their orientation, usually vertical; their limited apical growth; their peculiar branching, form, colour, contents, consistency; and their spore-production. According to the characters of the last, we might theoretically divide them into conidiophores, sporangiophores, gametophores, oidiophores, &c.; but since the two latter rarely occur, and more than one kind of spore or spore-case may occur on a sporophore, it is impossible to carry such a scheme fully into practice.

A simple sporophore may be merely a single short hypha, the end of which stops growing and becomes cut off as a conidium by the formation of a septum, which then splits and allows the conidium to fall. More generally the hypha below the septum grows forwards again, and repeats this process several times before the terminal conidium falls, and so a chain of conidia results, the oldest of which terminates the series (*Erysiphe*); when the primary branch has thus formed a basipetal series, branches may arise from below and again repeat this process, thus forming a tuft (*Penicillium*). Or the primary hypha may first swell at its apex, and put forth a series of short peg-like branches (*sterigmata*) from the increased surface thus provided, each of which develops a similar basipetal chain of conidia (*Aspergillus*), and various combinations of these processes result in the development of numerous varieties of exquisitely branched sporophores of this type (*Botrytis*, *Botryosporium*, *Verticillium*, &c.).

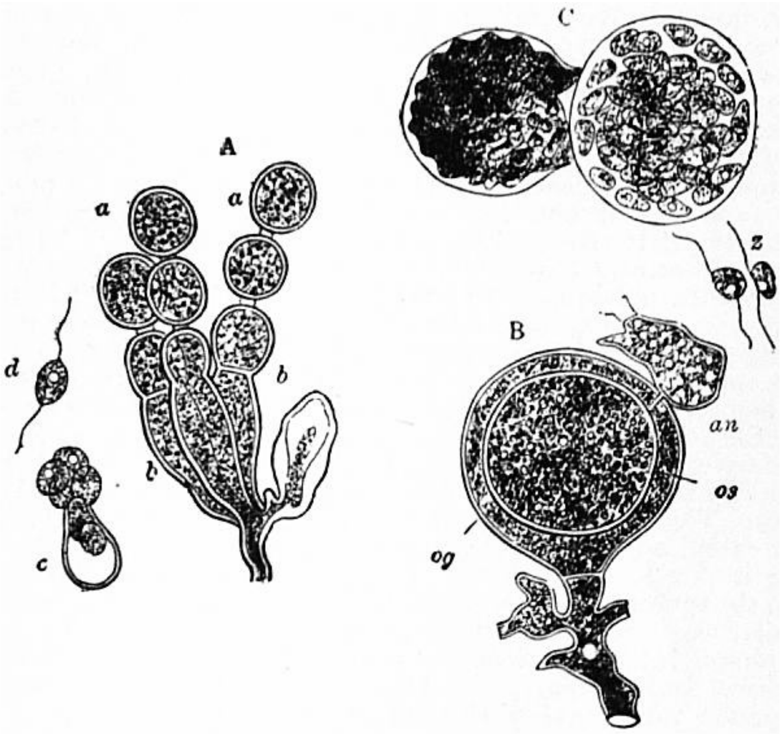

Fig. 3.—*Cystopus candidus*.

A. *a*, Conidia.
b, Conidiophores.
c, Conidium emitting zoospores.
d, Free zoospore.
B. *og*, Oogonium.
os, Oosphere.
an, Antheridium.
C. Formation of zoospores by oospores.
z, Free zoospores.
(After De Bary.)

A second type is developed as follows: the primary hypha forms a septum below its apex as before, and the terminal conidium, thus abstricted, puts out a branch at its apex, which starts as a mere point and rapidly swells to a second conidium; this repeats the process, and so on, so that we now have a chain of conidia developed in acropetal succession, the oldest being below, and, as in *Penicillium*, &c., branches put forth lower down may repeat the process (*Hormodendron*). In all these cases we may speak of simple conidiophores. The simple sporophore does not necessarily terminate in conidia, however. In *Mucor*, for example, the end of the primary hypha swells into a spheroidal head (sporangium), the protoplasm of which undergoes segmentation into more or less numerous globular masses, each of which secretes an enveloping cell-wall and becomes a spore (endospore), and branched systems of sporangia may arise as before (*Thamnidium*). Such may be termed sporangiophores. In *Sporodinia* the branches give rise also to short branches, which meet and fuse their contents to form zygospores. In Peronospora, Saprolegnia, &c., the ends of the branches swell up into sporangia, which develop zoospores in their interior (zoosporangia), or their contents become oospheres, which may be fertilized by the contents of other branches (antheridia) and so form egg-cases (oogonia). Since in such cases the sporophore bears sexual cells, they may be conveniently termed gametophores.

Compound sporophores arise when any of the branched or unbranched types of spore-bearing hyphae described above ascend into the air in consort, and are more or less crowded into definite layers, cushions, columns or other complex masses. The same laws apply to the individual hyphae and their branches as to simple sporophores, and as long as the conidia, sporangia, gametes, &c., are borne on their external surfaces, it is quite consistent to speak of these as compound sporophores, &c., in the sense described, however complex they may become. Among the simplest cases are the sheet-like aggregates of sporogenous hyphae in *Puccinia*, *Uromyces*, &c., or of basidia in *Exobasidium*, *Corticium*, &c., or of asci in *Exoascus*, *Ascocorticium*, &c. In the former, where the layer is small, it is often termed a sorus, but where, as in the latter, the sporogenous layer is extensive, and spread out more or less sheet-like on the supporting tissues, it is more frequently termed a hymenium. Another simple case is that of the columnar aggregates of sporogenous hyphae in forms like *Stilbum*, *Coremium*, &c. These lead us to cases where the main mass of the sporophore 336
forms a supporting tissue of closely crowded or interwoven hyphae, the sporogenous terminal parts of the hyphae being found at the periphery or apical regions only. Here we have the cushion-like type (stroma) of *Nectria* and many Pyrenomycetes, the clavate “receptacle” of *Clavaria*, &c., passing into the complex forms met with in *Sparassis*, *Xylaria*, *Polyporei*, and *Agaricini*, &c. In these cases the compound sporophore is often termed the hymenophore, and its various parts demand special names (pileus, stipes, gills, pores, &c.) to denote peculiarities of distribution of the hymenium over the surface.

Other series of modifications arise in which the tissues corresponding to the stroma invest the sporogenous hyphal ends, and thus enclose the spores, asci, basidia, &c., in a cavity. In the simplest case the stroma, after bearing its crop of conidia or oidia, develops ascogenous branches in the loosened meshes of its interior (*e.g.* *Onygena*). Another simple case is where the plane or slightly convex surface of the stroma rises at its margins and overgrows the sporogenous hyphal ends, so that the spores, asci, &c., come to lie in the depression of a cavity—*e.g.* *Solenia*, *Cyphella*—and even simpler cases are met with in *Mortierella*, where the zygospore is invested by the overgrowth of a dense mat of closely branching hyphae, and in *Gymnoascus*, where a loose mat of similarly barren hyphae covers in the tufts of asci as they develop.

In such examples as the above we may regard the hymenium (*Solenia*, *Cyphella*), zygospores, or asci as truly invested by later growth, but in the vast majority of cases the processes which result in the enclosure of the spores, asci, &c., in a “fructification” are much more involved, inasmuch as the latter is developed in the interior of hyphal tissues, which are by no means obviously homologous with a stroma. Thus in *Penicillium*, *Eurotium*, *Erysiphe*, &c., hyphal ends which are the initials of ascogenous branches, are invested by closely packed branches at an early stage of development, and the asci develop inside what has by that time become a complete investment. Whether a true sexual process precedes these processes or not does not affect the present question, the point being that the resulting spheroidal "fructification" (cleistocarp, perithecium) has a definite wall of its own not directly comparable with a stroma. In other cases (*Hypomyces*, *Nectria*) the perithecia arise on an already mature stroma, while yet more numerous examples can be given (*Poronia*, *Hypoxylon*, *Claviceps*, &c.) where the perithecia originate below the surface of a stroma formed long before. Similarly with the various types of conidial or oidial "fructifications," termed pycnidia, spermogonia, aecidia, &c. In the simplest of these cases—*e.g. Fumago*—a single mycelial cell divides by septa in all three planes until a more or less solid clump results. Then a hollow appears in the centre owing to the more rapid extension of the outer parts, and into this hollow the cells lining it put forth short sporogenous branches, from the tips of which the spores (stylospores, conidia, spermatia) are abstricted. In a similar way are developed the pycnidia of *Cicinnobolus*, *Pleospora*, *Cucurbitaria*, *Leptosphaeria* and others. In other cases (*Diplodia*, *Aecidium*, &c.) conidial or oidial "fructifications" arise by a number of hyphae interweaving themselves into a knot, as if they were forming a Sclerotium. The outer parts of the mass then differentiate as a wall or investment, and the interior becomes a hollow, into which hyphal ends grow and abstrict the spores. Much more complicated are the processes in a large series of "fructifications," where the mycelium first develops a densely packed mass of hyphae, all alike, in which labyrinths of cavities subsequently form by separation of hyphae in the previously homogeneous mass, and the hymenium covers the walls of these cavities and passages as with a lining layer. Meanwhile differences in consistency appear in various strata, and a dense outer protective layer (peridium), soft gelatinous layers, and so on are formed, the whole eventually attaining great complexity—*e.g.* puff-balls, earth-stars and various *Phalloideae*.

Spore-Distribution.—Ordinary conidia and similarly abstricted dry spores are so minute, light and numerous that their dispersal is ensured by any current of air or water, and we also know that rats and other burrowing animals often carry them on their fur; similarly with birds, insects, slugs, worms, &c., on claws, feathers, proboscides, &c., or merely adherent to the slimy body. In addition to these accidental modes of dispersal, however, there is a series of interesting adaptations on the part of the fungus itself. Passing over the locomotor activity of zoospores (*Pythium*, *Peronospora*, *Saprolegnia*) we often find spores held under tension in sporangia (*Pilobolus*) or in asci (*Peziza*) until ripe, and then forcibly shot out by the sudden rupture of the sporangial wall under the pressure of liquid behind—mechanism comparable to that of a pop-gun, if we suppose air replaced by watery sap. Even a single conidium, held tense to the last moment by the elastic cell-wall, may be thus shot forward by a spurt of liquid under pressure in the hypha abstricting it (*e.g. Empusa*), and similarly with *basidiospores* (*Coprinus*, *Agaricus*, &c.). A more complicated case is illustrated by *Sphaerobolus*, where the entire mass of spores, enclosed in its own peridium, is suddenly shot up into the air like a bomb from a mortar by the elastic retroversion of a peculiar layer which, up to the last moment, surrounded the bomb, and then suddenly splits above, turns inside out, and drives the former as a projectile from a gun. Gelatinous or mucilaginous degenerations of cell-walls are frequently employed in the interests of spore dispersal. The mucilage surrounding endospores of *Mucor*, conidia of *Empusa*, &c., serves to gum the spore to animals. Such gums are formed abundantly in pycnidia, and, absorbing water, swell and carry out the spores in long tendrils, which emerge for days and dry as they reach the air, the glued spores gradually being set free by rain, wind, &c. In oidial chains (*Sclerotinia*) a minute double wedge of wall-substance arises in the middle lamella between each pair of contiguous oidia, and by its enlargement splits the separating lamella. These disjunctors serve as points of application for the elastic push of the swelling spore-ends, and as the connecting outer lamella of cell-wall suddenly gives way, the spores are jerked asunder. In many cases the slimy masses of spermatia (*Uredineae*), conidia (*Claviceps*), basidiospores (*Phallus*, *Coprinus*), &c., emit more or less powerful odours, which attract flies or other insects, and it has been shown that bees carry the fragrant oidia of *Sclerotinia* to the stigma of *Vaccinium* and infect it, and that flies carry away the foetid spores of *Phallus*, just as pollen is dispersed by such insects. Whether the strong odour of trimethylamine evolved by the spores of *Tilletia* attracts insects is not known.

The recent observations and exceedingly ingenious experiments of Falck have shown that the sporophores of the Basidiomycetes—especially the large sporophores of such forms as *Boletus*, *Polyporus*—contain quantities of reserve combustible material which are burnt up by the active metabolism occurring when the fruit-body is ripe. By this means the temperature of the sporophore is raised and the difference between it and the surrounding air may be one of several degrees. As a result convection currents are produced in the air which are sufficient to catch the basidiospores in their fall and carry them, away from the regions of comparative atmospheric stillness near the ground, to the upper air where more powerful air-currents can bring about their wide distribution.

Classification.—It has been accepted for some time now that the majority of the fungi proper fall into three main groups, the Phycomycetes, Ascomycetes and Basidiomycetes, the Schizomycetes and Myxomycetes (Mycetozoa) being considered as independent groups not coming under the true fungi.

The chief schemes of classification put forward in detail have been those of P.A. Saccardo (1882-1892), of Oskar Brefeld and Von Tavel (1892), of P.E.L. Van Tieghem (1893) and of J. Schroeter (1892). The scheme of Brefeld, which was based on the view that the Ascomycetes and Basidiomycetes were completely asexual and that these two groups had been derived from one division (Zygomycetes) of the Phycomycetes, has been very widely accepted. The recent work of the last twelve years has shown, however, that the two higher groups of fungi exhibit distinct sexuality, of either a normal or reduced type, and has also rendered very doubtful the view of the origin of these two groups from the Phycomycetes. The real difficulty of classification of the fungi lies in the polyphyletic nature of the group. There is very little doubt that the primitive fungi have been derived by degradation from the lower algae. It appears, however, that such a degradation has occurred not only once in evolution but on several occasions, so that we have in the Phycomycetes not a series of naturally related forms, but groups which have arisen perfectly independently of one another from various groups of the algae. It is also possible in the absence of satisfactory intermediate forms that the Ascomycetes and Basidiomycetes have also been derived from the algae independently of the Phycomycetes, and perhaps of one another.

A natural classification on these lines would obviously be very complicated, so that in the present state of our knowledge it will be best to retain the three main groups mentioned above, bearing in mind that the Phycomycetes especially are far from being a natural group. The following gives a tabular survey of the scheme adopted in the present article:

A. Phycomycetes. Alga-like fungi with unicellular thallus and well-marked sexual organs.

Class I.—Oomycetes. Mycelium usually well developed, but sometimes poor or absent. Sexual reproduction by oogonia and antheridia; asexual reproduction by zoospores or conidia.

1. Monoblepharidineae. Mycelium present, antheridia with antherozoids, oogonium with single oosphere: Monoblepharidaceae.

2. Peronosporineae. Mycelium present; antheridia but no antherozoids; oogonia with one or more oospheres: Peronosporaceae, Saprolegniaceae.

3. Chytridineae. Mycelium poorly developed or absent; oogonia and antheridia (without antherozoids) known in some cases; zoospores common: Chytridiaceae. Ancylistaceae.

Class II.—Zygomycetes. Mycelium well developed; sexual reproduction by zygospores; asexual 337
reproduction by sporangia and conidia.

1. Mucorineae. Sexual reproduction as above, asexual by sporangia or conidia or both: Mucoraceae. Mortierellaceae, Chaetocladiaceae, Piptocephalidaceae.

2. Entomophthorineae. Sexual reproduction typical but with sometimes inequality of the fusing gametes (gametangia ?): Entomophthoraceae.

B. Higher Fungi. Fungi with segmental thallus; sexual reproduction sometimes with typical antheridia and oogonia (ascogonia) but usually much reduced.

Class I.—Ustilaginales. Forms with septate thallus, and reproduction by chlamydospores which on germination produce sporidia; sexuality doubtful.

Class II.—Ascomycetes. Thallus septate; spores developed in special type of sporangium, the ascus, the number of spores being usually eight. Sexual reproduction sometimes typical, usually reduced.

Exoascineae, Saccharomycetineae, Perisporinea, Discomycetes, Pyrenomycetes, Tuberineae, Laboulbeniineae.

Class III.—Basidiales. Thallus septate. Conidia (basidiospores) borne in fours on a special conidiophore, the basidium. Sexual reproduction always much reduced.

1. Uredineae. Life-history in some cases very complex and with well-marked sexual process and alternation of generations, in others much reduced; basidium (promycelium) derived usually from a thick-walled spore (teleutospore).

2. Basidiomycetes. Life-history always very simple, no well-marked alternation of generations; basidium borne directly on the mycelium.

(A) Protobasidiomycetes. Basidia septate. Auriculariaceae, Pilacreaceae, Tremellinaceae.

(B) Autobasidiomycetes. Basidia non-septate. Hymenomycetes, Gasteromycetes.

A. Phycomycetes.—Most of the recent work of importance in this group deals with the cytology of sexual reproduction and of spore-formation, and the effect of external conditions on the production of reproductive organs.

Monoblepharidaceae consists of a very small group of aquatic forms living on fallen twigs in ponds and ditches. Only one genus, *Monoblepharis*, can certainly be placed here, though a somewhat similar genus, *Myrioblepharis*, with a peculiar multiciliate zoospore like that of Vaucheria, is provisionally placed in the same group. *Monoblepharis* was first described by Cornu in 1871, but from that time until 1895 when Roland Thaxter described several species from America the genus was completely lost sight of. *Monoblepharis* has oogonia with single oospheres and antheridia developing a few amoeboid uniciliate antherozoids; these creep to the opening of the oogonium and then swim in. The resemblance between this genus and *Oedogonium* among the algae is very striking, as is also that of *Myrioblepharis* and *Vaucheria*.

Peronosporaceae are a group of endophytic parasites—about 100 species—of great importance as comprising the agents of “damping off” disease (*Pythium*), vine-mildew (*Plasmopara*), potato disease (Phytophthora), onion-mildew (*Peronospora*). *Pythium* is a semi-aquatic form attacking seedlings which are too plentifully supplied with water; its hyphae penetrate the cell-walls and rapidly destroy the watery tissues of the living plant; then the fungus lives in the dead remains. When the free ends of the hyphae emerge again into the air they swell up into spherical bodies which may either fall off and behave as conidia, each putting out a germ-tube and infecting the host; or the germ-tube itself swells up into a zoosporangium which develops a number of zoospores. In the rotting tissues branches of the older mycelium similarly swell up and form antheridia and oogonia (fig. 4). The contents of the antheridium are not set free, but that organ penetrates the oogonium by means of a narrow outgrowth, the fertilizing tube, and a male nucleus then passes over into the single oosphere, which at first multinucleate becomes uninucleate before fertilization. *Pythium* is of interest as illustrating the dependence of zoospore-formation on conditions and the indeterminate nature of conidia. The other genera are more purely parasitic; the mycelium usually sends haustoria into the cells of the host and puts out branched, aerial conidiophores through the stomata, the branches of which abstrict numerous “conidia”; these either germinate directly or their contents break up into zoospores (fig. 5). The development of the “conidia” as true conidial spores or as zoosporangia may occur in one and the same species (*Cystopus candidus*, *Phytophthora infestans*) as in *Pythium* described above; in other cases the direct conidial germination is characteristic of genera—*e.g. Peronospora*; while others emit zoospores—*e.g. Plasmopara*, &c. In *Cystopus* (*Albugo*) the “conidia” are abstricted in basipetal chain-like series from the ends of hyphae which come to the surface in tufts and break through the epidermis as white pustules. Each “conidium” contains numerous nuclei and is really a zoosporangium, as after dispersal it breaks up into a number of zoospores. The Peronosporaceae reproduce themselves sexually by means of antheridia and oogonia as described in *Pythium*. In *Cystopus Bliti* the oosphere contains numerous nuclei, and all the male nuclei from the antheridium pass into it, the male and female nuclei then fusing in pairs. We thus have a process of “multiple fertilization”; the oosphere really represents a large number of undifferentiated gametes and has been termed a coenogamete. Between *Cystopus Bliti* on the one hand and *Pythium de Baryanum* on the other a number of cytologically intermediate forms are known. The oospore on germination usually gives origin to a zoosporangium, but may form directly a germ tube which infects the host.

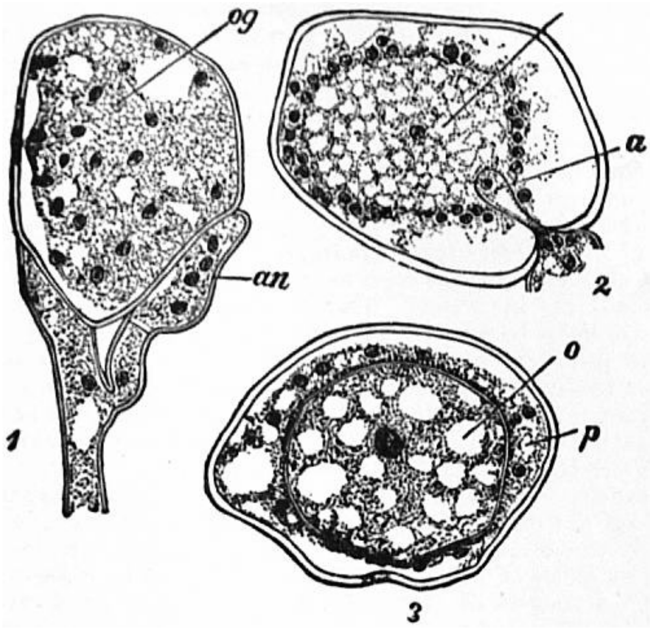

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FIG. 4.—Fertilization of the Peronosporeae. After Wager.

1, *Peronospora parasitica*. Young multinucleate oogonium (*og*) and antheridium (*an*).
2, *Albugo candida*. Oogonium with the central uninucleate oosphere and the fertilizing tube (*a*) of the antheridium which introduces the male nucleus.
3, The same. Fertilized egg-cell (*o*) surrounded by the periplasm (*p*).

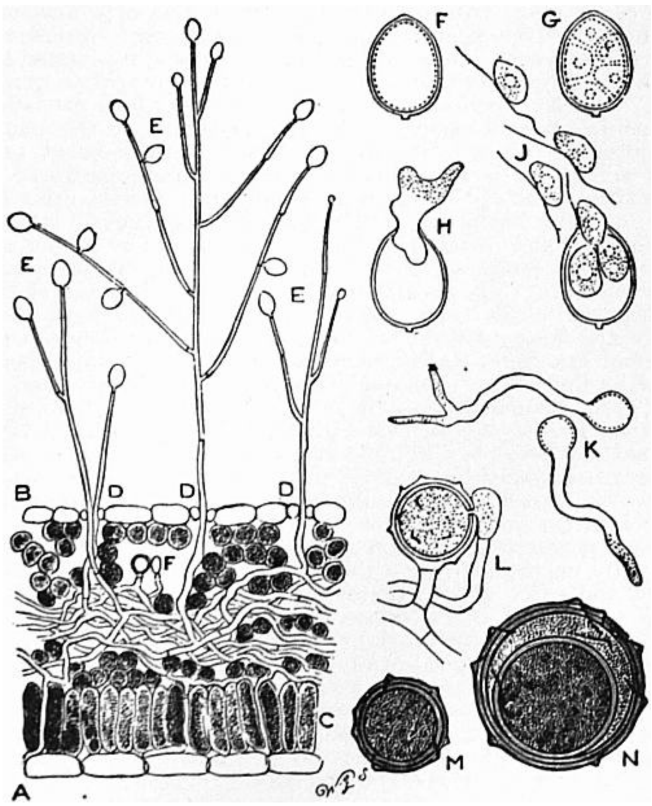

FIG. 5.—*Phytophthora infestans*. Fungus of Potato Disease.

A, B, Section of Leaf of Potato with sporangiophores of *Phytophthora infestans* passing through the stomata D, on the under surface of the
K, Germination of the zoospores formed in the sporangia.
L, M, N, Fertilization of the oogonium and development of the oospore in *Peronospora*.

leaf.
E, Sporangia.
F, G, H, J, Further development of the sporangia.

Saprolegniaceae are aquatic forms found growing usually on dead insects lying in water but occasionally on living fish (*e.g.* the salmon disease associated with *Saprolegnia ferax*). The chief genera are *Saprolegnia, Achlya, Pythiopsis, Dictyuchus, Aplanes.* Motile zoospores which escape from the zoosporangium are present except in Aplanes. The sexual reproduction shows all transitions between forms which are normally sexual, like the Peronosporaceae, to forms in which no antheridium is developed and the oospheres develop parthenogenetically. The oogonia, unlike the Peronosporaceae, contain more than one oosphere. Klebs has shown that the development of zoosporangia or of oogonia and pollinodia respectively in *Saprolegnia* is dependent on the external conditions; so long as a continued stream of suitable food-material is ensured the mycelium grows on without forming reproductive organs, but directly the supplies of nitrogenous and carbonaceous food fall below a certain degree of concentration sporangia are developed. Further reduction of the supplies of food effects the formation of oogonia. This explains the sequence of events in the case of a *Saprolegnia*-mycelium radiating from a dead fly in water. Those parts nearest the fly and best supplied develop barren hyphae only; in a zone at the periphery, where the products of putrefaction dissolved in the water form a dilute but easily accessible supply, the zoosporangia are developed in abundance; oogonia, however, are only formed in the depths of this radiating mycelium, where the supplies of available food materials are least abundant. 338

Chytridineae.—These parasitic and minute, chiefly aquatic, forms may be looked upon as degenerate Oomycetes, since a sexual process and feeble unicellular mycelium occur in some; or they may be regarded as series of primitive forms leading up to higher members. There is no means of deciding the question. They are usually included in Oomycetes, but their simple structure, minute size, usually uniciliate zoospores, and their negative characters would justify their retention as a separate group. It contains less than 200 species, chiefly parasitic on or in algae and other water-plants or animals, of various kinds, or in other fungi, seedlings, pollen and higher plants. They are often devoid of hyphae, or put forth fine protoplasmic filaments into the cells of their hosts. After absorbing the cell-contents of the latter, which it does in a few hours or days, the fungus puts out a sporangium, the contents of which break up into numerous minute swarm-spores, usually one-ciliate, rarely two-ciliate. Any one of these soon comes to rest on a host-cell, and either pierces it and empties its contents into its cavity, where the further development occurs (*Olpidium*), or merely sends in delicate protoplasmic filaments (*Rhizophydium*) or a short hyphal tube of, at most, two or three cells, which acts as a haustorium, the further development taking place outside the cell-wall of the host (*Chytridium*). In some cases resting spores are formed inside the host (*Chytridium*), and give rise to zoosporangia on germination. In a few species a sexual process is described, consisting in the conjugation of similar cells (*Zygochytrium*) or the union of two dissimilar ones (*Polyphagus*). In the development of distinct antheridial and oogonial cells the allied Ancylistineae show close alliances to *Pythium* and the Oomycetes. On the other hand, the uniciliate zoospores of *Polyphagus* have slightly amoeboid movements, and in this and the pseudopodium-like nature of the protoplasmic processes, such forms suggest resemblances to the Myxomycetes. Opinions differ as to whether the Chytridineae are degraded or primitive forms, and the group still needs critical revision. Many new forms will doubtless be discovered, as they are rarely collected on account of their minuteness. Some forms cause damping off of seedlings—*e.g. Olpidium Brassicae*; others discoloured spots and even tumour-like swellings—*e.g. Synchytium Scabiosae*, *S. Succisae*, *Urophlyctis*, &c., on higher plants. Analogies have been pointed out between Chytridiaceae and unicellular algae, such as Chlorosphaeraceae, Protococcaceae, "Palmellaceae," &c., some of which are parasitic, and suggestions may be entertained as to possible origin from such algae.

The *Zygomycetes*, of which about 200 species are described, are especially important from a theoretical standpoint, since they furnished the series whence Brefeld derived the vast majority of the fungi. They are characterized especially by the zygospores, but the asexual organs (sporangia) exhibit interesting series of changes, beginning with the typical sporangium of *Mucor* containing numerous endospores, passing to cases where, as in *Thamnidium*, these are accompanied with more numerous small sporangia (sporangioles) containing few spores, and thence to *Chaetocladium* and *Piptocephalis*, where the sporangioles form but one spore and fall and germinate as a whole; that is to say, the monosporous sporangium has become a conidium, and Brefeld regarded these and similar series of changes as explaining the relation of ascus to conidium in higher fungi. According to his view, the ascus is in effect the sporangium with several spores, the conidium the sporangiole with but one spore, and that not loose but fused with the sporangiole wall. On this basis, with other interesting morphological comparisons, Brefeld erected his hypothesis, now untenable, that the Ascomycetes and Basidiomycetes diverge from the Zygomycetes, the former having particularly specialized the ascus (sporangial) mode of reproduction, the latter having specialized the conidial (indehiscent one-spored sporangiole) mode. In addition to sporangia and the conidial spores referred to, some Mucorini show a peculiar mode of vegetative reproduction by means of gemmae or chlamydospores—*i.e.* short segments of the hyphae become stored with fatty reserves and act as spores. The gemmae formed on submerged Mucors may bud like a yeast, and even bring about alcoholic fermentation in a saccharine solution.

The segments of the hyphae in this group usually contain several nuclei. At the time of sporangial formation the protoplasm with numerous nuclei streams into the swollen end of the sporangiophore and there becomes cut off by a cell-wall to form the sporangium. The protoplasm then becomes cut up by a series of clefts into a number of smaller and smaller pieces which are unicellular in *Pilobolus*, multicellular in *Sporodinia*. These then become surrounded by a cell-wall and form the spores. This mode of spore-formation is totally different from that in the ascus; hence one of the difficulties of the acceptance of Brefeld's view of the homology of ascus and sporangium. The cytology of zygospore-formation is not known in detail; the so-called gametes which fuse are multinucleate and are no doubt of the nature of gametangia. The fate of these nuclei is doubtful, probably they fuse in pairs (fig. 6).

Blakeslee has lately made some very important observations of the Zygomycetes. It is well known that while in some forms, *e.g. Spordinia*, zygospores are easily obtained, in others, *e.g.* most species of *Mucor*, they are very erratic in their appearance. This has now been explained by Blakeslee, who finds that the Mucorinae can be divided into two groups, termed homothallic and heterothallic respectively. In the first group zygospores can arise by the union of branches from the *same* mycelium and so can be produced by the growth from a single spore; this group includes *Spordinia grandis*, *Spinellus fusiger*, some species of *Mucor*, &c. The majority of forms, however, fall into the heterothallic group, in which the association of branches from two mycelia *different in nature* is necessary for the formation of zygospores. These structures cannot then be produced from the product of a single spore nor even from the thalli derived from *any* two spores. The two kinds of thalli Blakeslee considers to have a differentiation of the nature of sex and he distinguishes them as (+) and (−) forms; the former being usually distinguished by a somewhat greater luxuriance of growth.

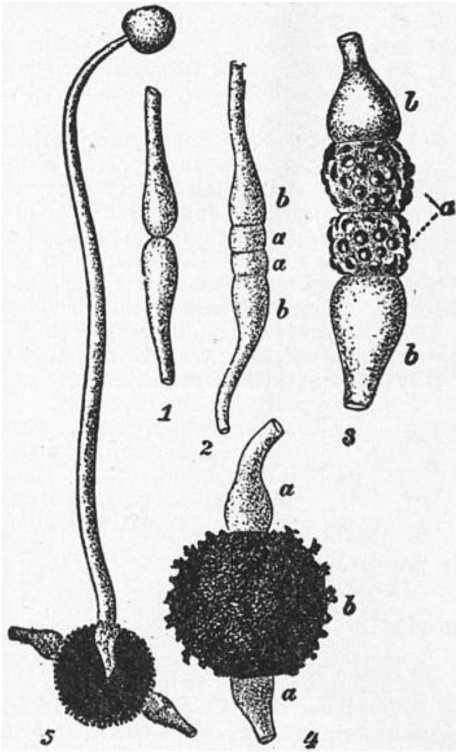

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Fig. 6.—*Mucor Mucedo.* Different stages in the formation and germination of the zygospore. (After Brefeld, 1-4. 5 from v. Tavel, *Pilze.*)

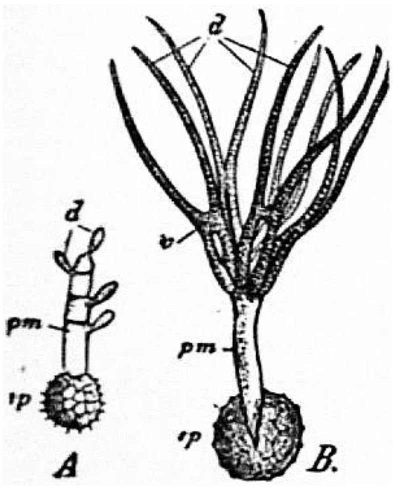
1, Two conjugating branches in contact.
2, Septation of the conjugating cells (*a*) from the suspensors (*b*).
3, More advanced stage, the conjugating cells (*a*) are still distinct from one another; the warty thickenings of their walls have commenced to form.
4, Ripe zygospore (*b*) between the suspensors (*a*).
5, Germinating zygospore with a germ-tube bearing a sporangium.

The classification of the Mucorini depends on the prevalence and characters of the conidia, and of the sporangia and zygospores—*e.g.* the presence or absence of a columella in the former, the formation of an investment round the latter. Most genera are saprophytes, but some—*Chaetocladium*, *Piptocephalis*—are parasites on other Mucorini, and one or two are associated casually with the rotting of tomatoes and other fruits, bulbs, &c., the fleshy parts of which are rapidly destroyed if once the hyphae gain entrance. Even more important is the question of mycosis in man and other animals, referred to species of *Mucor*, and investigated by Lucet and Costantin. Klebs has concluded that transpiration is the important factor in determining the formation of sporangia, while zygote-development depends on totally different conditions; these results have been called in question by Falck.

The *Entomophthoraceae* contain three genera, *Empusa*, *Entomophthora* and *Basidiobolus*. The two first genera consist of forms which are parasitic on insects. *Empusa Muscae* causes the well-known epidemic in house-flies during the autumn; the dead, affected flies are often found attached to the window surrounded by a white halo of conidia. *B. ranarum* is found in the alimentary canal of the frog and growing on its excrement. In these three genera the conidia are cast off with a jerk somewhat in the same way as the sporangium of *Pilobolus*.

B. Higher Fungi.—Now that Brefeld’s view of the origin of these forms from the Zygomycetes has been 339
overthrown, the relationship of the higher and lower forms of fungi is left in obscurity. The term *Eumycetes* is sometimes applied to this group to distinguish them from the Phycomycetes, but as the same name is also applied to the fungi as a whole to differentiate them from the Mycetozoa and Bacteria, the term had best be dropped. The Higher Fungi fall into three groups: the *Ustilaginales*, of doubtful position, and the two very sharply marked groups *Basidiales* and *Ascomycetes*.

I. *Ustilaginales.*—This includes two families Ustilaginaceae (smuts) and Tilletiaceae (bunts). The bunts and smuts which damage our grain and fodder plants comprise about 400 species of internal parasites, found in all countries on herbaceous plants, and especially on Monocotyledons. They are remarkable for their dark spores developed in gall-like excrescences on the leaves, stems, &c., or in the fruits of the host. The discovery of the yeast-conidia of these fungi, and their thorough investigation by Brefeld, have thrown new lights on the group, as also have the results elucidating the nature of the ordinary dark spores—smuts, bunt, &c.—which by their mode of origin and development are chlamydospores. When the latter germinate a slender “promycelium” is put out; in *Ustilago* and its allies this is transversely septate, and bears lateral conidia (sporidia); in *Tilletia* and its allies non-septate, and bears a terminal tuft of conidia (sporidia) (fig. 7). Brefeld regarded the promycelium as a kind of *basidium*, bearing lateral or terminal conidia (comparable to *basidiospores*), but since the number of basidiospores is not fixed, and the basidium has not yet assumed very definite morphological characters, Brefeld termed the group *Hemibasidii*, and regarded them as a half-way stage in the evolution of the true Basidiomycetes from Phycomycetes, the *Tilletia* type leading to the true basidium (Autobasidium), the *Ustilago* type to the protobasidium, with lateral spores; but this view is based on



From Vine’s *Students’ Text Book of Botany*, by permission of Swan Sonnenschein & Co.

Fig. 7.—Germinating resting-gonidia.

A, of *Ustilago receptaculorum*; B, of *Tilletia Caries*.

sp, The gonidium.
pm, The promycelium.
d, The sporidia: in B the sporidia have coalesced in pairs at *v*.

very poor evidence, so that it is best to place these forms as a separate group, the *Ustilaginales*. The yeast-conidia, which bud off from the conidia or their resulting mycelium when sown in nutrient solutions, are developed in successive crops by budding exactly as in the yeast plant, but they cannot ferment sugar solutions. It is the rapid spread of these yeast-conidia in manure and soil waters which makes it so difficult to get rid of smuts, &c., in the fields, and they, like the ordinary conidia, readily infect the seedling wheat, oats, barley or other cereals. Infection in these cases occurs in the seedling at the place where root and shoot meet, and the infecting hypha having entered the plant goes on living in it and growing up with it as if it had no parasitic action at all. When the flowers form, however, the mycelium sends hyphae into the young ovaries and rapidly replaces the stores of sugar and starch, &c., which would have gone to make the grain, by the soot-like mass of spores so well known as smut, &c. These spores adhere to the grain, and unless destroyed, by “steeping” or other treatment, are sown with it, and again produce sporidia and yeast-conidia which infect the seedlings. In other species the infection occurs through the style of the flower, but the fungus after reaching the ovule develops no further during that year but remains dormant in the embryo of the seed. On germination, however, the fungus behaves in the same way as one which has entered in the seedling stage. The cytology of these forms is very little known; Dangeard states that there is a fusion of two nuclei in the chlamydospore, but this requires confirmation. Apart from this observation there is no other trace of sexuality in the group.

II. *Ascomycetes.*—This, except in the case of a few of the simpler forms, is a very sharply marked group characterized by a special type of sporangium, the ascus. In the development of the ascus we find two nuclei at the base which fuse together to form the single nucleus of the young *ascus*. The single nucleus divides by three successive divisions to form eight nuclei lying free in the protoplasm of the ascus. Then by a special method, described first by Harper, a mass of protoplasm is cut out round each nucleus; thus eight uninucleate ascospores are formed by free-cell formation. The protoplasm remaining over is termed *epiplasm* and often contains glycogen (fig. 8). In some cases nuclear division is carried further before spore-formation occurs, and the number of spores is then 16, 32 and 64, &c.; in a few cases the number of spores is less than eight by abortion of some of the eight nuclei. The ascus is thus one of the most sharply characterized structures among the fungi.

In some forms we find definite male and female sexual organs (*Sphaerotheca*, *Pyronema*, &c.), in others the antheridium is abortive or absent, but the ascogonium (oogonium) is still present and the female nuclei fuse in pairs (*Lachnea stercorea*, *Humaria granulata*, *Ascobolus furfuraceus*); while in other forms ascogonium and antheridium are both absent and fusion occurs between vegetative nuclei (*Humaria rutilans*, and probably the majority of other forms). In other cases the sexual fusion is apparently absent altogether, as in *Exoascus*. In the first case (fig. 9) we have a true sexual process, while in the second and third cases we have a *reduced* sexual process in which the fusion of other nuclei has replaced the fusion of the normal male and female nuclei. It is to be noted that all the forms exhibit the fusion of nuclei in the ascus, so that those with the normal or reduced sexual process described above have two nuclear fusions in their life-history. The advantage or significance of the second (ascus) fusion is not clearly understood.

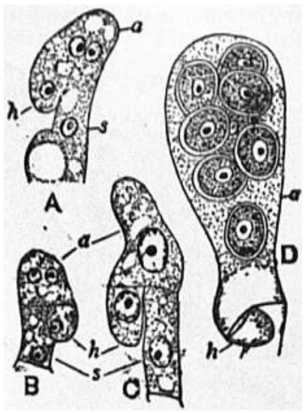


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Fig. 8.—Development of the Ascus.

A-C, *Pyronema confluens*. (After Harper.)
D, Young ascus of *Boudiera* with eight spores. (After Claussen.)

The group of the Hemiasci was founded by Brefeld to include forms which were supposed to be a connecting link between Phycomycetes and Ascomycetes. As mentioned before, the connexion between these two groups is very doubtful, and the derivation of the ascus from an ordinary sporangium of the Zygomycetes cannot be accepted. The majority of the forms which were formerly included in this group have been shown to be either true Phycomycetes (like *Ascoidea*) or true Ascomycetes (like *Thelebolus*). *Eremascus* and *Dipodascus*, which are often placed among the Hemiasci, possibly do not belong to the Ascomycetes series at all.

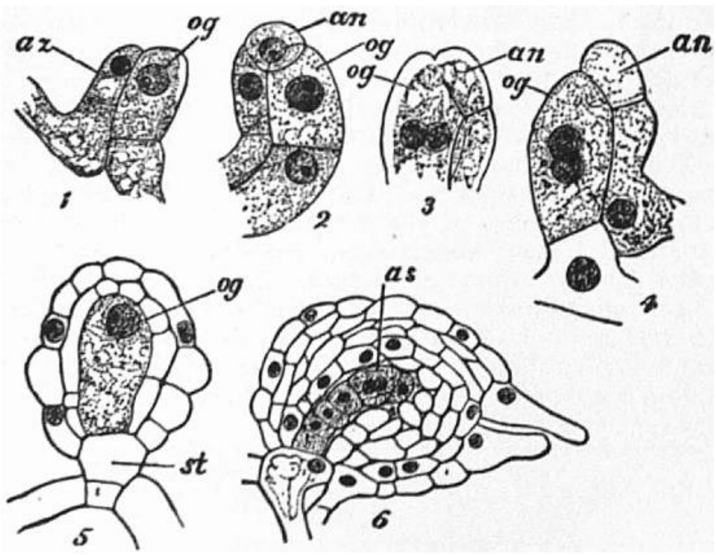

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Fig. 9.—*Sphaerotheca Castagnei*. Fertilization and Development of the Perithecium. (After Harper.)

1, Oogonium (*og*) with the antheridial branch (*az*) applied to its surface
2, Separation of antheridium (*an*).
3, Passage of the antheridial nucleus towards that of the oogonium.
4, Union of the nuclei.
5, Fertilized oogonium surrounded by two layers of hyphae derived from the stalk-cell (*st*).
6, The multicellular ascogonium derived by division from the oogonium; the terminal cell with the two nuclei (*as*) gives rise to the ascus.

Exoascaceae are a small group of doubtful extent here used to include *Exoascus*, *Taphrina*, *Ascorticium* and *Endomyces*. The mycelium is very much reduced in extent. The asci are borne directly on the mycelium and are therefore fully exposed, being devoid from the beginning of any investment. The *Taphrineae*, which include *Exoascus* and *Taphrina*, are important parasites—*e.g.* pocket-plums and witches' brooms on birches, &c., are due to their action (fig. 10). *Exoascus* and *Ascorticium* present interesting parallels to *Exobasidium* and *Corticium* among the Basidiomycetes.

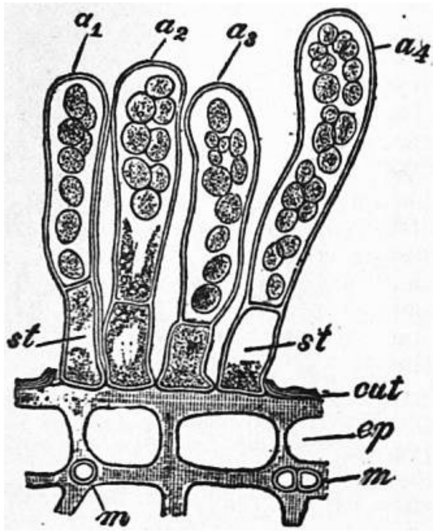

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Fig. 10.—*Taphrina Pruni*. Transverse section through the epidermis of an infected plum. Four ripe asci, a_1, a_2, with eight spores, a_3, a_4, with yeast-like conidia abstricted from the spores. After Sadebeck.

st, Stalk-cells of the asci.
m, Filaments of the mycelium cut transversely.
cut, Cuticle.
sp, Epidermis.

Saccharomycetaceae include the well-known yeasts which belong mainly to the genus *Saccharomyces*. They are characterized by their unicellular nature, their power of rapid budding, their capacity for fermenting various sugars, and their power of forming endogenous spores. The 340 sporangium with its endogenous spores has been compared with an ascus, and on these grounds the group is placed among the Ascomycetes—a very doubtful association. The group has attained an importance of late even beyond that to which it was brought by Pasteur's researches on alcoholic fermentation, chiefly owing to the exact results of the investigations of Hansen, who first applied the methods of pure cultures to the study of these organisms, and showed that many of the inconsistencies hitherto existing in the literature were due to the coexistence in the cultures of several species or races of yeasts morphologically almost indistinguishable, but physiologically very different. About fifty species of *Saccharomyces* are described more or less completely, but since many of these cannot be distinguished by the microscope, and some have been found to develop physiological races or varieties under special conditions of growth, the limits are still far too ill-defined for complete botanical treatment of the genus. A typical yeast is able to develop new cells by budding when submerged in a saccharine solution, and to ferment the sugar—*i.e.* so to break up its molecules that, apart from small quantities used for its own substance, masses of it out of all proportion to the mass of yeast used become resolved into other bodies, such as carbon dioxide and alcohol, the process requiring little or no oxygen. Brefeld regards the budding process as the formation of conidia. Under other conditions, of which the temperature is an important one, the nucleus in the yeast-cell divides, and each daughter-nucleus again, and four spores are formed in the mother cell, a process obviously comparable to the typical development of ascospores in an ascus. Under yet other conditions the quiescent yeast-cells floating on the surface of the fermented liquor grow out into elongated sausage-shaped or cylindrical cells and branching cell-series, which mat together into mycelium-like veils. At the bottom of the fermented liquor the cells often obtain fatty contents and thick walls, and behave as resting cells (chlamydospores). The characters employed by experts for determining a species of yeast are the sum of its peculiarities as regards form and size: the shapes, colours, consistency, &c., of the colonies grown on certain definite media; the optimum temperature for spore-formation, and for the development of the "veils"; and the behaviour as regards the various sugars.

The following summary of some of the principal characteristics of half-a-dozen species will serve to show how such peculiarities can be utilized for systematic purposes:

Species.	Optimum Temperature for		Characters of			Sugars Fermented and Products, &c.
	Spores.	Veils.	Fermentation.	Cells.	Spores.	
S. cereviseae I.	30°	20°-28°	High	Rounded	Globoid	Inverts maltose and saccharose and form alcohol 4-6 vol. %.
S. Pastorianus I	27°-5°	26°-28°	Low	Rounded	Globoid	
S. ellipsoideus	25°	33°-34°	Low	Rounded	Globoid	
S. anomalus	28°-31°	?	High	Elliptical	Hat-shaped	Ditto, and evolves a fragrant ether.
S. Ludwigii	30°-31°	?	?	Elongated	Globoid	Will not invert maltose.
S. membranaefaciens	30°	?	High	Elongated	Globoid	Inverts neither maltose nor saccharos

Two questions of great theoretical importance have been raised over and over again in connexion with yeasts, namely, (1) the morphological one as to whether yeasts are merely degraded forms of higher fungi, as would seem implied by their tendency to form elongated, hypha-like cells in the veils, and their development of "ascospores" as well as by the wide occurrence of yeast-like "sprouting forms" in other fungi (*e.g. Mucor*, Exoasci, Ustilagineae, higher Ascomycetes and Basidiomycetes); and (2) the question as to the physiological nature and meaning of fermentation. With regard to the first question no satisfactory proof has as yet been given that Saccharomycetes are derivable by culture from any higher form, the recent statements to that effect not having been confirmed. At the same time there are strong grounds for insisting on the resemblances between *Endomyces*, a hyphal fungus bearing yeast-like asci, and such a form as *Saccharomyces anomalus*. Concerning the second question, the recent investigations of Buchner and others have shown that a ferment (zymase) can be extracted from yeast-cells which causes sugar to break up into carbon dioxide and alcohol. It has since been shown by Buchner and Albert that yeast-cells which have been killed by alcohol and ether, or with acetone, still retain the enzyme. Such material is far more active than the zymase obtained originally by Buchner from the expressed juice of yeast-cells. Thus alcoholic fermentation is brought into line with the other fermentations.

Schizosaccharomyces includes a few species in which the cells do not “bud” but become elongated and then divide transversely. In the formation of sporangia two cells fuse together by means of outgrowths, in a manner very similar to that of *Spirogyra*; sometimes, however, the wall between two cells merely breaks down. The fused cell becomes a sporangium, and in it eight spores are developed. In certain cases single cells develop parthenogenetically, without fusion, each cell producing, however, only four spores. In *Zygosaccharomyces* described by Barker (1901) we have a form of the usual sprouting type, but here again there is a fusion of two cells to form a sporangium.

Cytology.—The study of the nucleus of yeast-cells is rendered difficult by the presence of other deeply staining granules termed by Guillermond *metachromatic granules*. These have often been mistaken for nuclei and have to be carefully distinguished by differential stains. In the process of budding the nucleus divides apparently by a process of direct division. In the formation of spores the nucleus of the cell divides, the protoplasm collects round the nuclei to form the spores by free-cell formation; the protoplasm (epiplasm) not used in this process becomes disorganized. A fusion of nuclei was originally described by Jansens and Leblanc, but it was observed neither by Wager nor Guillermond and is probably absent. In *Schizosaccharomyces* and *Zygosaccharomyces*, however, we have a fusion of nuclei in connexion with the conjugation of cells which precedes sporangium-formation. The theory may be put forward that the ordinary forms have been derived from sexual forms like *Schizosaccharomyces* and *Zygosaccharomyces* by a loss of sexuality, the sporangium being formed parthenogenetically without any nuclear fusion. This suggests a possible relationship to *Eremascus*, which can only doubtfully be placed in the Ascomycetes (*vide supra*).

Carpoascomycetes.—The other divisions of the Ascomycetes may be distinguished as Carpoascomycetes because they do not bear the asci free on the mycelium but enclosed in definite fruit bodies or ascocarps. The ascocarps can be distinguished into two portions, a mass of sterile or vegetative hyphae forming the main mass of the fruit body, and surrounding the fertile ascogenous hyphae which bear at their ends the asci. When the ascogonium (female organ) is present the ascogenous hyphae arise from it, with or without its previous fusion with an antheridium. In other cases the ascogenous hyphae arise directly from the vegetative hyphae. In connexion with this condition of reduction a fusion of nuclei has been observed in *Humaria rutilans* and is probably of frequent occurrence. The asci may be derived from the terminal cell of the branches of the ascogenous hyphae, but usually they are derived from the penultimate cell, the tip curving over to form the so-called crozier. By this means the ascus cell is brought uppermost, and after the fusion of the two nuclei it develops enormously and produces the ascospores. The ascospores escape from the asci in various ways, sometimes by a special ejaculation-mechanism. The Ascomycetes, at least the Carpoascomycetes, exhibit a well-marked alternation of sexual and asexual generations. The ordinary mycelium is the gametophyte since it bears the ascogonia and antheridia when present; the ascogenous hyphae with their asci represent the sporophyte since they are derived from the fertilized ascogonium. The matter is complicated by the apogamous transition from gametophyte to sporophyte in the absence of the ascogonium; also by the fact that there are normally two fusions in the life-history as mentioned earlier. If there are two fusions one would expect two reductions, and Harper has suggested that the division of the nuclei into eight in the ascus, instead of into four spores as in most reduction processes, is associated with a *double* reduction process in the ascus. Miss Fraser in *Humaria rutilans* finds two reductions: a normal synaptic reduction in the first nuclear division of the ascus, and a peculiar reduction division termed *brachymeiosis* in the third ascus division.

Various types of ascocarp are characteristic of the different divisions of the Carpoascomycetes: the cleistothecium, apothecium and perithecium.

Perisporineae.—This includes two chief families, Erysiphaceae and Perisporiaceae. They are characterized 341
by an ascocarp without any opening to the exterior, the ascospores being set free by the decay or rupture of the ascocarp wall; such a fruit-body is termed a *cleistothecium* (cleistocarp). The Erysiphaceae are a sharply marked group of forms which live as parasites. They form a superficial mycelium on the surface of the plant, the hyphae not usually penetrating the tissues but merely sending haustoria into the epidermal cells. Only in rare cases is the mycelium intercellular. Owing to their appearance they go by the popular name of mildews. *Sphaerotheca Humuli* is the well known hop-mildew, *Sphaerotheca Mors-Uvae* is the gooseberry mildew, the recent advent of which has led to special legislation in Great Britain to prevent its spreading, as when rampant it makes the culture of gooseberries impossible. *Erysiphe*, *Uncinula* and *Phyllactinia* are other well-known genera. The form of the fruit body, the difference and the nature of special outgrowths upon it—the appendages—are characteristic of the various genera. Besides peritheca the members of the Erysiphaceae possess conidia borne in simple chains. De Bary brought forward very strong evidence for the origin of the ascocarp in *Sphaerotheca* and *Erysiphe* by a sexual process, but Harper in 1895 was the first to prove conclusively, by the observation of the nuclear fusion, that there was a definite fertilization in *Sphaerotheca Humuli* by the fusion of a male (antheridial) nucleus with a female, ascogonial (oogonial) nucleus. Since then Harper has shown that the same process occurs in *Erysiphe* and *Phyllactinia*.

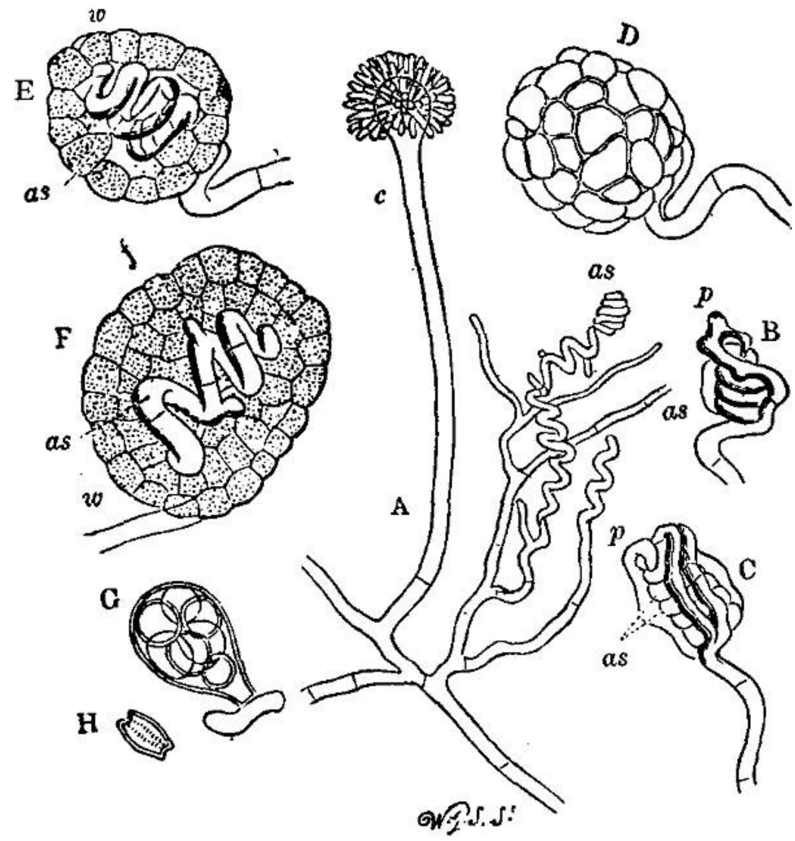
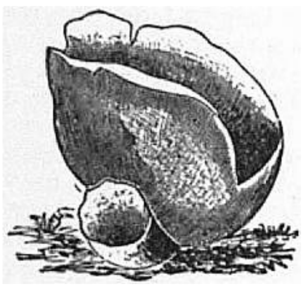


Fig. 11.—Development of *Eurotium repens*. (After De Bary.)

A, Small portion of mycelium with conidiophore (*c*), and archicarp (*as*).
B, The spiral archicarp (*as*), with the antheridium (*p*).
D, The same, beginning to be surrounded by the hyphae forming the perithecium wall.
D, The perithecium.
E, F, Sections of young perithecia.
w, Parietal cells.
f, Pseudo-parenchyma.
as, Ascogonium.
G, An ascus.
H, An ascospore.

The Perisporiaceae are saprophytic forms, the two chief genera being *Aspergillus* and *Penicillium*. The blue-green mould *P. crustaceum* and the green mould *A. herbariorium* (= *Eurotium herbariorum*) are extraordinarily widely distributed, moulds being found on almost any food-material which is exposed to the air. They have characteristic conidiophores bearing numerous conidia, and also cleistothecia which are spherical in form and yellowish in colour. The latter arise from the crown of a spirally coiled archicarp (bearing an ascogonium at its end) and a straight antheridium. Vegetative hyphae then grow up and surround these and enclose them in a continuous sheath of plectenchyma (fig. 11). It has lately been shown by Fraser and Chambers that in *Eurotium* both ascogonium and antheridium contain a number of nuclei (*i.e.* are coenogametes), but that the antheridium disorganizes without passing its contents into the ascogonium. There is apparently a reduced sexual process by the fusion of the ascogonial (female) nuclei in pairs. *Aspergillus Oryzae* plays an important part in saccharifying the starch of rice, maize, &c., by means of the abundant diastase it secretes, and, in symbiosis with a yeast which ferments the sugar formed, has long been used by the Japanese for the preparation of the alcoholic liquor saké. The process has now been successfully introduced into European commerce.



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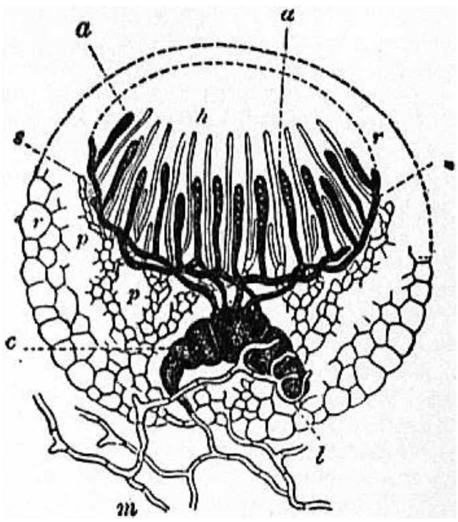

Fig. 13.—*Ascobolus furfuraceus.* Diagrammatic section of the fructification. (After Janczewski.)

m, Mycelium.

Fig. 12.—*Peziza aurantiaca.* (After Krombholz, nat. size.)

c, Archicarp.
l, Pollinodium.
s, Ascogenous filaments.
a, Asri.
r, *p*, The sterile tissue from which the paraphyses *h* spring.

Discomycetes.—Used in its widest sense this includes the Hysteriaceae, Phacidiaceae, Helvellaceae, &c. The group is characterized in general by the possession of an ascocarp which, though usually a completely closed structure during the earlier stages of development, at maturity opens out to form a bowl or saucer-shaped organ, thus completely exposing the layer of asci which forms the hymenium. Such an ascocarp goes by the name of *apothecium*. Owing to the shape of the fruit-body many of these forms are known as cup-fungi, the cup or apothecium often attaining a large size, sometimes several inches across (fig. 12). Functional male and female organs have been shown to exist in *Pyronema* and *Boudiera*; in *Lachnea stercorea* both ascogonia and antheridia are present, but the antheridium is non-functional, the ascogonial (female) nuclei fusing in pairs; this is also the case in *Humaria granulata* and *Ascobolus furfuraceus*, where the antheridium is entirely absent. In *H. rutilans*, however, both sexual organs are absent and the ascogenous hyphae arise apogamously from the ordinary hyphae of the mycelim. In all these cases the ascogonium and antheridium contain numerous nuclei; they are to be looked upon as gametangia in which there is no differentiation of gametes, and since they act as single gametes they are termed coenogametes. In some forms as in *Ascobolus* the ascogonium is multicellular, the various cells communicating by pores in the transverse walls (fig. 13).

In the Helvellaceae there is no apothecium but a large irregular fruit body which at maturity bears the asci on its surface. The development is only slightly known, but there is some evidence for believing that the fruit-body is closed in its very early stages.

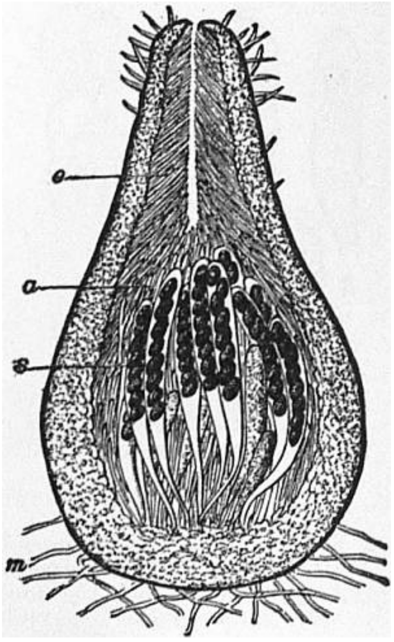

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Fig. 14.—Perithecium of Podospora fimiseda in longitudinal section. After v. Tavel.

s, Asci.
a, Paraphyses.
e, Periphyses.
m, Mycelial hyphae.

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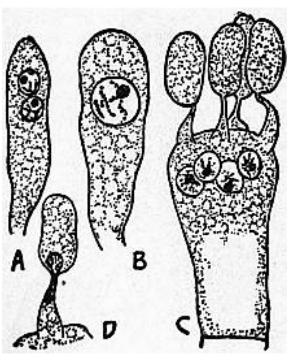
The genus *Peziza* (in its widest sense) may be taken as the type of the group. Most of them grow on living plants or on dead vegetable remains, very often on fallen wood; a number, however, are found growing on earth which is rich in humus. The genus *Sclerotinia* may be mentioned here; a number of forms have been investigated by Woronin. The conidia are fragrant and are carried by bees to the stigma of the bilberry; here they germinate with the pollen and the hyphae pass with the pollen tubes down the style; the former infect the ovules and produce sclerotia, therein reducing the fruits to a mummified condition. From the sclerotia later the apothecium develops. One species, *S. heteroica*, is *heteroecious*; the ascospores infecting the leaves of *Vaccinium uliginosum*, while the conidia which then arise infect only *Ledum palustre*. This is the only case of heteroecism known in the vegetable kingdom outside the Uredineae.

Pyrenomycetes.—This is an extraordinarily large and varied group of forms which mostly live parasitically or saprophytically on vegetable tissue, but a few are parasitic on insect-larvae. The group is characterized by a special type of ascocarp, the *perithecium*. This is typically of a flask-shaped form opening with a small pore at the top. The asci live at the bottom often mixed with paraphyses, while the upper “neck” of the flask is lined with special hyphae, the periphyses, which aid in the ejection of the spores (fig. 14). The simpler forms bear the perithecia directly on the mycelium, but the more highly developed forms often bear them on a special mycelial development—the stroma, which is often of large size and special shape and colour, and of dense consistence. The cytological details of development of the perithecia are not well known; most of them appear to develop their ascogenous hyphae in an apogamous way without any connexion with an ascogonium. Besides the special ascocarps, accessory reproductive organs are known in the majority of cases in the form of conidia.

Tuberineae.—These are a small group of fungi including the well-known truffles. They are found living saprophytically (in part parasitically) underground in forests. The asci are developed in the large dense fruit bodies (cleistothecia) and the spores escape by the decay of the wall. The fruit-body is of complicated structure, but its early stages of development are not known. Many of the fruit-bodies have a pleasant flavour and are eaten under the name of truffles (*Tuber brumale* and other species). The exact life-history of the truffle is not known.



Laboulbeniineae are a group of about 150 species of fungi found on insects, especially beetles, and principally known from the researches of Thaxter in America. The plant is a small, dark brown, erect structure (receptacle) of a few cells, and 1-10 mm. high, attached to the insect by the lowermost end (foot), and easily mistaken for a hair or similar appendage of the insect. The receptacle ends above in appendages, each consisting of one or a few cells, some of which are the male organs, others the female organs, and others again may be barren hairs. The male organ (antheridium) consists of a few cells, the terminal one of which either abstricts from its end, or emits from its interior the non-motile spermatia, reminding us of those of the Florideae. The female organ is essentially a flask-shaped structure; the neck of the flask growing out as the trichogyne, and the belly composed of an axial carpogenic cell surrounded by investing cells, From Strasburger’s *Lehrbuch der Botanik*, by permission of Gustav Fischer.

Fig. 15.—*Armillaria mellea.* (After Ruhland.)

A, Young basidium with the two primary nuclei.
B, After fusion of the two nuclei. *Hypholoma appendiculatum*.
C, A basidium before the four nuclei derived from the secondary nucleus of the basidium have passed into the four basidiospores.
D, Passage of a nucleus through the sterigma into the basidiospore.

and with one cell (trichophoric) between it and the trichogyne. These three elements—trichogyne, trichophoric cell, and carpogenic cell—are regarded as the procarp. The spermatia have been shown by Thaxter to fuse with the trichogyne, after which the axial cell below (carpogenic cell) undergoes divisions, and ultimately forms asci containing ascospores, while cells investing this form a perithecium, the whole structure reminding us essentially of the fructification of a Pyrenomycete. Many modifications in details occur, and the plants may be dioecious. No injury is done to the infested insects. It has lately been shown that there is a fusion of nuclei in connexion with ascus formation, so that there can be no doubt of the position of this extraordinary group of plants among the Ascomycetes. The various cells of these organisms are connected by large pits which are traversed by thick protoplasmic threads connecting one cell with the next. In this point and in their method of fertilization the Laboulbeniineae suggest a possible relationship of Ascomycetes and the Red Algae.

Basidiales.—This very large group of plants is characterized by the possession of a special type of conidiophore—the basidium, which gives its name to the group. The basidium is a unicellular or multicellular structure from which four basidiospores arise as outgrowths; it starts as a binucleate structure, but soon, like the ascus, becomes uninucleate by the fusion of the two nuclei. Then two successive nuclear divisions occur resulting in the formation of four nuclei which later migrate respectively into the four basidiospores (fig. 15). The Basidiales are further characterized by the complete loss of normal sexuality, but at some time or other in the life-history there takes place an association of two nuclei in a cell; the two nuclei are derived from separate cells or possibly in some cases are sister nuclei of the same cell. The two nuclei when once associated are termed “conjugate” nuclei, and they always divide at the same time, a half of each passing into each cell. This conjugate condition is finally brought to a close by the nuclear fusion in the basidium. Between the nuclear association and the nuclear fusion in the basidium many thousands of cell generations may be intercalated. This nuclear association of equivalent nuclei apparently represents a reduced sexual process (like the fusion of female nuclei in *Humaria granulata* and of vegetative nuclei in *H. rutilans*, among the Ascomycetes) in which, however, the actual fusion (normally, in a sexual process, occurring immediately after association) is delayed until the formation of the basidium. During the tetrad division in the basidium nuclear reduction occurs. There is thus in all the Basidiales an alternation of generations, obscured, however, by the apogamous transition from the gametophyte to sporophyte. The sporophyte may be considered to begin at the stage of nuclear association and end with the nuclear reduction in the basidium.

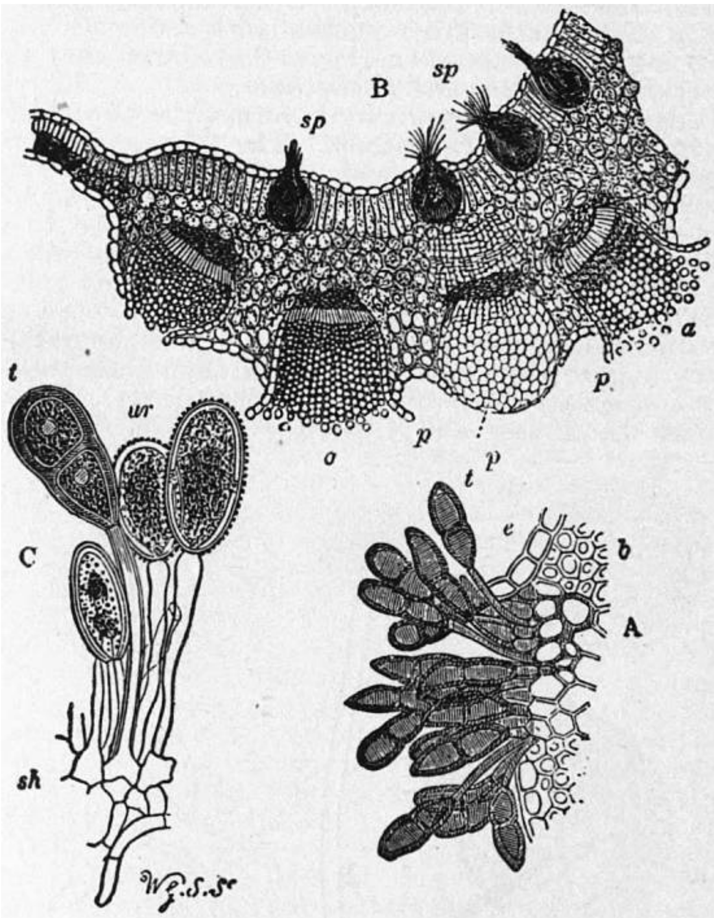


Fig. 16.—*Puccinia graminis.*

A, Mass of teleutospores (*t*) on a leaf of couch-grass.
e, Epidermis ruptured.
b, Sub-epidermal fibres. (After De Bary.)

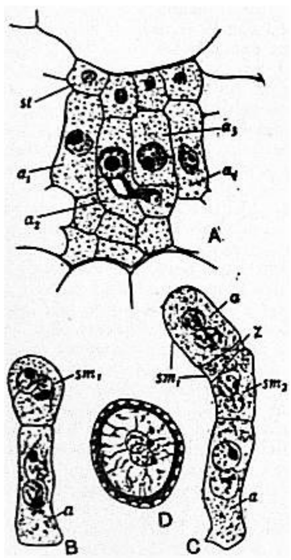
B, Part of vertical section through leaf of Berberis vulgaris, with *a*, aecidium fruits, *p*, peridium, and *sp*, spermogonia. (After Sachs.)
C, Mass of uredospores (*ur*), with one teleutospore (*t*).
sh, Sub-hymenial hyphae. (After De Bary.)

Uredineae.—This is a large group of about 2000 forms. They are all intercellular parasites living mostly on the leaves of higher plants. Owing to the presence of oily globules of an orange-yellow or rusty-red colour in their hyphae and spores they are termed Rust-Fungi. They are distinguished from the other fungi and the rest of the Basidiales by the great variety of the spores and the great elaboration of the life-history to be found in many cases. Five different kinds of spores may be present—teleutospores, sporidia (= basidiospores), aecidiospores, spermatia and uredospores (fig. 16). The teleutospore, with the sporidia which arise from it, is always present, and the division into genera is based chiefly on its characters. The teleutospore puts forth on germination a four-celled structure, the promycelium or basidium, and this bears later four sporidia or basidiospores, one on each cell. When the sporidia infect a plant the mycelium so produced gives origin to aecidiospores and spermatia; the aecidiospores on infection produce a mycelium which bears uredospores and later teleutospores. This is the life-history of the most complicated forms, of the so-called *eu* forms. In the *opsis* forms the uredospores are absent, the mycelium from the aecidiospores producing directly the teleutospores. In *brachy* and *hemi* the aecidiospores are absent, the mycelium from the sporidia giving origin directly to the uredospores; the former possess spermatia, in the latter they are absent. In *lepto* and *micro* forms both aecidiospores and uredospores are absent, the sporidia producing a mycelium which gives rise directly to teleutospores; in the *lepto* forms the teleutospores can germinate directly, in the *micro* forms only after a period of rest. We have thus a series showing a progressive reduction in the complexity of the life-history, the *lepto* and *micro* forms having a life-history like that of the Basidiomycetes. The *eu* and *opsis* forms may exhibit the remarkable phenomenon of heteroecism, *i.e.* the dependence of the fungus on two distinct host-plants for the completion of the life-history. Heteroecism is very common in this group and is now known in over one hundred and fifty species. In all cases of heteroecism the sporidia infect one host leading to the production of aecidiospores and spermatia (if present), while the aecidiospores are only able to infect another 343
host on which the uredospores (if present) and the teleutospores are developed. A few examples are appended:

Species.	Teleutospores on	Aecidiospores on
Coleosporium Senecionis	*Pinus*	*Senecio*
Melampsora Rostrupi	*Populus*	*Mecurialis*
Pucciniastrum Goeppertiana	*Vaccinium*	*Abies*
Gymnosporangium Sabinae	*Juniperus*	*Pyrus*
Uromyces Pisi	*Pisum, &c.*	*Euphorbia*
Puccinia graminis	*Triticum, &c.*	*Berberis*
P. dispersa	*Secale, &c.*	*Anchusa*
P. coronata	*Agrostis*	*Rhamnus*
P. Ari-Phalaridis	*Phalaris*	*Arum*
P. Caricis	*Carex*	*Urtica*
Cronartium Ribicola	*Ribes*	*Pinus*
Chrysomyxa Rhododendri	*Rhododendron*	*Picea*

Some of the Uredineae also exhibit the peculiarity of the development of biologic forms within a single morphological species, sometimes termed specialization of parasitism; this will be dealt with later under the section Physiology.

Cytology of Uredineae.—The study of the nuclear behaviour of the cells of the Uredineae has thrown great light on the question of sexuality. This group like the rest of the Basidiales exhibits an association of nuclei at some point in its life-history, but unlike the case of the Basidiomycetes the point of association in the Uredineae is very well defined in all those forms which possess aecidiospores. We find thus that in the *eu* and *opsis* forms the association of nuclei takes place at the base of the aecidium which produces the aecidiospores. There we find an association of nuclei either by the fusion of two similar cells as described by Christmann or by the migration of the nucleus of a vegetative cell into a special cell of the aecidium. After this association the nuclei continue in the conjugate condition so that the aecidiospores, the uredospore-bearing mycelium, the uredospores and the young teleutospores all contain two paired nuclei in their cells (fig. 17). Before the teleutospore reaches maturity the nuclei fuse, and the uninucleate condition then continues again until aecidium formation. In the *hemi*, *brachy*, *micro* and *lepto* forms, which possess no aecidium, we find that the association takes place at various points in the ordinary mycelium but always before the formation of the uredospores in the *hemi* and *brachy* forms, and before the formation of teleutospores in *micro* and *lepto* form. Whether the association of nuclei in the ordinary mycelium takes place by the migration of a nucleus from one cell to another or whether two daughter nuclei become conjugate in one cell, is not yet clear. The most reasonable interpretation of the spermatia is that they are abortive male cells. They have never been found to cause infection, and they have not the characters of conidia; the large size of their nuclei, the reduction of their cytoplasm and the absence of reserve material and their thin cell wall all point to their being male gametes. Although in the forms without aecidia the two generations are not sharply marked off from one another, we may look up the generation with single nuclei in the cells as the gametophyte and that with conjugate nuclei as the sporophyte. The subjoined diagram will indicate the relationship of the forms.



From Strasburger’s *Lehrbuch der Botanik*, by permission of Gustav Fischer.

Fig. 17.—*Phragmidium Violaceum.* (After Blackman.)

A, Portion of a young aecidium.
st, Sterile cell.
a, Fertile cells; at a_2 the passage of a nucleus from the adjoining cell is seen.
B, Formation of the first spore-mother-cell (*sm*), from the basal cell (*a*) of one of the

rows of spores.
C, A further stage in which from sm_1 the first aecidiospore (*a*) and the intercalary cell (*z*) have arisen.
sm_2, The second spore-mother-cell.
D, Ripe aecidiospore

Basidiomycetes.—This group is characterized by its greatly reduced life-history as compared with that of the *eu* forms among the Uredineae. All the forms have the same life-history as the *lepto* forms of that group, so that there is no longer any trace of sexual organs. There is also a further reduction in that the basidium is not derived from a teleutospore but is borne directly on the mycelium. Formerly, before the relationship of promycelium and basidium were understood, the Uredineae were considered as quite independent of the Basidiomycetes. Later, however, these Uredineae were placed as a mere subdivision of the Basidiomycetes. Although the Uredineae clearly lead on to the Basidiomycetes, yet owing to their retaining in many cases definite traces of sexual organs they are clearly a more primitive group. Their marked parasitic habit also separates them off, so that they are best included with the Basidiomycetes in a larger cohort which may be called Basidiales. Most of Basidiomycetes are characterized by the large sporophore on which the basidia with its basidiospores are borne.

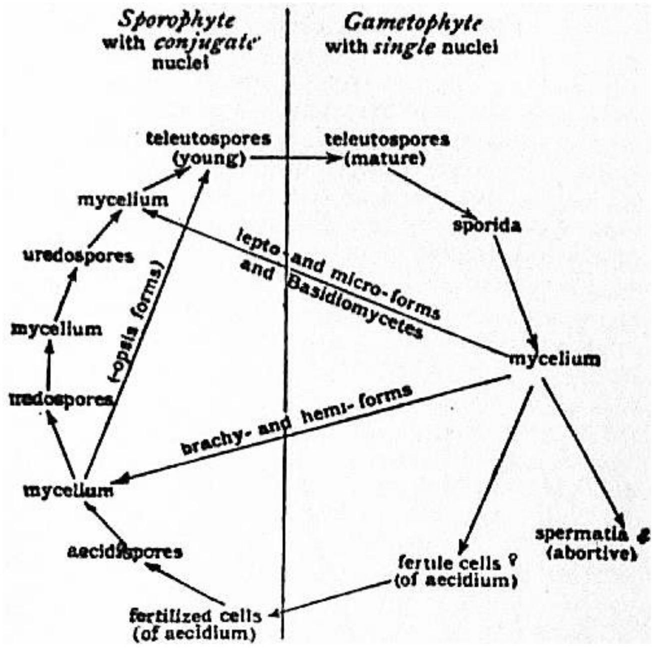


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FIG. 18.

It must be clearly borne in mind that though the Basidiomycetes show no traces of differentiated sexual organs yet, like the *micro* and *lepto* forms of the Uredineae, they still show (in the association of nuclei and later fusion of nuclei in the basidium), a reduced fertilization which denotes their derivation, through the Uredineae, from more typically sexual forms. No one has yet made out in any form the exact way in which the association of nuclei takes place in the group. The mycelium is always found to contain conjugate nuclei before the formation of basidia, but the point at which the conjugate condition arises seems very variable. Miss Nichols finds that it occurs very soon after the germination of the spore in *Coprinus*, but no fusion of cells or migration of nuclei was to be observed.

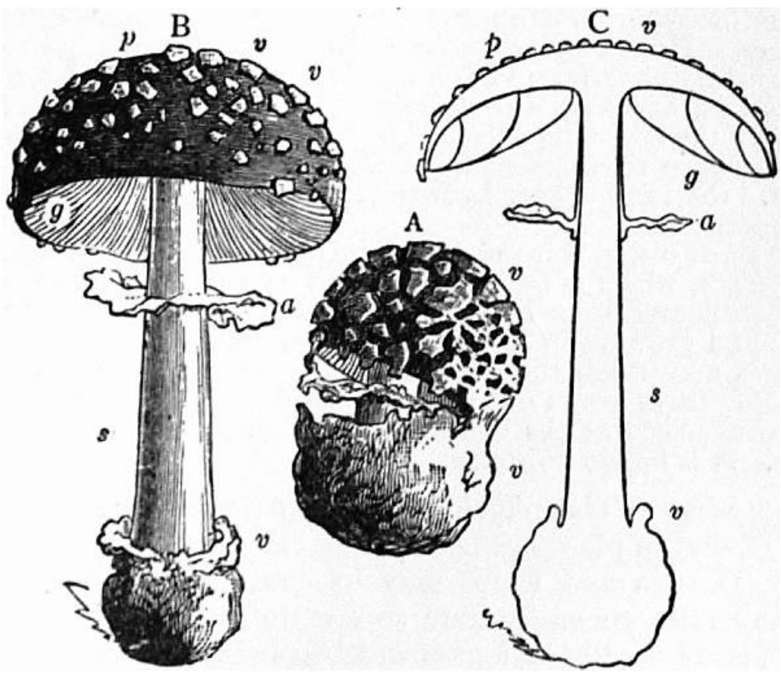

FIG. 19.—Amanita muscaria.

A, The young plant. *g*, The gills.

B, The mature plant.
C, Longitudinal section of mature plant.
p, The *pileus*.
a, The *annulus*, or remnant of *velum partiale*,
v, Remains of *volva* or *velum universale*.
s, The stalk.

Protobasidiomycetes.—This, by far the smaller division of Basidiomycetes, includes those forms which have a septate basidium. There are three families—Auriculariaceae, Pilacreaceae and Tremellinaceae. The first named contains a small number of forms with the basidium divided like the promycelium of the Uredineae. They are characterized by their gelatinous consistence and large size of their sporophore. *Hirneola* (*Auricularia*) *Auricula-Judae* is the well-known Jew’s Ear, so named from the resemblance of the sporophore to a human ear.

The Pilacreaceae are a family found by Brefeld to contain the genus *Pilacre*. *P. Petersii* has a transversely divided basidium as in *Auriculariaceae*, but the basidia are surrounded with a peridium-like sheath. The *Tremellinaceae* are characterized by the possession of basidia which are divided by two *vertical* walls at right angles to one another. From each of the four segments in the case of *Tremella* a long outgrowth arises which reaches to the surface of the hymenium and bears the basidiospores. In *Dacryomyces* only two outgrowths and two spores are produced. 344

Autobasidiomycetes.—In this by far the larger division of the Basidiomycetes the basidia are undivided and the four basidiospores are borne on short sterigmata nearly always at the apex of the basidium. The group may be divided into two main divisions, *Hymenomycetes* and *Gasteromycetes*.

Hymenomycetes are a very large group containing over 11,000 species, most of which live in soil rich in humus or on fallen wood or stems, a few only being parasites. In the simplest forms (*e.g. Exobasidium*) the basidia are borne directly on the ordinary mycelium, but in the majority of cases the basidia are found developed in layers (hymenium) on special sporophores of characteristic form in the various groups. In these sporophores (such as the well-known toadstools and mushrooms where the ordinary vegetative mycelium is underground) we have structures specially developed for bearing the basidiospores and protecting them from rain, &c., and for the distribution of the spores—see earlier part of article on distribution of spores (figs. 19 and 20). The underground mycelium in many cases spreads wider and wider each year, often in a circular manner, and the sporophores springing from it appear in the form of a ring—the so-called fairy rings. *Armillaria melleus* and *Polyporus annosus* are examples of parasitic forms which attack and destroy living trees, while *Merulius lacrymans* is the well-known “dry rot” fungus.

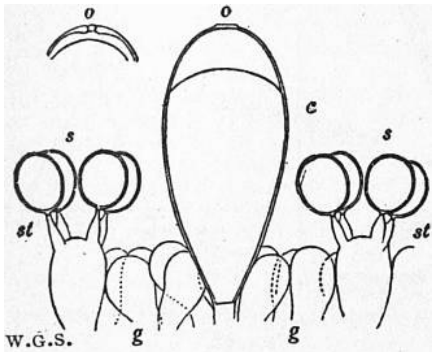

Fig. 20.—*Agaricus mucidus*. Portion of hymenium. *s*, Sporidia; *st*, sterigmata; *g*, sterile cells; *c*, cystidium, with operculum *o*.

Gasteromycetes are characterized by having closed sporophores or fruit-bodies which only open after the spores are ripe and then often merely by a small pore. The fruit-bodies are of very various shapes, showing a differentiation into an outer *peridium* and an inner spore-bearing mass, the *gleba*. The gleba is usually differentiated into a number of chambers which are lined directly by the hymenium (basidial layer), or else the chambers contain an interwoven mass of hyphae, the branches of which bear the basidia. By the breaking down of the inner tissues the spores often come to lie as a loose powdery mass in the interior of the hollow fruit-body, mixed sometimes with a capillitium. The best-known genera are *Bovista, Lycoperdon* (puff-ball) *Scleroderma, Geaster* (earth-star, *q.v.*). In the last-named genus the peridium is double and the outer layer becomes ruptured and spreads out in the form of star-shaped pieces; the inner layer, however, merely opens at the apex by a small pore.

The most complex members of the Gasteromycetes belong to the *Phalloideae*, which is sometimes placed as a distinct division of the Autobasidiomycetes. *Phallus impudicus*, the stink-horn, is occasionally found growing in woods in Britain. The fruit-body before it ruptures may reach the size of a hen’s egg and is white in colour; from this there grows out a hollow cylindrical structure which can be distinguished at the distance of several yards by its disgusting odour. It is highly poisonous.

Physiology.—The physiology of the fungi comes under the head of that of plants generally, and the works of Pfeffer, Sachs, Vines, Darwin and Klebs may be consulted for details. But we may refer generally here to certain phenomena peculiar to these plants, the life-actions of which are restricted and specialized by their peculiar dependence on organic supplies of carbon and nitrogen, so that most fungi resemble the colourless cells of higher plants in their nutrition. Like these they require water, small but indispensable quantities of salts of potassium, magnesium, sulphur and phosphorus, and supplies of carbonaceous and nitrogenous materials in different stages of complexity in the different cases. Like these, also, they respire oxygen, and are independent of light; and their various powers of growth, secretion, and general metabolism, irritability, and response to external factors show similar specific variations in both cases. It is quite a mistake to suppose that, apart from the chlorophyll function, the physiology of the fungus-cell is fundamentally different from that of ordinary plant-cells. Nevertheless, certain biological phenomena in fungi are especially pronounced, and of these the following require particular notice.

Parasitism. —Some fungi, though able to live as saprophytes, occasionally enter the body of living plants, and are thus termed facultative parasites. The occasion may be a wound (*e.g. Nectria*, *Dasyscypha*, &c.), or the enfeeblement of the tissues of the host, or invigoration of the fungus, the mycelium of which then becomes strong enough to overcome the host’s resistance (*Botrytis*). Many fungi, however, cannot complete their life-history apart from the host-plant. Such *obligate* parasites may be epiphytic (*Erysipheae*), the mycelium remaining on the outside and at most merely sending haustoria into the epidermal cells, or endophytic (*Uredineae*, *Ustilagineae*, &c.), when the mycelium is entirely inside the organs of the host. An epiphytic fungus is not necessarily a parasite, however, as many saprophytes (moulds, &c.) germinate and develop a loose mycelium on living leaves, but only enter and destroy the tissues after the leaf has fallen; in some cases, however, these saprophytic epiphytes can do harm by intercepting light and air from the leaf (*Fumago*, &c.), and such cases make it difficult to draw the line between saprophytism and parasitism. Endophytic parasites may be intracellular, when the fungus or its mycelium plunges into the cells and destroys their contents directly (*Olpidium*, *Lagenidium*, *Sclerotinia*, &c.), but they are far more frequently intercellular, at any rate while young, the mycelium growing in the lacunae between the cells (*Peronospora*, *Uredineae*) into which it may send short (*Cystopus*), or long and branched (*Peronospora Calotheca*) haustoria, or it extends in the middle lamella (*Ustilago*), or even in the solid substance of the cell-wall (*Botrytis*). No sharp lines can be drawn, however, since many mycelia are intercellular at first and subsequently become intracellular (*Ustilagineae*), and the various stages doubtless depend on the degrees of resistance which the host tissues are able to offer. Similar gradations are observed in the direct effect of the parasite on the host, which may be local (*Hemileia*) when the mycelium never extends far from the point of infection, or general (*Phytophthora*) when it runs throughout the plant. Destructive parasites rapidly ruin the whole plant-body (*Pythium*), whereas restrained parasites only tax the host slightly, and ill effects may not be visible for a long time, or only when the fungus is epidemic (*Rhytisma*). A parasite may be restricted during a long incubation-period, however, and rampant and destructive later (*Ustilago*). The latter fact, as well as the extraordinary fastidiousness, so to speak, of parasites in their choice of hosts or of organs for attack, point to reactions on the part of the host-plant, as well as capacities on that of the parasite, which may be partly explained in the light of what we now know regarding enzymes and chemotropism. Some parasites attack many hosts and almost any tissue or organ (*Botrytis cinerea*), others are restricted to one family (*Cystopus Candidus*) or genus (*Phytophthora infestans*) or even species (*Pucciniastrum Padi*), and it is customary to speak of root-parasites, leaf-parasites, &c., in expression of the fact that a given parasite occurs only on such organs—*e.g. Dematophora necatrix* on roots, *Calyptospora Goeppertiana* on stems, *Ustilago Scabiosae* in anthers, *Claviceps purpurea* in ovaries, &c. Associated with these relations are the specializations which parasites show in regard to the age of the host. Many parasites can enter a seedling, but are unable to attack the same host when older—*e.g. Pythium*, *Phytophthora omnivora*.

Chemotropism.—Taken in conjunction with Pfeffer's beautiful discovery that certain chemicals exert a distinct attractive influence on fungus hyphae (*chemotropism*), and the results of Miyoshi's experimental application of it, the phenomena of enzyme-secretion throw considerable light on the processes of infection and parasitism of fungi. Pfeffer showed that certain substances in definite concentrations cause the tips of hyphae to turn towards them; other substances, though not innutritious, repel them, as also do nutritious bodies if too highly concentrated. Marshall Ward showed that the hyphae of *Botrytis* pierce the cell-walls of a lily by secreting a cytase and dissolving a hole through the membrane. Miyoshi then demonstrated that if *Botrytis* is sown in a lamella of gelatine, and this lamella is superposed on another similar one to which a chemotropic substance is added, the tips of the hyphae at once turn from the former and enter the latter. If a thin cellulose membrane is interposed between the lamellae, the hyphae nevertheless turn chemotropically from the one lamella to the other and pierce the cellulose membrane in the process. The hyphae will also dissolve their way through a lamella of collodion, paraffin, parchment paper, elder-pith, or even cork or the wing of a fly, to do which it must excrete very different enzymes. If the membrane is of some impermeable substance, like gold leaf, the hyphae cannot dissolve its way through, but the tip finds the most minute pore and traverses the barrier by means of it, as it does a stoma on a leaf We may hence conclude that a parasitic hyphae pierces some plants or their stomata and refuses to enter others, because in the former case there are chemotropically attractive substances present which are absent from the latter, or are there replaced by repellent poisonous or protective substances such as enzymes or antitoxins.

Specialization of Parasitism.—The careful investigations of recent years have shown that in several groups of fungi we cannot be content to distinguish as units morphologically different species, but we are compelled to go deeper and analyse further the species. It has been shown especially in the *Uredineae* and *Erysiphaceae* that many forms which can hardly be distinguished morphologically, or which cannot be differentiated at all by structural characters, are not really homogeneous but consist of a number of forms which are sharply distinguishable by their infecting power. Eriksson found, for example, that the well-known species *Puccinia graminis* could be split up into a number of forms which though morphologically similar were physiologically distinct. He found that the species really consisted of six distinct races, each having a more or less narrow range of grasses on which it can live. The six races he named *P. graminis Secalis*, *Tritici*, *Avenae*, *Airae*, *Agrostis*, *Poae*. The first named will grow on rye and barley but not on wheat or oat. The form *Tritici* is the least sharply marked and will grow on wheat, barley, rye and oat but not on the other grasses. The form *Avenae* will grow on oat and many grasses but not on the other three cereals mentioned. The last three forms grow only on the genera *Aira*, *Agrostis* and *Poa* respectively. All these forms have of course their aecidium-stage on the barberry. The terms biologic forms, biological species, physiological species, physiological races, specialized forms have all been applied to these; perhaps the term biologic forms is the most satisfactory. A similar specialization has been observed by Marshall Ward in the *Puccinia* parasitic on species of *Bromus*, and by Neger, Marchal and especially Salmon in the Erysiphaceae. In the last-named family the single morphological species *Erysiphe graminis* is found growing on the cereals, barley, oat, wheat, rye and a number of wild grasses (such as *Poa*, *Bromus*, *Dactylis*). On each of these host-plants the fungus has become specialized so that the form on barley cannot infect the other three cereals or the wild grasses and so on. Just as the uredospores and aecidiospores both show these specialized characters in the case of *Puccinia graminis* so we find that both the conidia and ascospores of *E. graminis* show this phenomenon. Salmon has further shown in investigating the relation of *E. graminis* to various species of the genus, *Bromus*, that certain species may act as "bridging species," enabling the transfer of a biologic form to a host-plant which it cannot normally infect. Thus the biologic form on *B. racemosus* cannot infect *B. commutatus*. If, however, conidia from *B. racemosus* are sown on *B. hordaceus*, the conidia which develop on that plant are now able to infect *B. commutatus*; thus *B. hordaceus* acts as a bridging species. Salmon also found that injury of a leaf by mechanical means, by heat, by anaesthetics, &c., would affect the immunity of the plant and allow infection by conidia which was not able to enter a normal leaf. The effect of the abnormal conditions is probably to stop the production of, or weaken or destroy the protective enzymes or antitoxins, the presence of which normally confers immunity on the leaf. 345

Symbiosis.—The remarkable case of life in common first observed in lichens, where a fungus and an alga unite to form a compound organism—the lichen—totally different from either, has now been proved to be universal in these plants, and lichens are in all cases merely algae enmeshed in the interwoven hyphae of fungi (see LICHENS). This dualism, where the one constituent (alga) furnishes carbohydrates, and the other (fungus) ensures a supply of mineral matters, shade and moisture, has been termed *symbiosis*. Since then numerous other cases of symbiosis have been demonstrated. Many trees are found to have their smaller roots invaded by fungi and deformed by their action, but so far from these being injurious, experiments go to show that this mycorhiza (fungus-root) is necessary for the well-being of the tree. This is also the case with numerous other plants of moors and woodlands—*e.g.* Ericaceae, Pyrolaceae, Gentianaceae, Orchidaceae, ferns, &c. Recent experiments have shown that the difficulties of getting orchid seeds to germinate are due to the absence of the necessary fungus, which must be in readiness to infect the young seedling immediately after it emerges from the seed. The well-known failures with rhododendrons, heaths, &c., in ordinary garden soils are also explained by the need of the fungus-infected peat for their roots. The rôle of the fungus appears to be to supply materials from the leaf-mould around, in forms which ordinary root-hairs are incapable of providing for the plant; in return the latter supports the fungus at slight expense from its abundant stores of reserve materials. Numerous other cases of symbiosis have been discovered among the fungi of fermentation, of which those between *Aspergillus* and yeast in saké manufacture, and between yeasts and bacteria in kephir and in the ginger-beer plant are best worked out. For cases of symbiosis see Bacteriology.

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(H. M. W.; V. H. B.)

FUNJ (Funniyeh, Fung, Fungha), a very mixed negroid race, occupying parts of Sennar and the hilly country to the south between the White and Blue Niles. They traditionally come from west of the White Nile and are affiliated by some to the Kordofan Nubas, by others, more justifiably, to the negro Shilluks. These Funj, who became the dominant race in Sennar in the 15th century, almost everywhere assimilated the speech, religion and habits of the Arabs settled in that region. Until the 19th century they were one of the most powerful of African peoples in the eastern Sudan. About the end of the 15th century they overthrew the kingdom of Aloa, between the two Niles, and conquered the neighbouring peoples of the Sudan, Nubia and even Kordofan. The Funj had mixed much with the Arabs before their conquests, and had been converted to Islam. But they were still in many ways savages, for James Bruce (who traversed the district in 1772) says that their most famous king, Malek-el-Gahman, preferred human liver to any other food, and the Belgian traveller E. Pruyssenaere (1826-1864) found them still performing pagan rites on their sacred Mount Gula. Ernst Marno declared that as late as 1870 the most southern branch of the race, the Boruns, a non-Arabic speaking tribe, were cannibals. The Funj kings were content with levying tribute on their neighbours, and in this loose way Shendi, Berber and Dongola were once tributary. The Arab viziers gradually absorbed all power, the Funj sovereignty becoming nominal; and in 1821 the Egyptians easily destroyed the Funj domination. To-day the Funj are few, and represent no real type. They are a bright, hospitable folk. Many of them are skilful surgeons and go far afield in their work. The fellahin, indeed, call surgeons "Senaari" (men of Sennar). See further Sennar and Sudan (Anglo-Egyptian).

FUNKIA, in botany, a genus of rather handsome, hardy, herbaceous plants belonging to the natural order Liliaceae, and natives of China and Japan. They are tuberous, with broadly ovate or heart-shaped leaves and racemes of white or pale lilac, drooping, funnel-shaped flowers. They are useful for the borders of a shrubbery, the lawn or rock-work, or may be grown in pots for the greenhouse. The plants are propagated by dividing the crowns in autumn or when growth begins in spring.

FUNNEL (through an O. Fr. *founil*, found in Breton, from Lat. *infundibulum*, that through which anything is poured, from *fundere*, to pour), a vessel shaped like a cone having a small tube at the apex through which powder, liquid, &c., may be easily passed into another vessel with a small opening. The term is used in metal-casting of the hole through which the metal is poured into a mould, and in anatomy and zoology of an *infundibulum* or funnel-shaped organ. The word is thus used generally of any shaft or passage to convey light, air or smoke, as of the chimney of an engine or a steam-boat, or the flue of an ordinary chimney. It is also used of a shaft or channel in rocks, and in the decoying of wild-fowl is applied to the cone-shaped passage leading from a pond and covered with a net, a "funnel-net," into which the birds are decoyed.

FUR (connected with O. Fr. *forre*, a sheath or case; so "an outer covering"), the name specially given to the covering of the skin in certain animals which are natives of the colder climates, lying alongside of another and longer covering, called the overhair. The fur differs from the overhair, in that it is soft, silky, curly, downy and barbed lengthwise, while the overhair is straight, smooth and comparatively rigid. These properties of fur constitute its essential value for felting purposes, and mark its difference from wool and silk; the first, after some slight preparation by the aid of hot water, readily unites its fibres into a strong and compact mass; the others can best be managed by spinning and weaving.

On the living animal the overhair keeps the fur filaments apart, prevents their tendency to felt, and protects them from injury—thus securing to the animal an immunity from cold and storm; while, as a matter of fact, this very overhair, though of an humbler name, is most generally the beauty and pride of the pelt, and marks its chief value with the furrier. We arrive thus at two distinct and opposite uses and values of fur. Regarded as useful for felt it is denominated staple fur, while with respect to its use with and on the pelt it is called fancy fur.

History.—The manufacture of fur into a felt is of comparatively modern origin, while the use of fur pelts as a covering for the body, for the couch, or for the tent is coeval with the earliest history of all northern tribes and nations. Their use was not simply a barbarous expedient to defend man from the rigours of an arctic winter; woven wool alone cannot, in its most perfect form, accomplish this. The pelt or skin is requisite to keep out the piercing wind and driving storm, while the fur and overhair ward off the cold; and “furs” are as much a necessity to-day among more northern peoples as they ever were in the days of barbarism. With them the providing of this necessary covering became the first purpose of their toil; subsequently it grew into an object of barter and traffic, at first among themselves, and afterwards with their neighbours of more temperate climes; and with the latter it naturally became an article of fashion, of ornament and of luxury. This, in brief, has been the history of its use in China, Tatary, Russia, Siberia and North America, and at present the employment of fancy furs among civilized nations has grown to be more extensive than at any former period.

The supply of this demand in earlier times led to such severe competition as to terminate in tribal pillages and even national wars; and in modern times it has led to commercial ventures on the part of individuals and companies, the account of which, told in its plainest form, reads like the pages of romance. Furs have constituted the price of redemption for royal captives, the gifts of emperors and kings, and the peculiar badge of state functionaries. At the present day they vie with precious gems and gold as ornaments and garniture for wealth and fashion; but by their abundance, and the cheapness of some varieties, they have recently come within the reach of men of moderate incomes. The history of furs can be read in Marco Polo, as he grows eloquent with the description of the rich skins of the khan of Tatary; in the early fathers of the church, who lament their introduction into Rome and Byzantium as an evidence of barbaric and debasing luxury; in the political history of Russia, stretching out a powerful arm over Siberia to secure her rich treasures; in the story of the French occupation of Canada, and the ascent of the St Lawrence to Lake Superior, and the subsequent contest to retain possession against England; in the history of early settlements of New England, New York and Virginia; in Irving’s *Astoria*; in the records of the Hudson’s Bay Company; and in the annals of the fairs held at Nizhniy Novgorod and Leipzig. Here it may suffice to give some account of the present condition of the trade in fancy furs. The collection of skins is now chiefly a matter of private enterprise. Few, if any, monopolies exist.

Natural Supplies.—We are dependent upon the Carnivora, Rodentia, Ungulata and Marsupialia for our supplies of furs, the first two classes being by far of the greatest importance. The Carnivora include bears, wolverines, wolves, raccoons, foxes, sables, martens, skunks, kolinskis, fitch, fishers, ermines, cats, sea otters, fur seals, hair seals, lions, tigers, leopards, lynxes, jackals, &c. The Rodentia include beavers, nutrias, musk-rats or musquash, marmots, hamsters, chinchillas, hares, rabbits, squirrels, &c. The Ungulata include Persian, Astrachan, Crimean, Chinese and Tibet lambs, mouflon, guanaco, goats, ponies, &c. The Marsupialia 347
include opossums, wallabies and kangaroos. These, of course, could be subdivided, but for general purposes of the fur trade the above is deemed sufficient.

The question frequently arises, not only for those interested in the production of fur apparel, but for those who derive so much comfort and pleasure from its use, whether the supply of fur-bearing animals is likely to be exhausted. Although it is a fact that the demand is ever increasing, and that some of the rarer animals are decreasing in numbers, yet on the other hand some kinds of furs are occasionally neglected through vagaries of fashion, which give nature an opportunity to replenish their source. These respites are, however, becoming fewer every day, and what were formerly the most neglected kinds of furs are becoming more and more sought after. The supply of some of the most valuable, such as sable, silver and natural black fox, sea otter and ermine, which are all taken from animals of a more or less shy nature, does very gradually decrease with persistent hunting and the encroachment of man upon the districts where they live, but the climate of these vast regions is so cold and inhospitable that the probabilities of man ever permanently inhabiting them in numbers sufficient to scare away or exterminate the fur-bearing wild animals is unlikely. Besides these there are many useful, though commonplace, fur-bearing animals like mink, musquash, skunk, raccoon, opossum, hamster, rabbit, hares and moles, that thrive by depredations upon cultivated land. Some of these are reared upon extensive wild farms. In addition there are domestic fur-bearing animals, such as Persian, Astrachan and Chinese lambs, and goats, easily bred and available.

With regard to the rearing of the Persian lamb, there is a prevalent idea that the skins of the unborn lamb are frequently used; this, however, is a mistake. A few such skins have been taken, but they are too delicate to be of any service. The youngest, known as “broadtails,” are killed when a few days old, but for the well-developed curly fur, the lambs must be six or seven weeks old. During these weeks their bodies are covered with leather so that the fur may develop in close, light and clean curls. The experiment has been tried of rearing rare, wild, fur-bearing animals in captivity, and although climatic conditions and food have been precisely as in their natural environment, the fur has been poor in quality and bad in colour, totally unlike that taken from animals in the wild state. The sensation of fear or the restriction of movement and the obtaining of food without exertion evidently prevent the normal development of the creature.

In mountainous districts in the more temperate zones some good supplies are found. Chinchillas and nutrias are obtained from South America, whence come also civet cats, jaguars, ocelots and pumas. Opossums and wallabies, good useful furs, come from Australia and New Zealand. The martens, foxes and otters imported from southern Europe and southern Asia, are very mixed in quality, and the majority are poor compared with those of Canada and the north.

Certain characteristics In the skin reveal to the expert from what section of territory they come, but in classifying them it is considered sufficient to mention territories only.

Some of the poorer sorts of furs, such as hamster, marmot, Chinese goats and lambs, Tatar ponies, weasels, kaluga, various monkeys, antelopes, foxes, otters, jackals and others from the warmer zones, which until recently were neglected on account of their inferior quality of colour, by the better class of the trade, are now being deftly dressed or dyed in Europe and America, and good effects are produced, although the lack of quality when compared with the better furs from colder climates which possess full top hair, close underwool and supple leathers, is readily manifest. It is only the pressure of increasing demand that makes marketable hard pelts with harsh brittle hair of nondescript hue, and these would, naturally, be the last to attract the notice of dealers.

As it is impossible that we shall ever discover any new fur-bearing animals other than those we know, it behoves responsible authorities to enforce close seasons and restrictions, as to the sex and age, in the killing for the purpose of equalizing the numbers of the catches. As evidence of indiscriminate slaughter the case of the American buffaloes may be cited. At one time thousands of buffalo skins were obtainable and provided material for most useful coats and rugs for rough wear in cold regions, but to-day only a herd or so of the animals remain, and in captivity.

The majority of animals taken for their fur are trapped or snared, the gun being avoided as much as possible in order that the coat may be quite undamaged. Many weary hours are spent in setting baits, traps and wires, and, frequently, when the hunter retraces his steps to collect the quarry it is only to find it gone, devoured by some large animal that has visited his traps before him. After the skins have been carefully removed—the sooner after death the better for the subsequent condition of the fur—they are lightly tacked out, pelt outwards, and, without being exposed to the sun or close contact with a fire, allowed to dry in a hut or shady place where there is some warmth or movement of air. With the exception of sealskins, which are pickled in brine, all raw skins come to the various trade markets simply dried like this.

Quality and Colour.—The best fur is obtained by killing animals when the winter is at its height and the colder the season the better its quality and colour. Fur skins taken out of season are indifferent, and the hair is liable to shed itself freely; a good furrier will, however, reject such faulty specimens in the manufacturing. The finest furs are obtained from the Arctic and northern regions, and the lower the latitude the less full and silky the fur, till, at the torrid zone, fur gives place to harsh hair without any underwool. The finest and closest wools are possessed by the amphibious Carnivora and Rodentia, viz. seals, otters, beavers, nutrias and musquash, the beauty of which is not seen until after the stiff water or top hairs are pulled out or otherwise removed. In this class of animal the underneath wool of the belly is thicker than that of the back, while the opposite is true of those found on the land. The sea otter, one of the richest and rarest of furs, especially for men’s wear, is an exception to this unhairing process, which it does not require, the hair being of the same length as the wool, silky and bright, quite the reverse of the case of other aquatic animals.

Of sealskins there are two distinct classes, the fur seals and the hair seals. The latter have no growth of fur under the stiff top hair and are killed, with few exceptions (generally of the marbled seals), on account of the oil and leather they yield. The best fur seals are found off the Alaska coast and down as far south as San Francisco.

It is found that in densely wooded districts furs are darker in colour than in exposed regions, and that the quality of wool and hair is softer and more silky than those from bare tracts of country, where nature exacts from its creatures greater efforts to secure food, thereby developing stronger limbs and a consequently coarser body covering.

As regards density of colour the skunk or black marten has the blackest fur, and some cats of the domestic kind, specially reared for their fur, are nearly black. Black bears have occasionally very black coats, but the majority have a brownish underwool. The natural black fox is a member of the silver fox family and is very rare, the skins bringing a high price. Most silver foxes have dark necks and in some the dark shade runs a quarter, half-way, or three-quarters, or even the whole length of the skin, but it is rather of a brownish hue. Some Russian sables are of a very dense bluish brown almost a black, which is the origin undoubtedly of the term “sables,” while some, from one district in particular, have a quantity of silver hairs, evenly interspersed in the fur, a peculiarity which has nothing to do with age. The best sea otters have very dark coats which are highly esteemed, a few with silver hairs in parts; where these are equally and evenly spread the skins are very valuable. Otters and beavers that run dark in the hair or wool are more valuable than the paler ones, the wools of which are frequently touched with a chemical to produce a golden shade. This is also done with nutrias after unhairing. The darker sorts of mink, musquash, raccoon and wolverine are more valuable than 348 the paler skins.

Collective Supplies and Sales.—There are ten large American and Canadian companies with extensive systems for gathering the annual hauls of skins from the far-scattered trappers. These are the Hudson’s Bay Co., Russian Fur Co., Alaska Commercial Co., North American Commercial Co., Russian Sealskin Co., Harmony Fur Co., Royal Greenland Fur Co., American Fur Co., Missouri Co. and Pacific Co. Most of the raw skins are forwarded to about half-a-dozen brokers in London, who roughly sort them in convenient lots, issuing catalogues to the traders of the world, and after due time for examination of the goods by intending purchasers, the lots are sold by public auction. The principal sales of general furs are held in London in January and March, smaller offerings being made in June and October; while the bulk of fur sealskins is sold separately in December. The Hudson’s Bay Co.’s sales take place before the others, and, as no reserves are placed on any lot, the results are taken as exactly indicating current values. While many buyers from America and Russia are personally in attendance at the sales, many more are represented by London and Leipzig agents who buy for them upon commission. In addition to the fur skins coming from North America vast numbers from Russia, Siberia, China, Japan, Australia and South America are offered during the same periods at public auction. Fairs are also held in Siberia, Russia and Germany for the distribution of fur skins as follows:—

January:	Frankfort-on-the-Oder	Small collection of provincial produce, such as otter, fox, fitch and marten.
February:	Irbit, Siberia	General Russian furs.

Easter:	Leipzig, Germany	General furs.
August:	Nizhniy Novgorod, Russia	Persian lamb and general furs.
August:	Kiakhta, Siberia	Chinese furs and ermine.
December:	Ishim, Siberia	Chiefly squirrels.

Of course there are many transactions, generally in the cheaper and coarser kinds of furs, used only in central Europe, Russia and Asia which in no way interest the London market, and there are many direct consignments of skins from collectors in America and Russia to London, New York and Leipzig merchants. But the bulk of the fine furs of the world is sold at the large public trade auction sales in London. The chief exceptions are the Persian and Astrachan lambs, which are bought at the Russian fairs, and are dressed and dyed in Leipzig, and the ermine and Russian squirrels, which are dressed and manufactured into linings either in Russia or Germany before offered for sale to the wholesale merchants or manufacturers.

The annual collection of fur skins varies considerably in quantity according to the demand and to the good or bad climatic conditions of the season; and it is impossible to give a complete record, as many skins are used in the country of their origin or exported direct to merchants. But a fairly exact statement of the numbers sold in the great public trade auction sales in London during the year 1905-1906 is herewith set out.

Year ending 31st of March 1906.	Total Number of Skins.
Badger	28,634
Badger, Japanese	6,026
Bear	18,576
Beaver	80,514
Cat, Civet	157,915
Cat, House	126,703
Cat, Wild	32,253
Chinchilla (La Plata), known also as Bastard	43,578
Chinchilla Peruvian finest	5,603
Deer, Chinese	124,355
Ermine	40,641
Fisher	5,949
Fitch	77,578
Fox, Blue	1,893
Fox, Cross	10,276
Fox, Grey	59,561
Fox, Japanese	81,429
Fox, Kit	4,023
Fox, Red	158,961
Fox, Silver	2,510
Fox, White	27,463
Goats, Chinese	261,190
Hares	41,256
Kangaroo	7,115
Kid, Chinese linings and skins equal to	5,080,047
Kolinsky	114,251
Lamb, Mongolian linings and skins equal to	214,072
Lamb, Slink linings and skins equal to	167,372
Lamb, Tibet linings and skins equal to	794,130
Leopard	3,574
Lynx	88,822
Marmot, linings and skins equal to	1,600,600
Marten, Baum	4,573
Marten, Japanese	16,461
Marten, Stone	12,939
Mink, Canadian and American	299,254
Mink, Japanese	360,373
Mouflon	23,594
Musk-rat or Musquash, Brown	5,126,339
Musk-rat or Musquash, Black	41,788
Nutria	82,474
Opossum, American	902,065
Opossum, Australian	4,161,685
Otter, River	21,235
Otter, Sea	522
Raccoon	310,712
Sable, Canadian and American	97,282
Sable, Japanese	556
Sable, Russian	26,399
Seals, Fur	77,000
Seals, Hair	31,943
Skunk	1,068,408
Squirrel	194,596
Squirrel Linings each averaging 126 skins	1,982,736
Tiger	392
Wallaby	60,956
Wolf	56,642
Wolverine	1,726
Wombat	193,625

A brief account of the different qualities of the pelts, with some general remarks as to their customary uses, follows. The prices quoted are subject to constant fluctuation and represent purely trade prices for bulk, and it should be explained that the very great variations are due to different sizes, qualities and colours, and moreover are only *first cost*, before skins are dressed and prepared. These preparations are in some cases expensive, and there is generally a considerable percentage of waste. The prices cannot be taken as a guide to the wholesale price of a single and finished skin, but simply as *relative* value.

The fullest and darkest skins of each kind are the most valuable, and, in cases of bluish grey or white, the fuller, clearer and brighter are the more expensive. A few albinos are found in every species, but whatever their value to a museum, they are of little commercial importance. Some odd lots of skins arrive designated simply as "sundries," so no classification is possible, and this will account for the absence of a few names of skins of which the imports are insignificant in quantity, or are received direct by the wholesale merchants.

Names, Qualities and Uses of Pelts.[1]

Astrachan.—See *Lambs*, below.

Badger.—Size 2 × 1 ft. American sorts have coarse thick underwool of a pale fawn or stone colour with a growth of longer black and white hairs, 3 or 4 in. long; a very durable but clumsy fur. The best skins are exported to France, Spain and Italy, and used for carriage rugs and military purposes. Asiatic, including Japanese, skins are more woolly. Russian and Prussian kinds are coarser and darker, and used mostly for brush trade. Value 6d. to 19s.

Bear, Australian.—See *Wombat*, below.

Bear, Black.—Size 6 × 3 ft. Fine dark brown underwool with bright black and flowing top hair 4 in. long. Cubs are nearly as long in the hair although only about half the size and not only softer and better, but have the advantage of being very much lighter in pelt. Widely distributed in North America, the best come from Canada, are costly and are used for military caps, boas, muffs, trimmings, carriage rugs and coachmen's capes, and the fur wears exceedingly well. Value 17s. 6d. to 86s. Those from East India and warm climates are harsh, poor and only fit for floor rugs.

Bear, Brown.—Size 6 × 3 ft. Similar in quality to the black, but far more limited in number; the colours range from light yellow to a rich dark brown. The best come from Hudson Bay territory and are valuable. Used for muffs, trimmings, boas, and carriage rugs. Inferior sorts, almost grizzly in effect and some very pale, are found in Europe and Asia and are mostly used locally. In India there is a species called Isabelline bear, which was formerly imported to Great Britain, but does not now arrive in any quantity worth mentioning. Value 10s. 6d. to 60s., Isabelline sort 10s. 6d. to 78s. 349

Bear, Grizzly.—Size 8 × 4 ft. Coarse hair, heavy pelt, mostly dark yellowish and brown colours, only found in western parts of United States, Russia and Siberia. Used as carriage rugs and floor rugs, most durable for latter purpose and of fine effect. They are about half the value of brown bear. Value 15s. to 54s.

Bear, Isabelline.—See *Bear*, *Brown*, above.

Bear, White.—Size 10 × 5 ft. The largest of all bears. Short close hair except on flanks, colour white to yellow. An inhabitant of the Arctic circle, best from Greenland. Used for floor rugs, very durable; and very white specimens are valuable. Value 20s. to 520s.

Beaver. Size 3 × 2 ft. The largest of rodents, it possesses a close underwool of bluish-brown hue, nearly an inch in depth, with coarse, bright, black or reddish-brown top hair, 3 in. long. Found widely in North America. After being unhaired the darkest wools are the most valuable, although many people prefer the bright, lighter brown tones. Used for collars, cuffs, boas, muffs, trimmings, coat linings and carriage aprons, and is of a most durable nature, in addition to having a rich and good appearance. Value 10s. to 39s. 6d.

Broadtail.—See *Lambs*, below.

Caracal.—A small lynx from India, the fur very poor, seldom imported.

Caracul.—See *Goats* and *Lambs*, below.

Cat, Civet.—Size 9 × 4½ in., short, thick and dark underwool with silky black top hair with irregular and unique white markings. It is similar to skunk, but is much lighter in weight, softer and less full, without any disagreeable odour. Used for coat linings it is very warm and durable. A few come from China, but the fur is yellowish-grey, slightly spotted and worth little. Value 1s. 1d. to 1s. 11d.

Cat, House, &c.—18 × 9 in., mostly black and dark brown, imported from Holland, Bavaria, America and Russia, where they are reared for their coats. The best, from Holland, are used for coat linings. Although in colour, weight and warmth they are excellent, the fur is apt to become loose and to fall off with friction of wear. The black are known as genet, although the true genet is a spotted wild cat. Wild sorts of the tabby order are coarser, and not so good and silky in effect as when domestically reared. Value of the black sorts 2d. to 3s. Wild 9d. to 14s. Some small wild cats, very poor flat fur of a pale fawn colour with yellow spots, are imported from Australia and used for linings. Value 5½d. to 1s. 1d.

Cheetah.—Size of a small leopard and similar in colour, but has black spots in lieu of rings. Only a few are now imported, which are used for mats. Value 2s. 6d. to 18s.

Chinchilla, Peruvian and Bolivian.—Size 12 × 7 in., fur 1 to 1¼ in. deep. Delicate blue-grey with black shadings, one of nature's most beautiful productions, though not a durable one. Used for ladies' coats, stoles, muffs, hats and trimmings. Yearly becoming scarcer and most costly. Value 8s. 6d. to 56s. 8d.

Chinchilla, La Plata, incorrectly named and known in the trade as "bastard chinchilla," size 9 × 4 in., in a similar species, but owing to lower altitudes and warmer climatic conditions of habitation is smaller, with shorter and less beautiful fur, the underwool colour being darker and the top colour less pure. Used exactly as the better kind, and the picked skins are most effective. As with the best sort it is not serviceable for constant wear. Value 4s. 2d. to 27s. 6d.

Chinchillone.—Size 13 × 8 in., obtained also from South America. Fur is longer and weaker and poorer and yellower than chinchilla. Probably a crossbred animal, very limited importation. Value 3s. 6d. to 16s. 8d.

Deer, Chinese and East Indian.—Small, light, pelted skins, the majority of which are used for mats. Reindeer and other varieties are of little interest for use other than trophy mats. Thousands are taken for the leather trade. Value of Chinese 1s. 2d. to 1s. 6d. each.

Dog.—The only dogs that are used in the fur trade in civilized countries are those imported from China, which are heavy and coarse, and only used in the cheaper trade, chiefly for rugs. Value 6d. to 1s.

Dog Wolf.—See *Wolf*, below.

Ermine.—Size 12 × 2½ in. Underwool short and even, with a shade longer top hair. Pelt light and close in texture, and durable. In the height of winter the colour is pure white with exception of the tip of tail, which is quite black. Supplies are obtained from Siberia and America. Best are from Ishim in Siberia. Used for cloak linings, stoles, muffs and trimmings, also for embellishment of British state, parliamentary and legal robes. When this fur is symmetrically spotted with black lamb pieces it is styled miniver, in which form it is used at the grand coronation functions of British sovereigns. Value 1s. 3d. to 8s. 6d.

Fisher.—Size 30 × 12 in., tail 12 to 18 in. long, the largest of the martens; has a dark shaded deep underwool with fine, glossy, dark and strong top hair 2 in. or more long. Best obtained from British America. The tails are almost black and make up most handsomely into trimmings, muffs, &c. Tails worked separately in these forms are as rich and fine and more durable than any other fur suitable for a like purpose. The fur of the skin itself is something like a dark silky raccoon, but is not as attractive as the tails. Value 12s. to 46s.

Fitch.—Size 12 × 3 in., of the marten species, also known as the pole cat. Yellow underwool 1/3 in. deep, black top hair, 1½ to 1¾ in. long, very fine and open in growth, and not close as in martens. Largest skins come from Denmark, Holland and Germany. The Russian are smaller, but more silky and, as now dyed, make a cheap and fair substitute for sable. They are excellent for linings of ladies’ coats, being of light weight and fairly strong in the pelt. English mayors’ and civic officials’ robes are frequently trimmed with this fur in lieu of sable. Value of the German variety 2s. to 5s. 6d. and of the Russian 7d. to 1s. 4d.

Fox, Blue.—Size 24 × 8 in. Underwool thick and long. Top hair fine and not so plentiful as in other foxes. Found in Alaska, Hudson Bay territory, Archangel and Greenland. Although called blue, the colour is a slaty or drab tone. Those from Archangel are more silky and of a smoky bluish colour and are the most valuable. These are scarce and consequently dear. The white foxes that are dyed smoke and celestial blue are brilliant and totally unlike the browner shades of this fox. Value 34s. to 195s.

Fox, Common.—The variation of size and quality is considerable, and the colour is anything from grey to red. In Great Britain the animal is now only regarded for the sport it provides. On the European continent, however, some hundreds of thousands of skins, principally German, Russian and Norwegian, are sold annually, for home use, and for dyeing and exportation, chiefly to the United States. The qualities do not compare with those species found in North America and the Arctic circle. The Asiatic, African and South American varieties are, with the exception of those taken in the mountains, poorly furred and usually brittle and therefore of no great service. No commercial value can be quoted.

Fox, Cross.—Size 20 × 7 in., are about as large as the silver and generally have a pale yellowish or orange tone with some silvery points and a darkish cross marking on the shoulders. Some are very similar to the pale red fox from the North-West of America and a few are exceptionally large. The darkest and best come from Labrador and Hudson Bay, and the ordinary sorts from the north-west of the United States and, as with silver and other kinds, the quality is inferior when taken from warmer latitudes. Value 10s. 6d. to 60s.

Fox, grey.—Size 27 × 10 in. Has a close dark drab underwool with yellowish grizzly, grey, regular and coarse top hair. The majority used for the trade come from Virginia and the southern and western parts of the United States. Those from the west are larger than the average, with more fur of a brighter tone. The fur is fairly serviceable for carriage rugs, the leather being stout, but its harshness of quality and nondescript colour does not contribute to make it a favourite. Value 9d. to 4s. 9d.

Fox, Japanese.—See *Fox, Red*, and *Raccoon*, below.

Fox, Kit.—Size 20 × 6 in. The underwool is short and soft, as is also the top hair, which is of very pale grey mixed with some yellowish-white hair. It is the smallest of foxes, and is found in Canada and the northern section of the United States. It is similar in colour and quality to the prairie fox and to many kinds from the warmer zones, such as from Turkey, eastern Asia and elsewhere. Value 1s. 3d. to 5s. 6d.

Fox, Red.—Size 24 × 8 in., though a few kinds are much larger. The underwool is long and soft and the hair plentiful and strong. It is found widely in the northern parts of America and in smaller numbers south of the United States, also in China, Japan and Australia. The colours vary from pale yellowish to a dark red, some being very brilliant. Those of Kamschatka are rich and fine in quality. Farther north, especially near the sea, the fur is coarse. Where the best coloured skins are not used for carriage rugs they are extensively dyed, and badger and other white hairs are inserted to resemble silver fox. They are also dyed a sable colour. The skins, being the strongest of foxes’, both in the fur and pelt, are serviceable. The preparations in imitation of the natural black and silver sorts are very good and attractive. Value 1s. to 41s.

Fox, Silver. Size 30 × 10 in. Underwool close and fine. Top hair black to silvery, 3 in. long. The fur upon the necks usually runs dark, almost black, and in some cases the fur is black half-way down the length of the skin, in rarer cases three-quarters of the length and, in the most exceptional instances, the whole length, and when this is the case they are known as “Natural Black Foxes” and fetch enormous prices. The even silvery sorts are highly esteemed, and the fur is one of the most effective and precious. The finest are taken in Labrador. The farther south they are found, the poorer and coarser the fur. The brush has invariably a white tip. Value £1 to £320.

Fox, White.—Size 20 × 7 in. Animals of this species are generally small in size and inhabit the extreme northern sections of Hudson Bay, Newfoundland, Greenland, Labrador and Siberia. The Canadian are silky in nature and inclined to a creamy colour, while the Siberian are more woolly and rather whiter. Those taken in central Asia near or in Chinese territory are poorer and yellowish. The underwool in all sorts is generally of a bluish-grey tone, but the top hair in the depth of winter is usually full enough in quantity to hide any such variation. Those skins in which the underwool is quite white are rare and much more expensive. In summer specimens of this species, as with other white furred animals, have slightly discoloured coats. The skins that are not perfectly white are dyed jet black, dark or light smoke, violet-blue, blue-grey, and also in imitation of the drab shades of the natural blue. Value 18s. to 66s. 350

Genet.—Size 10 × 4 in. The genet proper is a small white spotted cat found in Europe, but the quantity is too small to be of commercial interest. The name has been adopted for the black cats used so much in the trade. (See *Cats*, above.) Value 1s. to 6s. 6d.

Goats.—Size varies greatly. The European, Arabian and East Indian kinds are seldom used for rugs, the skins are chiefly dressed as leather for books and furniture, and the kids for boots and gloves, and the finer wool and hair are woven into various materials. Many from Russia are dyed black for floor and carriage rugs; the hair is brittle, with poor underwool and not very durable; the cost, however, is small. The Chinese export thousands of similar skins in black, grey and white, usually ready dressed and made into rugs of two skins each. A great many are dyed black and brown, in imitation of bear, and are used largely in the western parts of the United States and Canada for sleigh and carriage rugs. Many are used for their leather. Thousands of the kids are also dyed black and worked into cross-shaped pieces, in which shape they are largely exported to Germany, France, Great Britain and America, and sold by the retail as caracal, kid or caracul. The grey ones are in good demand for motor coats. The word caracul has been adopted from the Turkish and signifies black-eared. See also *Lambs, caracul*. Value of Chinese white 3s. 6d. to 6s. 6d.: grey, 4s. to 6s. 9d.

The Angora from the heights of central Asia Minor has curly, fleecy, silky, white wool, 4 to 7 in. long. The fur is not used in Great Britain, as formerly, and the greater quantity, known as mohair, is now imported for purposes of weaving. This species of goat was some years since introduced into Cape Colony, but its wool is not so good as the Asiatic breed. Good business, however, is done with the product, but chiefly for leather. Value 4s. to 12s. 6d.

The Mongolian goat has a very soft silk underwool, and after the long top hair is removed it is dressed and imported and erroneously named mouflon. The colour is a light fawn, but it is so pale that it lends itself to be dyed any colour. It was popular some years since in the cheaper trade, but it is not now much seen in England. Value 2s. to 6s.

The Tibet goat is similar to the Angora in the fineness of its wool, and many are used in the making of cashmere shawls. The Tibet lamb so largely imported and used for children’s wear is often miscalled Tibet goat. Value 3s. to 7s. 6d.

Guanaco.—Size 30 × 15 in. Is a species of goat found in Patagonia and other parts of South America. It has a very long neck and exceedingly soft woolly fur of a light reddish-fawn colour with very white flanks. It is usually imported in small quantities, native dressed, and ready made into rugs. The dressing is hard and brittle. If the skins are dressed in Europe they afford a very comfortable rug, though a very marked one in effect. They have a similar wool to the vicuna, but coarser and redder; both are largely used in South America. Value 1s. to 4s. 6d.

Hamster.—Size 8 × 3½ in. A destructive rodent, is found in great numbers in Russia and Germany. The fur is very flat and poor, of a yellowish pale brown with a little marking of black. Being of a light weight it is used for linings. Value 3d. to 1s.

Hare.—Size 24 × 9 in. The common hare of Europe does not much interest the furrier, the fur being chiefly used by makers of hatters’ felt. The white hares, however, of Russia, Siberia and other regions in the Arctic circle are very largely used in the cheaper trade of Europe, America and the British colonies. The fur is of the whitest when killed in winter, and that upon the flanks of the animal is very much longer than that upon its back. The flanks are usually cut off and made into muffs and stoles. The hair is, however, brittle and is not at all durable. This fur is dyed jet black and various shades of brown and grey, and manufactured into articles for the small drapers and for exportation. The North American hares are also dyed black and brown and used in the same way. Value of white 2d. to 5d.

Jackal.—Size 2 to 3 ft. long. Is found in India and north and south Africa. Indian are light brown and reddish, those from the Cape are dark grey and rather silvery. Few are imported. Fur generally poor and harsh, only suitable for carriage rugs. Value 1s. to 3s. 6d.

Jaguar.—Size 7 to 10 ft. long. Is found in Mexico and British Honduras. The markings are an irregular ring formation with a spot in the centre. Leopards have rings only and cheetahs solid spots. Suitable only for hearth-rugs. Supply very limited. Value 5s. to 45s.

Kaluga.—See *Souslik*, below.

Kangaroo.—The sizes vary considerably, some being huge, others quite small. The larger varieties, viz. the red and the great, do not usually interest furriers, the fur being harsh and poor without underwool. They are tanned for the leather trade. The sorts used for carriage aprons, coat linings and the outside of motor coats include: blue kangaroo, bush kangaroo, bridled kangaroo, wallaroo, yellow kangaroo, rock wallaby, swamp wallaby and short-tailed wallaby. Many of the swamp sort are dyed to imitate skunk and look well. Generally the colours are yellowish or brown. Some are dark brown as in the swamp, which being strong are suitable for motor coats. The rock wallabies are soft and woolly and often of a pretty bluish tone, and make moderately useful carriage rugs and perambulator aprons. The redder and browner sorts are also good for rugs as they are thick in the pelt. On the European continent many of these are dyed. The best of the lighter weights are frequently insufficiently strong in the hair to stand the friction of wear in a coat lining. Value, kangaroo 9d. to 3s., wallaby 1½ d. to 5s. 3d., wallaroo 1s. to 5s. 6d.

Kids.—See *Goats*, above.

Kolinsky.—Size 12 × 2½ in. Is one of the marten tribe. The underwool is short and rather weak, but regular, as is also the top hair; the colour is usually yellow. They have been successfully dyed and used as a substitute for sable. They are found in Siberia, Amoor, China and Japan, but the best are from Siberia. They are light in weight and therefore suitable for linings of coats. The tails are used for artists’ “sable” brushes. The fur has often been designated as red or Tatar sable. Value 1s. 6d. to 4s. 6d.

Lambs.—The sorts that primarily interest the fur trade in Europe and America are those from south Russia, Persia and Afghanistan, which are included under the following wholesale or retail commercial terms: Persian lamb, broadtail, astrachan, Shiraz, Bokharan and caracul lamb. With the public the general term astrachan is an old one, embracing all the above curly sorts; the flatter kinds, as broadtail and caracul lamb, have always been named separately. The Persian lambs, size 18 × 9 in., are the finest and the best of them. When dressed and dyed they should have regular, close and bright curl, varying from a small to a very large one, and if of equal size, regularity, tightness and brightness, the value is comparatively a matter of fancy. Those that are dull and loose, or very coarse and flat in the curl, are of far less market value.

All the above enumerated lambs are naturally a rusty black or brown, and with very few exceptions are dyed a jet black. Lustre, however, cannot be imparted unless the wool was originally of a silky nature. Broadtails, size 10 × 5 in., are the very young of the Persian sheep, and are killed before the wool has time to develop beyond the flat wavy state which can be best compared to a piece of moiré silk. They are naturally exceedingly light in weight, and those that are of an even pattern, possessing a lustrous sheen, are costly. There is, notwithstanding, a great demand for these from the fashionable world, as not only are they very effective, but being so flat in the wool the figure of the wearer can be shown as perfectly as in a garment made of silk. It cannot be regarded as an economical fur, as the pelt is too delicate to resist hard wear.

Persian Lamb	price 12s. 6d.	to 25s.
Broadtail	price 10s.	to 35s.

Astrachan, Shiraz and Bokharan lambs, size 22 by 9 in., are of a coarser, looser curl, and chiefly used for coat linings, while the Persians are used for outside of garments, collars, cuffs, stoles, muffs, hats and trimmings and gloves. The so-called caracul lambs, size 12 × 6 in., are the very young of the astrachan sheep, and the pick of them are almost as effective as broadtails, although less fine in the texture. See also remarks as to caracul kid under Goats, above.

Astrachan	price 1s.	to 5s. 6d.
Caracul Lamb	price 2s. 6d	to 10s. 6d.
Shiraz	price 4s. 6d	to 10s.
Bokharan	price 1s. 6d	to 3s. 6d.

Grey lambs, size 24 × 10 in., are obtained from the Crimea and known in the trade as “crimmers.” They are of a similar nature to the caracul lambs, but looser in curl, ranging from a very light to a dark grey. The best are the pale bluish greys, and are chiefly used for ladies’ coats, stoles, muffs and hats. Price 2s. to 6s. Mongolian lambs, size 24 × 15 in., are of a short wavy loose curl, creamy white colour, and are usually exported from China dressed, the majority being ready-made into cross-shaped coats or linings. They are used principally for linings of good evening wraps for ladies. Price 1s. to 2s. 6d. Slink lambs come from South America and China. The former are very small and generally those that are stillborn. They have a particularly thin pelt with very close wool of minute curl. The China sorts are much larger. The smallest are used for glove linings and the others for opera cloak linings. Price 1s. to 6s. 6d.

Leopard.—Size 3 to 6 ft. long. There are several kinds, the chief being the snow or ounce, Chinese, Bengal, Persian, East Indian and African. The first variety inhabit the Himalayas and are beautifully covered with a deep soft fur quite long compared to the flat harsh hair of the Bengal sort. The colours are pale orange and white with very dark markings, a strong contrast making a fine effect. Most artists prize these skins above all others. The Chinese are of a medium orange brown colour, but full in fur. The East Indian are less full and not so dark. The Bengal are dark and medium in colour, short and hard hair, but useful for floor rugs, as they do not hold the dust like the fuller and softer hair of the kinds previously named. They are also used for drummers’ aprons and saddle cloths in the Indian army. The African are small with pale lemon colour grounds very closely marked with black spots on the skin, the strong contrast making a pleasing effect. Occasionally, where something very marked is wanted, skating jackets and carriage aprons are made from the softest and 351
flattest of skins, but usually they are made into settee covers, floor rugs and foot muffs. Value 2s. to 40s.

Lion.—Size 5 to 6 ft. long. These skins are found in Africa, Arabia and part of India, and are every year becoming scarcer. They are only used for floor rugs, and the males are more highly esteemed on account of the set-off of the mane. Value, lions’ £10 to £100; lionesses’ £5 to £25.

Lynx.—Size 45 × 20 in. The underwool is thinner than fox, but the top hair is fine, silky and flowing, 4 in. long, of a pale grey, slightly mottled with fine streaks and dark spots. The fur upon the flanks is longer and white with very pronounced markings of dark spots, and this part of the skin is generally worked separately from the rest and is very effective for gown trimmings. Where the colour is of a sandy and reddish hue the value is far less than where it is of a bluish tone. They inhabit North America as far south as California, also Norway and Sweden. Those from the Hudson Bay district and Sweden are the best and are very similar. Those taken in Central Asia are mostly used locally. For attire the skins manufactured in Europe are generally dyed black or brown, in which state it has a similar appearance to dyed fox, but having less thick underwool and finer hair flows freely. The finest skins when dyed black are used very largely in America in place of the dyed black fox so fashionable for mourning wear in Great Britain and France. The British Hussar busbies are made of the dark brown lynx, and it is the free silky easy movement of the fur with the least disturbance in the atmosphere that gives it such a pleasing effect. It is used for rugs in its natural state and also in Turkey as trimmings for garments. Value 13s. 6d. to 56s.

Lynx Cat or Bay Lynx.—Is about half the size and depth of fur of a lynx proper, and inhabits the central United States. It is a flat and reddish fur compared to the lynx and is suitable for cheap carriage aprons. A few come from Canada and are of better quality. Value 5s. to 15s.

Marmot.—Size 18 × 12 in. Is a rodent and is found in considerable numbers in the south of Prussia. The fur is a yellowish brown and rather harsh and brittle and has no underwool. Since, however, the value of all good furs has advanced, dyers and manufacturers have made very successful efforts with this fur. The Viennese have been particularly successful, and their method has been to dye the skins a good brown and then not put in the dark stripes, which exist in sable and mink, until the garment or article is finished, thus obtaining as perfectly symmetrical effects as if the articles were made of small skins instead of large ones. Marmots are also found in North America, Canada and China; the best, however, come from Russia. It should always be a cheap fur, having so few good qualities to recommend it. Value 9d. to 2s. 6d.

Marten, American.—See *Sable*, below.

Marten, Baum.—Size 16 × 5 in. Is sometimes called the pine marten, and is found in quantity in the wooded and mountainous districts of Russia, Norway, Germany and Switzerland. It possesses a thick underwool with strong top hair, and ranges from a pale to a dark bluish brown. The best, from Norway, are very durable and of good appearance and an excellent substitute for American sable. The tails when split into two or three, with small strips of narrow tape so as to separate the otherwise dense fur, formerly made very handsome sets of trimmings, ties and muffs, and the probabilities are, as with other fashions, such use will have its period of revival. Value 6s. to 85s.

Marten, Black.—See *Skunk*, below.

Marten, Japanese.—Size 16 × 5 in. Is of a woolly nature with rather coarse top hair and quite yellow in colour. It is dyed for the cheap trade for boas and muffs, but it is not an attractive fur at the best of times. It lacks a silky, bright and fresh appearance, and therefore is unlikely to be in great demand, except where economy is an object. Value 6s. 6d. to 18s. 6d.

Marten, Stone.—Size and quality similar to the baum; the colour, however, of the underwool is a stony white and the top hair is very dark, almost black. They live in rocky and stony districts. Skins of a pale bluish tone are generally used in their natural state for stoles, boas and muffs, but the less clear coloured skins are dyed in beautiful shades similar in density to the dark and valuable sables from Russia, and are the most effective skins that can be purchased at a reasonable price. The tails have also been worked, in the manner explained with regard to the baum marten, as sets of trimmings and in other forms. Stone martens are found in Russia, Bosnia, Turkey, Greece, Germany, the Alps and France. The Bosnian and the French are the best in colour. The Asiatic sorts are less woolly, but being silky are useful when dyed. There are many from Afghanistan and India which are too poor to interest the European markets. Value 7s. 6d. to 26s.

Mink.—Size 16 × 5 in. Is of the amphibious class and is found throughout North America and in Russia, China and Japan. The underwool is short, close and even, as is also the top hair, which is very strong. The best skins are very dark and are obtained from Nova Scotia. In the central states of America the colour is a good brown, but in the north-west and south-west the fur is coarse and generally pale. It is very durable for linings, and is an economical substitute for sable for coats, capes, boas and trimmings. Values have greatly increased, and the fur possessing good qualities as to colour and durability will doubtless always be in good request. The Russian species is dark but flat and poor in quality, and the Chinese and Japanese are so pale that they are invariably dyed. These, however, are of very inferior nature. Value of American 3s. 3d. to 40s., Japanese 3d. to 2s. 3d.

Mole.—Size 3½ × 2½ in. Moles are plentiful in the British Isles and Europe, and owing to their lovely velvety coats of exquisite blue shade and to the dearness of other furs are much in demand. Though the fur is cheap in itself, the expense of dressing and working up these little skins is considerable, and they possess the unique charm of an exceptional colour with little weight of pelt; the quality of resistance to friction is, however, so slight as to make them expensive in wear. The best are the dark blue from the Fen district of Cambridgeshire in England. Value ½d. to 2d.

Mongolian Lambs.—See *Lambs*, above.

Monkey, Black.—Size 18 × 10 in. Among the species of monkeys only one interests to any extent the fur trade, and that is the black monkey taken on the west coast of Africa (*Colobus satanas*). The hair is very long, very black and bright with no underwool, and the white pelt of the base of the hair, by reason of the great contrast of colour, is very noticeable. The skins were in 1850 very fashionable in England for stoles, muffs and trimmings, and in America also as recently as 1890. They are now mostly bought for Germany and the continent. Value 6d. to 1s. 6d.

Mouflon.—Size 30 × 15 in. Is a sheep found in Russia and Corsica and now very little in demand, and but few are imported into Great Britain. Many Mongolian goats with the long hairs pulled out are sold as mouflon. Value 4s. to 10s. 6d.

Musk-Ox.—Size 6 × 3 ft. These animals have a dense coat of fine, long brown wool, with very long dark brown hair on the head, flanks and tail, and, in the centre, a peculiar pale oval marking. There is no other fur that is so thick, and it is eminently suitable for sleighing rugs, for which purpose it is highly prized in Canada. The musk-ox inhabits the north part of Greenland and part of Canada, but in very limited numbers. Value 10s. to 130s.

Musquash or Musk-Rat, Brown and Black Russian.—Size 12 × 8 in. A very prolific rodent of the amphibious class obtained from Canada and the United States, similar in habit to the English vole, with a fairly thick and even brown underwool and rather strong top dark hair of medium density. It is a very useful fur for men’s coat linings and ladies’ driving or motoring coats, being warm, durable and not too heavy. If the colour were less motley and the joins between the skins could be made less noticeable, it would be largely in demand for stoles, ties and muffs. As it is, this fur is only used for these smaller articles for the cheaper trade. It has, however, of later years been “unhaired,” the underwool clipped very even and then dyed seal colour, in which way very useful and attractive garments are supplied at less than half the cost of the cheaper sealskins. They do not wear as well, however, as the pelt and the wool are not of a strength comparable to those of sealskin. With care, however, such a garment lasts sufficiently long to warrant the present outlay. Value 5½d. to 1s. 9d.

There is a so-called black variety found in Delaware and New Jersey, but the number is very small compared to the brown species. They are excellent for men’s coat linings and the outside of ladies’ coats, for stoles, muffs, collars and cuffs. Value 10d. to 3s. 7d.

The Russian musquash is very small, 7 × 4 in., and is limited in numbers compared to the brown. Only a few thousands are imported to London. It is of a very pretty silvery-blue shade of even wool with very little silky top hair, having silvery-white sides and altogether a very marked effect. The odour, however, even after dressing is rather pungent of musk, which is generally an objection. Value 4s. to 6s. 6d.

Nutria.—Size 20 × 12 in. Is a rodent known in natural history as the coypu, about half the size of a beaver, and when unhaired has not more than half, generally less, the depth of fur, which is also not so close. Formerly the fur was only used for hatters’ felt, but with the rise in prices of furs these skins have been more carefully removed and—with improved dressing, unhairing and silvering processes—the best provides a very effective and suitable fur for ladies’ coats, capes, stoles, muffs, hats and gloves, while the lower qualities make very useful, light-weighted and inexpensive linings for men’s or women’s driving coats. It is also dyed sealskin colour, but its woolly nature renders it less effective than the more silky musquash. They are obtained from the northern part of South America. Value is. 6d. to 6s. 6d.

Ocelot.—Size 36 × 13 in. Is of the nature of a leopard and prettily marked with stripes and oblong spots. Only a few are now imported from South America for carriage aprons or mats. The numbers are very limited. Value 1s. to 2s. 6d.

Opossum, American.—Size 18 × 10 in. Is a marsupial, a class with this exception not met with out of Australia. The underwool is of a very close frizzy nature, and nearly white, with long bluish grey mixed with some black top hair. It is only found in the central sections of the United States. About 1870 in England it was dyed dark brown or black and used for boas, muffs and trimmings, but until recently has been neglected on the continent. With, however, recent experiments in brown and skunk coloured dyes, it bids fair to become a popular fur. Value 2½d. to 5s. 6d.

Opossum, Australian.—Size 16 × 8 in. Is a totally different nature of fur to the American. Although it has wool and top hair, the latter is so sparse and fine that the coat may be considered as one of close even wool. 352
The colour varies according to the district of origin, from a blue grey to yellow with reddish tones. Those from the neighbourhood of Sydney are light clear blue, while those from Victoria are dark iron grey and stronger in the wool. These animals are most prolific and evidently increasing in numbers. Their fur is pretty, warm and as yet inexpensive, and is useful for rugs, coat linings, stoles, muffs, trimmings and perambulator aprons. The worst coloured ones are frequently dyed black and brown. The most pleasing natural grey come from Adelaide. The reddest are the cheapest. Value 3¾d. to 3s. 6d.

Opossum, Ringtailed.—Size 7 × 4 in. Has a very short close and dark grey wool, some being almost black. There are but a few thousands imported, and being so flat they are only of use for coat linings, but they are very warm and light in weight. Value 6d. to 10d.

Opossum, Tasmanian (grey and black).—Size 20 × 10 in. Is of a similar description, but darker and stronger in the wool and larger. Besides these there are some very rich brown skins which were formerly in such request in Europe, especially Russia, that undue killing occurred until 1899, when the government stopped for a time the taking of any of this class. They are excellent for carriage aprons, being not only very light in weight and warm, but handsome. Value 2s. 6d. to 8s. 6d.

Otter, River.—The size varies considerably, as does the underwool and the top hair, according to the country of origin. There are few rivers in the world where they do not live. But it is in the colder northern regions that they are found in the greatest numbers and with the best fur or underwool, the top hair, which, with the exception of the scarce and very rich dark brown specimens they have in common with most aquatic animals, is pulled out before the skins are manufactured. Most of the best river otter comes from Canada and the United States and averages 36 × 18 in. in size. Skins from Germany and China are smaller, and shorter in the wool. The colours of the under wools of river otters vary, some being very dark, others almost yellow. Both as a fur and as a pelt it is extremely strong, but owing to its short and close wool it is usually made up for the linings, collars and cuffs of men’s coats. A large number of skins, after unhairing, is dyed seal colour and used in America. Those from hot climates are very poor in quality. Value 28s. to 118s.

Otter, Sea.—Size 50 × 25 in. Possesses one of the most beautiful of coats. Unlike other aquatic animals the skin undergoes no process of unhairing, the fur being of a rich dense silky wool with the softest and shortest of water hairs. The colours vary from pale grey brown to a rich black, and many have even or uneven sprinkling of white or silvery-white hairs. The blacker the wool and the more regular the silver points, the more valuable the skin. Sea otters are, unfortunately, decreasing in numbers, while the demand is increasing. The fur is most highly esteemed in Russia and China; in the latter country it is used to trim mandarins’ state robes. In Europe and America it is much used for collar, long facings and cuffs of a gentleman’s coat; such a set may cost from £200 to £600, and in all probability will soon cost more. Taking into consideration the size, it is not so costly as the natural black fox, or the darkest Russian sable, which is now the most expensive of all. The smaller and young sea otters of a grey or brown colour are of small value compared to the large dark and silvery ones. Value £10 to £220. A single skin has been known to fetch £400.

Ounce.—See *Leopard*, above.

Persian Lambs.—See *Lambs*, above.

Platypus.—Size 12 × 8 in. One of the most singular of fur-bearing animals, being the link between bird and beast. It has fur similar to otter, is of aquatic habits, being web-footed with spurs of a cock and the bill of a duck. The skins are not obtained in any numbers, but being brought over by travellers as curiosities and used for muffs, collars and cuffs, &c., they are included here for reference. Value 2s. to 3s. 6d.

Pony or *Tatar Foal*.—Size 36 × 20 in. These skins are of comparatively recent importation to the civilized world. They are obtained from the young of the numerous herds of wild horses that roam over the plains of Turkestan. The coat is usually a shade of brown, sometimes greyish, fairly bright and with a suggestion of waviness. Useful for motor coats. Value 3s. to 10s. 6d.

Puma.—Size 4½ × 3 ft. Is a native of South America, similar to a lion in habits and colour of coat. The hair and pelt is, however, of less strength, and only a few are now used for floor rugs. Value 5s. to 10s.

Raccoon.—Size 20 × 12 in. Is an animal varying considerably in size and in quality and colour of fur, according to the part of North America in which it is found. In common parlance, it may be described as a species of wild dog with close affinity to the bear. The underwool is 1 to 1½ in. deep, pale brown, with long top hairs of a dark and silvery-grey mixture of a grizzly type, the best having a bluish tone and the cheapest a yellowish or reddish-brown. A limited number of very dark and black sorts exist and are highly valued for trimmings. The very finest skins are chiefly used for stoles and muffs, and the general run for coachmen’s capes and carriage rugs, which are very handsome when the tails, which are marked with rings of dark and light fur alternately, are left on. Raccoons are used in enormous quantities in Canada for men’s coats, the fur outside. The poorer qualities are extensively bought and made up in a similar way for Austria-Hungary and Germany. These make excellent linings for coats or footsacks for open driving in very cold climates. The worst coloured skins are dyed black or brown and are used for British military busbies, or caps, stoles, boas, muffs and coachmen’s capes. The best skins come from the northern parts of the United States. A smaller and poorer species inhabits South America, and a very few are found in the north of India, but these do not interest the European trade. From Japan a similar animal is obtained in smaller quantities with very good but longer fur, of yellowish motley light-brown shades. It is more often imported and sold as Japanese fox, but its resemblance to the fur of the American raccoon is so marked as to surely identify it. When dyed dark blue or skunk colour it is good-looking and is sold widely in Europe. Raccoon skins are also frequently unhaired, and if the underwool is of good quality the effect is similar to beaver. It is the most useful fur for use in America or Russia, having a full quantity of fur which will retain heat. Value 10d. to 26s.

Sable, American and Canadian.—Size 17 × 5 in. The skins are sold in the trade sale as martens, but as there are many that are of a very dark colour and the majority are almost as silky as the Russian sable, the retail trade has for generations back applied the term of sable to this fur. The prevailing colour is a medium brown, and many are quite yellow. The dyeing of these very pale skins has been for so long well executed that it has been possible to make very good useful and effective articles of them at a moderate price compared to Russian sable. The finest skins are found in the East Main and the Esquimaux Bay, in the Hudson’s Bay Company’s districts, and the poorest in Alaska. They are not found very far south of the northern boundary of the United States. The best skins are excellent in quality, colour and effect, and wear well. Value 27s. 3d. to 290s.

Sable, Chinese and Japanese.—Size 14 × 4½ in. These are similar to the Amur skins previously referred to, but of much poorer quality and generally only suitable for linings. The very palest skins are dyed and made by the Chinese into mandarins’ coats, in which form they are found in the London trade sales, but being overdressed they are inclined to be loose in the hair and the colour of the dye is not good. The Japanese kind are imported raw, but are few in numbers, very pale and require dyeing. Value 15s. to 150s.

Sable, Russian.—Size 15 × 5 in. These skins belong to a species of marten, very similar to the European and American, but much more silky in the nature of their fur. They have long been known as “sables,” doubtless owing to the density of colour to which many of them attain, and they have always been held in the highest esteem by connoisseurs as possessing a combination of rare qualities. The underwool is close, fine and very soft, the top hair is regular, fine, silky and flowing, varying from 1½ to 2½ in. in depth. In colour they range from a pale stony or yellowish shade to a rich dark brown, almost black with a bluish tone. The pelts are exceedingly fine and close in texture and, although of little weight, are very durable, and articles made of them produce a sensation of warmth immediately they are put upon the body.

The Yakutsk, Okhotsk and Kamschatka sorts are good, the last being the largest and fullest furred, but of less density of colour than the others. Many from other districts are pale or yellowish brown, and those from Saghalien are poor in quality. The most valuable are the darkest from Yakutsk in Siberia, particularly those that have silvery hairs evenly distributed over the skin. These however are exceedingly scarce, and when a number are required to match for a large garment, considerable time may be necessary to collect them. This class of skin is the most expensive fur in the world, reckoning values by a square foot unit.

The Amur skins are paler, but often of a pretty bluish stony tone with many frequently interspersed silvery hairs. The quality too is lower, that is, the fur is not so close or deep, but they are very effective, particularly for close-fitting garments, as they possess the least appearance of bulk. The paler skins from all districts in Siberia are now cleverly coloured or “topped,” that is, just the tips of the hair are stained dark, and it is only an expert who can detect them from perfectly natural shades. If this colouring process is properly executed it remains fairly fast. Notwithstanding the reported rights of the Russian imperial authorities over some regions with respect to these and other valuable fur-bearing animals, there are in addition to the numbers regularly sent to the trade auction sales in London many good parcels of raw skins to be easily bought direct, provided price is not the first consideration. Value 25s. to 980s.

Seal, Fur.—Sizes range from 24 × 15 in. to 55 × 25 in., the width being taken at the widest part of the skin after preparation. The centre of the skin between the fins is very narrow and the skins taper at each end, particularly at the tail. The very small pups are of a beautiful quality, but too tiny to make into garments, and, as the aim of a good furrier is to avoid all lateral or cross seams, skins are selected that are the length of the garment that is to be made. The most useful skins for coats are the large pups 42 in. long, and the quality is very good and uniform. The largest skins, known in the trade as “wigs,” which range up to 8 ft. in length, are uneven and weak in the fur, and hunters do not seek to obtain them. The supply of the best sort is chiefly from the North Pacific, viz. Pribilof Islands, Alaska, north-west coast of America, Copper Island of the Aleutian 353
group near to Kamschatka, Robben Island and Japan. Other kinds are taken from the South Pacific and South Atlantic Oceans, around Cape Horn, the Falkland Islands up to Lobos Islands at the entrance of the La Plata river, off the Cape of Good Hope and Crozet Isles. With, however, the exception of the pick of the Lobos Island seals the fur of the southern sea seals is very poor and only suitable for the cheapest market. Formerly many skins were obtained from New Zealand and Australia, but the importation is now small and the quality not good. The preparation of seal skin occupies a longer time than any other fur skin, but its fine rich effect when finished and its many properties of warmth and durability well repay it. Value 10s. to 232s.

Seal, Hair.—There are several varieties of these seals in the seas stretching north from Scotland, around Newfoundland, Greenland and the north-west coast of America, and they are far more numerous than fur seals. Generally they have coarse rigid hair and none possess any underwool. They are taken principally for the oil and leather they yield. Some of the better haired sorts are dyed black and brown and used for men’s motor coats when quite a waterproof garment is wanted, and they are used also for this quality in China. The young of the Greenland seals are called whitecoats on account of the early growth being of a yellowish white colour; the hair is ¾ to 1 in. long, and at this early stage of their life is soft compared to that of the older seals. These fur skins are dyed black or dark brown and are used for military caps and hearth-rugs. Value 2s. to 15s. There are fewer hair seals in the southern than in the northern seas.

Sheep.—Vary much in size and in quality of wool. Many of the domestic kind in central and northern Europe and Canada are used for drivers’ and peasants’ coat linings, &c. In Great Britain many coats of the home-reared sheep, having wools two and a half to five inches long, are dyed various colours and used as floor rugs. Skins with very short wool are dyed black and used for military saddle-cloths. The bulk, however, is used in the wool trade. The Hungarian peasants are very fond of their natural brown sheep coats, the leather side of which is not lined, but embellished by a very close fancy embroidery, worked upon the leather itself; these garments are reversible, the fur being worn inside when the weather is cold. Chinese sheep are largely used for cheap rugs. Value of English sheep from 3s. to 10s.

SKUNK or BLACK MARTEN.—Size 15 × 8 in. The underwool is full and fairly close with glossy, flowing top hair about 2½ in. long. The majority have two stripes of white hair, extending the whole length of the skin, but these are cut out by the manufacturing furrier and sold to the dealers in pieces for exportation. The animals are found widely spread throughout North and South America. The skins which are of the greatest interest to the European trade are those from North America, the South American species being small, coarse and generally brown. The best skins come from Ohio and New York. If it were not for its disagreeable odour, skunk would be worth much more than the usual market value, as it is naturally the blackest fur, silky in appearance and most durable. The improved dressing processes have to a large extent removed the naturally pungent scent. The fur is excellent for stoles, boas, collars, cuffs, muffs and trimmings. Value 1s. 6d. to 11s.

SOUSLIK.—Size 7 in. × 2¼. Is a small rodent found in the south of Russia and also in parts of America. It has very short hair and is a poor fur even for the cheapest linings, which is the only use to which the skin could be put. It is known as kaluga when imported in ready-made linings from Russia where the skins are dressed and worked in an inferior way. Value 1d. to 3d.

SQUIRREL.—Size 10 × 5 in. This measurement refers to the Russian and Siberian sorts, which are the only kind imported for the fur. The numerous other species are too poor in their coats to attract notice from fur dealers. The back of the Russian squirrel has an even close fur varying from a clear bluish-grey to a reddish-brown, the bellies in the former being of a flat quality and white, in the latter yellowish. The backs are worked into linings separately, as are the bellies or "locks." The pelts, although very light, are tough and durable, hence their good reputation for linings for ladies' walking or driving coats. The best skins also provide excellent material for coats, capes, stoles, ties, collars, cuffs, gloves, muffs, hoods and light-weight carriage aprons. The tails are dark and very small, and when required for ends of boas three or four are made as one. Value per skin from 2½d. to 1s. 1d.

TIBET LAMB.—Size 27 × 13 in. These pretty animals have a long, very fine, silky and curly fleece of a creamy white. The majority are consigned to the trade auction sales in London ready dressed and worked into cross-shaped coats, and the remainder, a fourth of the total, come as dressed skins. They are excellent for trimmings of evening mantles and for children's ties, muffs and perambulator aprons. The fur is too long and bulky for linings. Value per skin from 4s. 6d. to 8s. 6d.

TIGER.—Size varies considerably, largest about 10 ft. from nose to root of tail. Tigers are found throughout India, Turkestan, China, Mongolia and the East Indies. The coats of the Bengal kind are short and of a dark orange brown with black stripes, those from east or further India are similar in colour, but longer in the hair, while those from north of the Himalayas and the mountains of China are not only huge in size, but have a very long soft hair of delicate orange brown with very white flanks, and marked generally with the blackest of stripes. The last are of a noble appearance and exceedingly scarce. They all make handsome floor rugs.

Value of the Indian	from £3 to £15.
Value of the Chinese	from £10 to £65.

VICUNA is a species of long-necked sheep native to South America, bearing some resemblance to the guanaco, but the fur is shorter, closer and much finer. The colour is a pale golden-brown and the fur is held in great repute in South America for carriage rugs. The supply is evidently small as the prices are high. There is scarcely a commercial quotation in London, few coming in except from private sources. 2s. 6d. to 5s. 6d. may be considered as the average value.

WALLABY.—See *Kangaroo*, above.

WALLAROO.—See *Kangaroo*, above.

WOLF.—Size 50 × 25 in. Is closely allied to the dog tribe and, like the jackals, is found through a wide range of the world,—North and South America, Europe and Asia. Good supplies are available from North America and Siberia and a very few from China. The best are the full furred ones of a very pale bluish-grey with fine flowing black top hair, which are obtained from the Hudson Bay district. Those from the United States and Asia are harsher in quality and browner. A few black American specimens come into the market, but usually the quality is poor compared to the lighter furred animal. The Siberian is smaller than the North American and the Russian still smaller. Besides the wolf proper a large number of prairie or dog wolves from America and Asia are used for cheaper rugs. In size they are less than half that of a large wolf and are of a motley sandy colour. Numbers of the Russian are retained for home use. The finest wolves are very light weighted and most suitable for carriage aprons, in fact, ideal for the purpose, though lacking the strength of some other furs.

Wolves	value 2s. 6d.	to 64s.
Dog wolves	value 1s.	to 2s. 6d.

WOLVERINE.—Size 16 × 18 in. Is native to America, Siberia, Russia and Scandinavia and generally partakes of the nature of a bear. The underwool is full and thick with strong and bright top hair about 2½ in. long. The colour is of two or three shades of brown in one skin, the centre being an oval dark saddle, edged as it were with quite a pale tone and merging to a darker one towards the flanks. This peculiar character alone stamps it as a distinguished fur, in addition to which it has the excellent advantage of being the most durable fur for carriage aprons, as well as the richest in colour. It is not prolific, added to which it is very difficult to match a number of skins in quality as well as colour. Hence it is an expensive fur, but its excellent qualities make it valuable. The darkest of the least coarse skins are worth the most. Prices from 6s. to 37s.

WOMBAT, KOALA or Australian Bear.—Size 20 × 12 in. Has light grey or brown close thick wool half an inch deep without any top hair, with a rather thick spongy pelt. It is quite inexpensive and only suitable for cheap rough coats, carriage rugs, perambulator aprons and linings for footbags. The coats are largely used in western America and Canada. Value 3d. to 1s. 8½d.

Preparing and Dressing.—A furrier or skin merchant must possess a good eye for colour to be successful, the difference in value on this subtle matter solely (in the rarer precious sorts, especially sables, natural

black, silver and blue fox, sea otters, chinchillas, fine mink, &c.) being so considerable that not only a practised but an intuitive sense of colour is necessary to accurately determine the exact merits of every skin. In addition to this a knowledge is required of what the condition of a pelt should be; a good judge knows by experience whether a skin will turn out soft and strong, after dressing, and whether the hair is in the best condition of strength and beauty. The dressing of the pelt or skin that is to be preserved for fur is totally different to the making of leather; in the latter tannic acid is used, but never should be with a fur skin, as is so often done by natives of districts where a regular fur trade is not carried on. The results of applying tannic acid are to harden the pelt and discolour and weaken the fur. The best methods for dressing fur skins are those of a tawer or currier, the aim being to retain all the natural oil in the pelt, in order to preserve the natural colour of the fur, and to render the pelt as supple as possible. Generally the skins are placed in an alkali bath, then by hand with a blunt wooden instrument the moisture of the pelt is worked out and it is drawn carefully to and fro over a straight, dull-edged knife to remove any superfluous flesh and unevenness. Special grease is then rubbed in and the skin placed in a machine which softly and continuously beats in the softening mixture, after which it is put into a slowly revolving drum, fitted with wooden paddles, partly filled with various kinds of fine hard sawdust according to the nature of the furs dealt with. This process with a moderate degree of heat thoroughly cleans it of external greasy matter, and all that is necessary before 354
manufacturing is to gently tap the fur upon a leather cushion stuffed with horsehair with smooth canes of a flexibility suited to the strength of the fur. After dressing most skins alter in shape and decrease in size.

With regard to the merits of European dressing, it may be fairly taken that English, German and French dressers have specialities of excellence. In England, for instance, the dressing of sables, martens, foxes, otters, seals, bears, lions, tigers and leopards is first rate; while with skunk, mink, musquash, chinchillas, beavers, lambs and squirrels, the Germans show better results, particularly in the last. The pelt after the German dressing is dry, soft and white, which is due to a finishing process where meal is used, thus they compare favourably with the moister and consequently heavier English finish. In France they do well with cheaper skins, such as musquash, rabbit and hare, which they dye in addition to dressing. Russian dressing is seldom reliable; not only is there an unpleasant odour, but in damp weather the pelts often become clammy, which is due to the saline matter in the dressing mixture. Chinese dressing is white and supple, but contains much powder, which is disagreeable and difficult to get rid of, and in many instances the skin is rendered so thin that the roots of the fur are weakened, which means that it is liable to shed itself freely, when subject to ordinary friction in handling or wearing. American and Canadian dressing is gradually improving, but hitherto their results have been inferior to the older European methods.

In the case of seal and beaver skins the process is a much more difficult one, as the water or hard top hairs have to be removed by hand after the pelt has been carefully rendered moist and warm. With seal skins the process is longer than with any other fur preparation and the series of processes engage many specialists, each man being constantly kept upon one section of the work. The skins arrive simply salted. After being purchased at the auction sales they are washed, then stretched upon a hoop, when all blubber and unnecessary flesh is removed, and the pelt is reduced to an equal thickness, but not so thin as it is finally rendered. Subsequently the hard top hairs are taken out as in the case of otters and beavers and the whole thoroughly cleaned in the revolving drums. The close underwool, which is of a slightly wavy nature and mostly of a pale drab colour, is then dyed by repeated applications of a rich dark brown colour, one coat after another, each being allowed to thoroughly dry before the next is put on, till the effect is almost a lustrous black on the top. The whole is again put through the cleaning process and evenly reduced in thickness by revolving emery wheels, and eventually finished off in the palest buff colour.

The English dye for seals is to-day undoubtedly the best; its constituents are more or less of a trade secret, but the principal ingredients comprise gall nuts, copper dust, camphor and antimony, and it would appear after years of careful watching that the atmosphere and particularly the water of London are partly responsible for good and lasting results. The Paris dyers do excellent work in this direction, but the colour is not so durable, probably owing to a less pure water. In America of late, strides have been made in seal dyeing, but preference is still given to London work. In Paris, too, they obtain beautiful results in the "topping" or colouring Russian sables and the Germans are particularly successful in dyeing Persian lambs black and foxes in all blue, grey, black and smoke colours and in the insertion of white hairs in imitation of the real silver fox. Small quantities of good beaver are dyed in Russia occasionally, and white hairs put in so well that an effect similar to sea otter is obtained.

The process of inserting white hairs is called in the trade "pointing, "and is either done by stitching them in with a needle or by adhesive caoutchouc.

The Viennese are successful in dyeing marmot well, and their cleverness in colouring it with a series of stripes to represent the natural markings of sable which has been done after the garments have been made, so as to obtain symmetry of lines, has secured for them a large trade among the dealers of cheap furs in England and the continent.

Manufacturing Methods and Specialities.—In the olden times the Skinners' Company of the city of London was an association of furriers and skin dressers established under royal charter granted by Edward III. At that period the chief concern of the body was to prevent buyers from being imposed upon by sellers who were much given to offering old furs as new; a century later the Skinners' Company received other charters empowering them to inspect not only warehouses and open markets, but workrooms. In 1667 they were given power to scrutinize the preparing of rabbit or cony wool for the wool trade and the registration of the then customary seven years' apprenticeship. To-day all these privileges and powers are in abeyance, and the interest that they took in the fur trade has been gradually transferred to the leather-dressing craft.

The work done by English furriers was generally good, but since about 1865 has considerably improved on account of the influx of German workmen, who have long been celebrated for excellent fur work, being In their own country obliged to satisfy officially appointed experts and to obtain a certificate of capacity before they can be there employed. The French influence upon the trade has been, and still is, primarily one of style and combination of colour, bad judgment in which will mar the beauty of the most valuable furs. It is a recognized law among high-class furriers that furs should be simply arranged, that is, that an article should consist of one fur or of two furs of a suitable contrast, to which lace may be in some cases added with advantage. As illustrative of this, it may be explained that any brown tone of fur such as sable, marten, mink, black marten, beaver, nutria, &c., will go well upon black or very dark-brown furs, while those of a white or grey nature, such as ermine, white lamb, chinchilla, blue fox, silver fox, opossum, grey squirrel, grey lamb, will set well upon seal or black furs, as Persian lamb, broadtail, astrachan, caracul lamb, &c. White is also permissible upon some light browns and greys, but brown motley colours and greys should never be in contrast. One neutralizes the other and the effect is bad. The qualities, too have to be considered—the fulness of one, the flatness of the other, or the coarseness or fineness of the furs. The introduction of a third fur in the same garment or indiscriminate selection of colours of silk linings, braids, buttons, &c., often spoils an otherwise good article.

With regard to the natural colours of furs, the browns that command the highest prices are those that are of a bluish rather than a reddish tendency. With greys it is those that are bluish, not yellow, and with white those that are purest, and with black the most dense, that are most esteemed and that are the rarest.

Perhaps for ingenuity and the latest methods of manipulating skins in the manufacturing of furs the Americans lead the way, but as fur cutters are more or less of a roving and cosmopolitan character the larger fur businesses in London, Berlin, Vienna, St Petersburg, Paris and New York are guided by the same thorough and comparatively advanced principles.

During the period just mentioned the tailors’ methods of scientific pattern cutting have been adopted by the leading furriers in place of the old chance methods of fur cutters, so that to-day a fur garment may be as accurately and gracefully fitted as plush or velvet, and with all good houses a material pattern is fitted and approved before the skins are cut.

Through the advent of German and American fur sewing-machines since about 1890 fur work has been done better and cheaper. There are, however, certain parts of a garment, such as the putting in of sleeves and placing on of collars, &c., that can only be sewn by hand. For straight seams the machines are excellent, making as neat a seam as is found in glove work, unless, of course, the pelts are especially heavy, such as bears and sheep rugs.

A very great feature of German and Russian work is the fur linings called rotondes, sacques or plates, which are made for their home use and exportation chiefly to Great Britain, America and France.

In Weissenfels, near Leipzig, the dressing of Russian grey squirrel and the making it into linings is a gigantic industry, and is the principal support of the place. After the dressing process the backs of the squirrels are made up separately from the under and thinner white and grey parts, the first being known as 355
squirrel-back and the other as squirrel-lock linings. A few linings are made from entire skins and others are made from the quite white pieces, which in some instances are spotted with the black ear tips of the animals to resemble ermine. The smaller and uneven pieces of heads and legs are made up into linings, so there is absolutely no waste. Similar work is done in Russia on almost as extensive a scale, but neither the dressing nor the work is so good as the German.

The majority of heads, gills or throats, sides or flanks, paws and pieces of skins cut up in the fur workshops of Great Britain, America and France, weighing many tons, are chiefly exported to Leipzig, and made up in neighbouring countries and Greece, where labour can be obtained at an alarmingly low rate. Although the sewing, which is necessarily done by hand, the sections being of so unequal and tortuous a character, is rather roughly executed, the matching of colours and qualities is excellent. The enormous quantities of pieces admit of good selection and where odd colours prevail in a lining it is dyed. Many squirrel-lock linings are dyed blue and brown and used for the outside of cheap garments. They are of little weight, warm and effective, but not of great durability.

The principal linings are as follows: Sable sides, sable heads and paws, sable gills, mink sides, heads and gills, marten sides, heads and gills, Persian lamb pieces and paws, caracul lamb pieces or paws, musquash sides and heads, nutria sides, genet pieces, raccoon sides or flanks, fox sides, kolinski whole skins, and small rodents as kaluga and hamster. The white stripes cut out of skunks are made into rugs.

Another great source of inexpensive furs is China, and for many years past enormous quantities of dressed furs, many of which are made up in the form of linings and Chinese loose-shaped garments, have been imported by England, Germany and France for the lower class of business; the garments are only regarded as so much fur and are reworked. With, however, the exception of the best white Tibet lambs, the majority of Chinese furs can only be regarded as inferior material. While the work is often cleverly done as to matching and manipulation of the pelt which is very soft, there are great objections in the odour and the brittleness or weakness of the fur. One of the most remarkable results of the European intervention in the Boxer rising in China (1900) was the absurd price paid for so-called “loot” of furs, particularly in mandarins’ coats of dyed and natural fox skins and pieces, and natural ermine, poor in quality and yellowish in colour; from three to ten times their value was paid for them when at the same time huge parcels of similar quality were warehoused in the London docks, because purchasers could not be found for them.

With regard to Japanese furs, there is little to commend them. The best are a species of raccoon usually sold as fox, and, being of close long quality of fur, they are serviceable for boas, collars, muffs and carriage aprons. The sables, martens, minks and otters are poor in quality, and all of a very yellow colour and they are generally dyed for the cheap trade. A small number of very pretty guanaco and vicuna carriage rugs are imported into Europe, and many come through travellers and private sources, but generally they are so badly dressed that they are quite brittle upon the leather side. Similar remarks are applicable to opossum rugs made in Australia. From South Africa a quantity of jackal, hyena, fox, leopard and sheep karosses, *i.e.* a peculiarly shaped rug or covering used by native chiefs, is privately brought over. The skins are invariably tanned and beautifully sewn, the furs are generally flat in quality and not very strong in the hair, and are retained’ more as curiosities than for use as a warm covering.

Hatters’ Furs and Cloths and Shawls.—The hat trade is largely interested in the fur piece trade, the best felt hats being made from beaver and musquash wool and the cheaper sorts from nutria, hare and rabbit wools. For weaving, the most valuable pieces are mohair taken from the angora and vicuna. They are limited in quantity and costly, and the trade depends upon various sorts of other sheep and goat wools for the bulk of its productions.

Frauds and Imitations.—The opportunities for cheating in the fur trade are very considerable, and most serious frauds have been perpetrated in the selling of sables that have been coloured or “topped”; that is, just the tips of the hairs stained dark to represent more expensive skins. It is only by years of experience that some of these colourings can be detected. Where the skins are heavily dyed it is comparatively easy to see the difference between a natural and a dyed colour, as the underwool and top hair become almost alike and the leather is also dark, whereas in natural skins the base of the underwool is much paler than the top, or of a different colour, and the leather Is white unless finished in a pale reddish tone as is sometimes the case when mahogany sawdust is used in the final cleaning. As has been explained, sable is a term applied for centuries past to the darker sorts of the Russian Siberian martens, and for years past the same term has been bestowed by the retail trade upon the American and Canadian martens. The baum and stone martens caught in France, the north of Turkey and Norway are of the same family, but coarser in underwool and the top hair is less in quantity and not so silky. The kolinski, or as it is sometimes styled Tatar sable, is the animal, the tail of which supplies hair for artists’ brushes. This is also of the marten species and has been frequently offered, when dyed dark, as have baum and stone martens, as Russian sables. Hares, too, are dyed a sable colour and advertised as sable. The fur, apart from a clumsy appearance, is so brittle, however, as to be of scarcely any service whatever.

Among the principal imitations of other furs is musquash, out of which the top hair has been pulled and the undergrowth of wool clipped and dyed exactly the same colour as is used for seal, which is then offered as seal or red river seal. Its durability, however, is far less than that of seal. Rabbit is prepared and dyed and frequently offered as “electric sealskin.” Nutria also is prepared to represent sealskin, and in its natural colour, after the long hairs are plucked out, it is sold as otter or beaver. The wool is, however, poor compared to the otter and beaver, and the pelt thin and in no way comparable to them in strength. White hares are frequently sold as white fox, but the fur is weak, brittle and exceedingly poor compared to fox and possesses no thick underwool. Foxes, too, and badger are dyed a brownish black, and white hairs inserted to imitate silver fox, but the white hairs are too coarse and the colour too dense to mislead any one who knows the real article. But if sold upon its own merits, pointed fox is a durable fur.

Garments made of sealskin pieces and Persian lamb pieces are frequently sold as if they were made of solid skins, the term “pieces” being simply suppressed. The London Chamber of Commerce have issued to the British trade a notice that any misleading term in advertising and all attempts at deception are illegal, and offenders are liable under the Merchandise Marks Act 1887.

The most usual misnaming of manufactured furs is as follow:—

Musquash, pulled and dyed	Sold as seal.
Nutria, pulled and dyed	Sold as seal.
Nutria, pulled and natural	Sold as beaver.
Rabbit, sheared and dyed	Sold as seal or electric seal.
Otter, pulled and dyed	Sold as seal.
Marmot, dyed	Sold as mink or sable.
Fitch, dyed	Sold as sable.
Rabbit, dyed	Sold as sable or French sable.
Hare, dyed	Sold as sable, or fox, or lynx.
Musquash, dyed	Sold as mink or sable.
Wallaby, dyed	Sold as skunk.
White Rabbit	Sold as ermine.
White Rabbit, dyed	Sold as chinchilla.
White Hare, dyed or natural	Sold as fox, foxaline, and other similar names.
Goat, dyed	Sold as bear, leopard, &c.
Dyed manufactured articles of all kinds	Sold as “natural.”
White hairs inserted in foxes and sables	Sold as real or natural furs.
Kids	Sold as lamb or broadtails.
American sable	Sold as real Russian sable.
Mink	Sold as sable.

The Preservation of Furs.—For many years raw sealskins have been preserved in cold storage, but it is only 356
within a recent period, owing to the difficulty there was in obtaining the necessary perfectly dry atmosphere, that dressed and made-up furs have been preserved by freezing. Furs kept in such a condition are not only immune from the ravages of the larvae of moth, but all the natural oils in the pelt and fur are conserved, so that its colour and life are prolonged, and the natural deterioration is arrested. Sunlight has a tendency to bleach furs and to encourage the development of moth eggs, therefore continued exposure is to be avoided. When furs are wetted by rain they should be well shaken and allowed to dry in a current of air without exposure to sun or open fire.

Where a freezing store for furs is not accessible, furs should be well shaken and afterwards packed in linen and kept in a perfectly cool dry place, and examined in the summer at periods of not less than five weeks. Naphthalene and the usual malodorous powders are not only very disagreeable, but quite useless. Any chemical that is strong enough to destroy the life in a moth egg would also be sufficiently potent to injure the fur itself. In England moth life is practically continuous all the year round, that is, as regards those moths that attack furs, though the destructive element exists to a far greater extent during spring and summer.

Comparative Durability of Various Furs and Weight of Unlined Skins per Square Foot.

The following estimates of durability refer to the use of fur when made up “hair outside” in garments or stoles, not as a lining. The durability of fur used as linings, which is affected by other conditions, is set forth separately. Otter, with its water hairs removed, the strongest of furs for external use, is, in this table, taken as the standard at 100 and other furs marked accordingly:—

The Precious Furs.

	Points of Durability.	[2] Weight in oz. per sq. ft.
Sable	60	2½
Sea	75	3
Fox, Silver or Black	40	3
Fox, White	20	3
Ermine	25	1¼
Chinchilla	15	1½
Sea-otter (for stoles or collars)	100	4¼

The Less Valuable Furs.

	Points of Durability.	Weight in oz. per sq. ft.
Sable “topped,” *i.e.* top hairs coloured	55	2½
Sable tinted, *i.e.* fur all coloured.	50	2½
Baum Marten, natural	65	2¾
Baum Marten, tinted	45	2¾
Stone Marten	40	2¾
Nutria	27	3¼
Musquash, natural	37	3¼
Musquash, water hairs removed, sheared and seal finished.	33	3¼
Skunk	70	2¾
Mink	70	3¼
Lynx, natural	25	2¾
Lynx, tinted black	20	2¾
Marmot, tinted	10	3
Fox, tinted black	25	3
Fox, tinted blue	20	3
Opossum	37	3
Otter (with water hairs)	100	4
Otter (water hairs removed)	95	$3\frac{15}{16}$
Beaver (water hairs cut level with fur)	90	4
Beaver (water hairs removed)	85	$3\frac{15}{16}$
Moleskin	7	1¾
Persian Lamb	65	3¼
Grey Lamb	30	3¼
Broadtail	15	2¼
Caracul Kid	10	3¼
Caracul Lamb	15	3¼
Squirrel	25	1¾
Hare	5	1¾
Rabbit	5	2¼

Quantities of Fur needed, in Square Feet.

The “Paris Model” figure is the basis of these estimates for ladies’ garments, the standard measurements being height 5 ft. 6 in., waist 23 in., bust 38 in.

	Sq. Ft. (approximate).
Straight stole ½ length (just below the waist line)	2¾
Straight stole ¾ length (just below the knee)	3¾
Stole, broad enough at the neck to cover the top of arm ¾ length	5
The same, full length (to hem of skirt)	6
Eton jacket, without collar	13
Plain cape, 15 in. long	6½
Deep cape, 30 in. long	15
Full cape with broad stole front, ¾ length	15
Inverness cape (to knee)	25
Double-breasted, straight, semi-fitting coat, covering hips	16
Double-breasted sacque jacket, 36 in. long, full sleeves	20
Same, 30 in. long	18
Same, 22 in. long	15
Long, full, shawl cape with points at back and front, well below knee	15
Shorter shawl cape	16
Motoring or driving coat, ¾ length	22
Motoring or driving coat, full length	27

Weight and Durability of Furs for Men’s Coat Linings.

Otter with the water hairs removed, the strongest fur suited for linings, is here taken as the standard.

	Points of Durability.	Weight in oz. per sq. ft.
Otter (the water hairs removed)	100	$3\frac{15}{16}$
Beaver (the water hairs removed)	90	$3\frac{15}{16}$
Mink	90	3¼
Sealskin	75	3
Raccoon	75	4½
Persian lamb or astrachan	70	3¼
Sable	65	2½
Musquash	55	3½
Nutria	40	3¼
Grey Opossum	40	3
Wallaby	30	3¾
Squirrel	30	1¾
Hamster	15	1¼
Rabbit	10	2¼

Durability and Weight of Linings for Ladies’ Coats or Wraps.

Sable gills, the strongest fur suited for ladies’ linings, is taken as the standard.

	Points of Durability.	Weight in oz. per sq. ft.
Sable gills	100	2⅞
Sable	85	2½
Sable paws	64	1⅝
Ermine	57	1¼
Squirrel back	50	1¾
Squirrel heads	36	2½
Squirrel lock	21	$1\frac{3}{16}$
Hamster	10	1¼
Rabbit	7	2¼

Durability and Weight of Motoring Furs made up with Fur outside.

Otter with the water hairs, the strongest fur suited for motoring garments, is taken as the standard.

	Points of Durability.	Weight in oz. per sq. ft.
Otter (with water hairs)	100	4
Sealskin, marble	80	3
”Hair Sealskin” (tinted) with water hairs (a special variety of seal)	75	3¼
Raccoon	65	4½
Russian Pony	35	2⅝

Durability and Weight of Furs for Rugs and Foot-sacks.

	Points of Durability.	Weight in oz. per sq. ft.
Wolverine	100	6
Bear (black or brown natural)	94	7
Bear (tinted black)	88	7½
Beaver	88	4
Raccoon	77	4½
Opossum	61	3
Wolf	50	6½
Jackal	27	4½
Australian Bear	16	6
Goat	11	4⅙

Wolverine, the strongest fur suited for rugs and foot-sacks, is taken as the standard.

For a rug about 20 to 25 sq. ft. of fur are needed, for a foot-sack 14½.

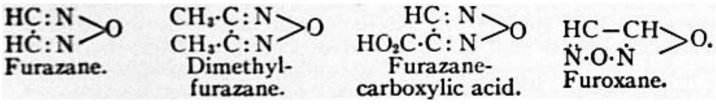
(W. S. P.)

1 The measurements given are from nose to root of tail of average large sizes after the dressing process, which has a shrinking tendency. The depths of fur quoted are the greatest, but there are plenty of good useful skins possessing a lesser depth.

2 Stout, old-fashioned boxcloth is almost the only cloth that (after a soft, heavy lining has been added to it) affords even two-thirds as much protection against cold as does fur. It weighs 4.273 oz. per sq. ft. more than the heaviest of coat-furs, and is so rigid as to be uncomfortable, while the subtileness of fur makes it “kind” to the body.

FURAZANES (*furo*—a.a′—*diazoles*), organic compounds obtained by heating the glyoximes (dioximes of ortho-diketones) with alkalis or ammonia. Dimethylfurazane is prepared by heating dimethylglyoxime with excess of ammonia for six hours at 165° C. (L. Wolff, *Ber.*, 1895, 28, p. 70). It is a liquid (at ordinary temperature) which boils at 156° C. (744 mm.). Potassium permanganate oxidizes it first to methylfurazane-carboxylic acid and then to furazanedicarboxylic acid. Methyl-ethylfurazane and diphenylfurazane are also known. By warming oxyfurazane acetic acid with excess of potassium permanganate to 100° C. oxyfurazanecarboxylic acid is obtained (A. Hantzsch and J. Urbahn, *Ber.*, 1895, 28, p. 764). It crystallizes in prisms, which melt at 175° C. Furazanecarboxylic acid is prepared by the action of a large excess of potassium permanganate on a hot solution of furazanepropionic acid. It melts at 107º C, and dissolves in caustic soda, with a deep yellow colour and formation of nitrosocyanacetic acid (L. Wolff and P.F. Ganz, *Ber.*, 1891, 24, p. 1167). Furoxane is an oxide of furazane, considered by H. Wieland to be identical with glyoxime peroxide; Kekulés dibromnitroacetonitrile is dibromfuroxane.

The formulae of the compounds above mentioned are:



FURETIÈRE, ANTOINE (1619-1688), French scholar and miscellaneous writer, was born in Paris on the 28th of December 1619. He first studied law, and practised for a time as an advocate, but eventually took orders and after various preferments became abbé of Chalivoy in the diocese of Bourges in 1662. In his leisure moments he devoted himself to letters, and in virtue of his satires—*Nouvelle Allégorique, ou histoire des derniers troubles arrivés au royaume d’éloquence* (1658); *Voyage de Mercure* (1653)—he was admitted a member of the French Academy in 1662. That learned body had long promised a complete dictionary of the French tongue; and when they heard that Furetière was on the point of issuing a work of a similar nature, they interfered, alleging that he had purloined from their stores, and that they possessed the exclusive privilege of publishing such a book. After much bitter recrimination on both sides the offender was expelled in 1685; but for this act of injustice he took a severe revenge in his satire, *Couches de l’académie* (Amsterdam, 1687). His *Dictionnaire universel* was posthumously published in 1690 (Rotterdam, 2 vols.). It was afterwards revised and improved by the Protestant jurist, Henri Basnage de Beauval (1656-1710), who published his edition (3 vols.) in 1701; and it was only superseded by the compilation known as the *Dictionnaire de Trévoux* (Paris, 3 vols., 1704; 7th ed., 8 vols., 1771), which was in fact little more than a reimpression of Basnage’s edition. Furetière is perhaps even better known as the author of *Le Roman bourgeois* (1666). It cast ridicule on the fashionable romances of Mlle de Scudéry and of La Calprenède, and is of interest as descriptive of the everyday life of his times. There is no element of burlesque, as in Scarron’s *Roman comique*, but the author contents himself with stringing together a number of episodes and portraits, obviously drawn from life, without much attempt at sequence. The book was edited in 1854 by Edward Fournier and Charles Asselineau and by P. Jannet.

The *Fureteriana*, which appeared in Paris eight years after Furetière’s death, which took place on the 14th of May 1688, is a collection of but little value.

FURFOOZ, a village some 10 m. from Dinant in the Ardennes, Belgium. Three caves containing prehistoric remains were here excavated in 1872. Of these the *Trou de Frontal* is the most famous. In it were found human skeletons with brachycephalic skulls, associated with animal bones, those of the reindeer being particularly plentiful. Among the skeletons was discovered an oval vase of pottery. The Furfooz type of mankind is believed to date from the close of the Quaternary age. G. de Mortillet dates the type in the Robenhausen epoch of the Neolithic period. His theory is that the bones are those of men of that period buried in what had been a cave-dwelling of the Madelenian epoch.

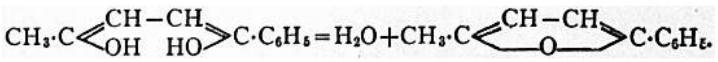
FURFURANE, or Furane, C_4H_4O, a colourless liquid boiling at 32° C., found in the distillation products of pine wood. It was first synthetically prepared by H. Limpricht (*Ann.*, 1873, 165, p. 281) by distilling barium mucate with soda lime, pyromucic acid $C_4H_3O{\cdot}CO_2H$ being formed, which, on further loss of carbon dioxide, yielded furfurane. A. Henniger (*Ann. chim. phys.*, 1886 [2], 7, p. 220), by distilling erthyrite with formic acid, obtained a dihydrofurfurane

$$C_4H_6(OH)_4 + 2H_2CO_2 = C_4H_6O + CO + CO_2 + 4H_2O,$$

which, on treatment with phosphorus pentachloride, yielded furfurane. Furfurane is insoluble in water and possesses a characteristic smell. It does not react with sodium or with phenylhydrazine, but yields dye-stuffs with isatin and phenanthrenequinone. It reacts violently with hydrochloric acid, producing a brown amorphous substance. Methyl and phenyl derivatives have been prepared by C. Paal (*Ber.*, 1884, 17, p. 915). Paal prepared acetonyl acetophenone by condensing sodium acetoacetate with phenacylbromide, and this substance on dehydration yields αα′-phenylmethylfurfurane, the acetonyl acetophenone probably reacting in the tautomeric "enolic" form,

$$CH_3{\cdot}CO{\cdot}CHNa{\cdot}COOR + C_6H_5{\cdot}CO{\cdot}CH_2Br = CH_3{\cdot}CO{\cdot}CH(CH_2COC_6H_5){\cdot}COOR.$$

This ester readily hydrolyses, and the acid formed yields acetonyl acetophenone (by loss of carbon dioxide), which then on dehydration yields the furfurane derivative, thus



L. Knorr (*Ber.*, 1889, 22, p. 158) obtained diacetosuccinic ester by condensing sodium acetoacetate with iodine, and by dehydrating the ester he prepared αα′-dimethylfurfurane ββ′-dicarboxylic acid (carbopyrotritaric acid), which on distillation yields αα′-dimethylfurfurane as a liquid boiling at 94° C. Paal also obtained this compound by using monochloracetone in the place of phenacylbromide. By the distillation of mucic acid or isosaccharic acid, furfurane-α-carboxylic acid (pyromucic acid), $C_4H_3O{\cdot}CO_2H$, is obtained; it crystallizes in needles or leaflets, and melts at 134° C.

Furfurol (furol), $C_4H_3O{\cdot}CHO$, is the aldehyde of pyromucic acid, and is formed on distilling bran, sugar, wood and most carbohydrates with dilute sulphuric acid, or by distilling the pentoses with hydrochloric acid. It is a colourless liquid which boils at 162° C., and is moderately soluble in water; it turns brown on exposure to air and has a characteristic aromatic smell. It shows all the usual properties of an aldehyde, forming a bisulphite compound, an oxime and a hydrazone; whilst it can be reduced to the corresponding furfuryl alcohol by means of sodium amalgam, and oxidized to pyromucic acid by means of silver oxide. It also shows all the condensation reactions of benzaldehyde (*q.v.*); condensing with aldehydes and ketones in the presence of caustic soda to form more complex aldehydes and ketones with unsaturated side chains, such as 358
furfuracrolein, $C_4H_3O{\cdot}CH{:}CH{\cdot}CHO$, and furfuracetone, $C_4H_3O{\cdot}CH{:}CH{\cdot}CO{\cdot}CH_3$. With alcoholic potassium cyanide It changes to furoin, $C_4H_3O{\cdot}CHOH{\cdot}CO{\cdot}C_4H_3O$, which can be oxidized to furil, $C_4H_3O{\cdot}CO{\cdot}CO{\cdot}C_4H_3O$, whilst alcoholic potash converts it into furfuryl alcohol. With fatty acids and acid anhydrides it gives the "Perkin" reaction (see Cinnamic Acid). Furfurol is shown to have its aldehydic group in the *a* position, by conversion into furfurpropionic acid, $C_4H_3O{\cdot}CH_2{\cdot}CH_2{\cdot}CO_2H$, which on oxidation by bromine water and subsequent reduction of the oxidized product is converted into *n*-pimelic acid, $HO_2C(CH_2)_5CO_2H$. Furfurol in minute quantities can be detected by the red colour it forms with a solution of aniline acetate.

Furfurane—αα′-dicarboxylic acid or dehydromucic acid, $C_4H_2O(CO_2H)_2$, is formed when mucic acid is heated with hydrochloric acid at 100° C. On being heated, it loses carbon dioxide and gives pyromucic acid. By digesting acetoacetic ester with sodium succinate and acetic anhydride, methronic acid, $C_8H_8O_5$, is obtained; for the constitution of this acid, see L. Knorr, *Ber.*, 1889, 22, p. 152, and R. Fittig, *Ann.*, 1889, 259, p. 166.

Di- and tetrahydrofurfurane compounds are also known (see A. Lipp, *Ber.*, 1889, 22, p. 1196; W.H. Perkin, junr. *Journ. Chem. Soc.*, 1899, 57, p. 944; and S. Ruhemann, *ibid.*, 1896, 69, p. 1383).

FURIES (Lat. *Furiae*, also called Dirae), in Roman mythology an adaptation of the Greek Erinyes (*q.v.*), with whom they are generally identical. A special aspect of them in Virgil is that of agents employed by the higher gods to stir up mischief, strife and hatred upon earth. Mention may here be made of an old Italian deity Furina (or Furrina), whose worship fell early into disuse, and who was almost forgotten in the time of Varro. By the mythologists of Cicero's time the name was connected with the verb *furere* and the noun *furia*, which in the plural (not being used in the singular in this sense) was accepted as the equivalent of the Greek Erinyes. But it is more probably related to *furvus*, *fuscus*, and signifies one of the spirits of darkness, who watched over men's lives and haunted their abodes. This goddess had her own special priest, a grove across the Tiber where Gaius Gracchus was slain, and a festival on the 25th of July. Authorities differ as to the existence of more than one goddess called Furina, and their identity with the Forinae mentioned in two inscriptions found at Rome (*C.I.L.* vi. 422 and 10,200).

FURLONG (from the O. Eng. *furlang*, *i.e.* "furrow-long"), a measure of length, originally the length of a furrow in the "common field" system. As the field in this system was generally taken to be a square, 10 acres in extent, and as the acre varied in different districts and at different times, the "furlong" also varied. The side of a square containing 10 statute acres is 220 yds. or 40 poles, which was the usually accepted length of the furlong. This is also the length of $\frac{1}{8}$th of the statute mile. "Furlong" was as early as the 9th century used to translate the Latin *stadium*, $\frac{1}{8}$th of the Roman mile.

FURNACE, a contrivance for the production and utilization of heat by the combustion of fuel. The word is common to all the Romance tongues, appearing in more or less modified forms of the Latin *fornax*. But in all those languages the word has a more extended meaning than in English, as it covers every variety of heating apparatus; while here, in addition to furnaces proper, we distinguish other varieties as *ovens*, *stoves* and *kilns*. The first of these, in the form *Ofen*, is used in German as a general term like the French *four*; but in English it has been restricted to those apparatus in which only a moderate temperature, usually below a red heat, is produced in a close chamber. Our bakers' ovens, hot-air ovens or stoves, annealing ovens for glass or metal, &c., would all be called *fours* in French and *Öfen* in German, in common with furnaces of all kinds. Stove, an equivalent of oven, is from the German *Stube*, *i.e.* a heated room, and is commonly so understood; but is also applied to open fire-places, which appears to be somewhat of a departure from the original signification.

Furnaces are constructed according to many different patterns with varying degrees of complexity in arrangement; but all may be considered as combining three essential parts, namely, the fire-place in which the fuel is consumed, the heated chamber, laboratory, hearth or working bed, as it is variously called, where the heat is applied to the special work for which the furnace is designed, and the apparatus for producing rapid combustion by the supply of air under pressure to the fire. In the simplest cases the functions of two or more of these parts may be combined into one, as in the smith's forge, where the fire-place and heating chamber are united, the iron being placed among the coals, only the air for burning being supplied under pressure from a blowing engine by a second special contrivance, the tuyere, tuiron, twyer or blast-pipe; but in the more refined modern furnaces, where great economy of fuel is an object, the different functions are distributed over separate and distinct apparatus, the fuel being converted into gas in one, dried in another, and heated in a third, before arriving at the point of combustion in the working chamber of the furnace proper.

Furnaces may be classified according as the products of combustion are employed (1) only for heating purposes, or (2) both for heating and bringing about some chemical change. The furnaces employed for steam-raising or for heating buildings are invariably of the first type (see Boiler and Heating), while those employed in metallurgy are generally of the second. The essential difference in construction is that in the first class the substances heated do not come into contact with either the fuel or the furnace gases, whereas in the second they do. Metallurgical furnaces of the first class are termed crucible, muffle or retort furnaces, and of the second shaft and reverberatory furnaces. The following is a detailed subdivision:—

(1) Fuel and substance in contact.

- (*a*) Height of furnace greater than diameter = shaft furnaces.
  - (α) No blast = kilns.
  - (β) With blast = blast furnaces.
- (*b*) Height not much greater than diameter = hearth furnaces.

(2) Substance heated by products of combustion = reverberatory furnaces.

- (*a*) Charge not melted = roasting or calcining furnaces.
- (*b*) Charge melted = melting furnaces.

(3) Substance is not directly heated by the fuel or by the products of combustion.

- (*a*) Heating chamber fixed and forming part of furnace = muffle furnaces.
- (*b*) Crucible furnaces.
- (*c*) Retort furnaces.

Another classification may be based upon the nature of the heating agent, according as it is coal (or some similar combustible) oil, gas or electricity. In this article the general principles of metallurgical furnaces will be treated; the subject of gas- and oil-heated furnaces is treated in the article Fuel, and of the electric furnace in the article Electrometallurgy. For special furnaces reference should be made to the articles on the industry concerned, *e.g.* Glass, Gas, § Manufacture, &c.

Shaft, Blast and Hearth Furnaces.—The blast furnace in its simplest form is among the oldest, if not the oldest, of metallurgical contrivances. In the old copper-smelting district of Arabia Petraea, clay blast-pipes dating back to the earlier dynasties of ancient Egypt have been found buried in slag heaps; and in India the native smiths and iron-workers continue to use furnaces of similar types. These, when reduced to their most simple expression, are mere basin-shaped hollows in the ground, containing ignited charcoal and the substances to be heated, the fire being urged by a blast of air blown in through one or more nozzles from a bellows at or near the top. They are essentially the same as the smith's forge. This class of furnace is usually known as an open fire or hearth, and is represented in a more advanced stage of development by the Catalan, German and Walloon forges formerly used in the production of malleable iron.

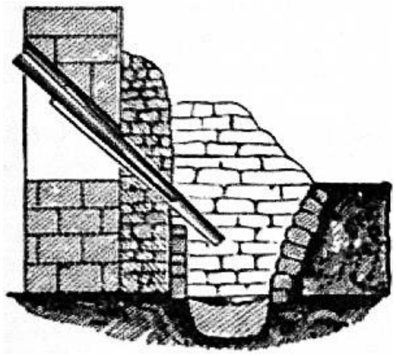


Fig. 1.—Elevation of Catalan Forge.

Fig. 1 represents a Catalan forge. The cavity in the ground is represented by a pit of square or rectangular section lined with brick or stone of a kind not readily acted on by heat, about 1½ or 2 ft. deep, usually somewhat larger above than below, with a tuyere or blast-pipe of copper penetrating one of the walls near the top, with a considerable downward inclination, so that the air meets the fuel some way down. In iron-smelting the ore is laid in a heap upon the fuel (charcoal) filling up the hearth, and is gradually brought to the metallic state by the reducing action of the carbon monoxide formed at the tuyere. The metal sinks through the ignited fuel, forming, in the hearth, a spongy mass or ball, which is lifted out by the smelters at the end of each operation, and carried to the forge hammer. The earthy matters form a fusible glass or slag melt, and collect at the lowest point of the hearth, whence they are removed by opening a hole pierced through the front wall at the bottom. The active portion of such a furnace is essentially that above the blast-pipe, the function of the lower part being merely the collection of the reduced metal; the fire may therefore be regarded as burning in an unconfined space, with the waste of a large amount of its heating power. By continuing the walls of the hearth above the tuyere, into a shaft or stack either of the same or some other section, we obtain a furnace of increased capacity, but with no greater power of consuming fuel, in which the material to be treated can be heated up gradually by loading it into the stack, alternately with layers of fuel, the charge descending regularly to the point of combustion, and absorbing a proportion of the heat of the flame that went to waste in the open fire. This principle is capable of very wide extension, the blast furnace being mainly limited in height by the strength the column of materials or "burden" has to resist crushing, under the weight due to the head adopted, and the power of the blowing engine to supply blast of sufficient density to overcome the resistance of the closely packed materials to the free passage of the spent gases. The consuming power of the furnace or the rate at which it can burn the fuel supplied is measured by the number of tuyeres and their section.

The development of blast furnaces is practically the development of iron-smelting. The profile has been very much varied at different times. The earliest examples were square or rectangular in horizontal section, but the general tendency of modern practice is to substitute round sections, their construction being facilitated by the use of specially moulded bricks which have entirely superseded the sandstone blocks formerly used. The vertical section, on the other hand, is subject to considerable variation according to the work to which the furnace is applied. Where the operation is simply one of fusion, as in the iron-founder's cupola, in which there is no very great change in volume in the materials on their descent to the tuyeres, the stack is nearly or quite straight-sided; but when, as is the case with the smelting of iron ores with limestone flux, a large proportion of volatile matter has to be removed in the process, a wall of varying inclination is used, so that the body of the furnace is formed of two dissimilar truncated cones, joined by their bases, the lower one passing downwards into a short, nearly cylindrical, position. For further consideration of this subject see Iron and Steel.

Hearth furnaces are employed in certain metallurgical operations, *e.g.* in the air-reduction process for smelting lead ores. The principle is essentially that of the Catalan forge. Such furnaces are very wasteful, and have little to recommend them (see Schnabel, Metallurgy, 1905, vol. 1. p. 409).

Reverberatory Furnaces.—Blast furnaces are, from the intimate contact between the burden to be smelted and the fuel, the least wasteful of heat; but their use supposes the possibility of obtaining fuel of good quality and free from sulphur or other substances likely to deteriorate the metal produced. In all cases, therefore, where it is desired to do the work out of contact with the solid fuel, the operation of burning or heat-producing must be performed in a special fire-place or combustion chamber, the body of flame and heated gas being afterwards made to act upon the surface of the material exposed in a broad thin layer in the working bed or laboratory of the furnace by reverberation from the low vaulted roof covering the bed. Such furnaces are known by the general name of reverberatory or reverbatory furnaces, also as air or wind furnaces, to distinguish them from those worked with compressed air or blast.

Originally the term cupola was used for the reverberatory furnace, but in the course of time it has changed its meaning, and is now given to a small blast furnace such as that used by iron-founders—reverberatory smelting furnaces in the same trade being called air furnaces.

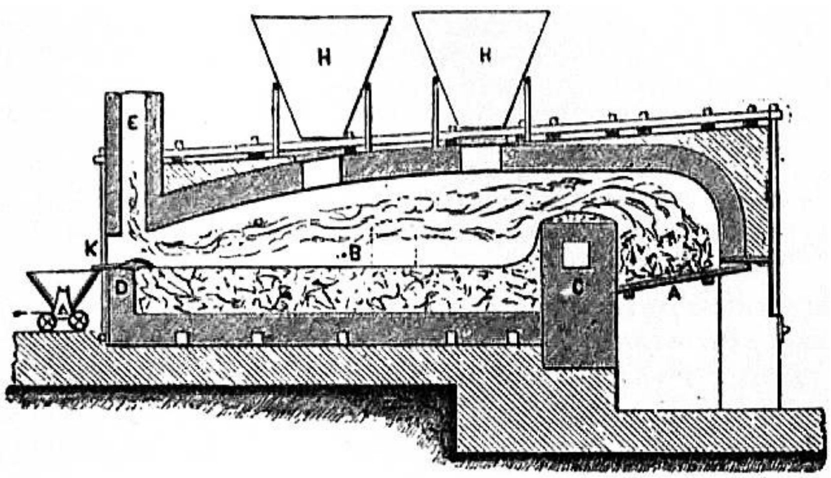


Fig. 2.—Longitudinal section of Reverberatory Furnace.

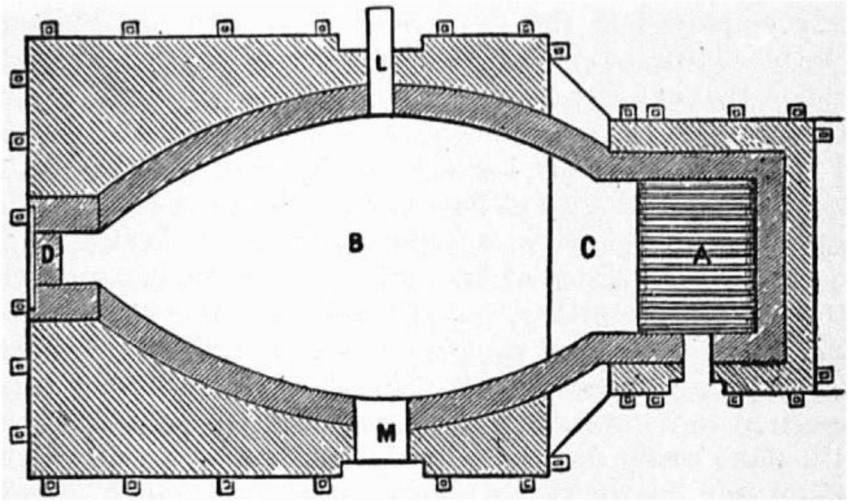


Fig. 3.—Reverberatory Furnace (horizontal section).

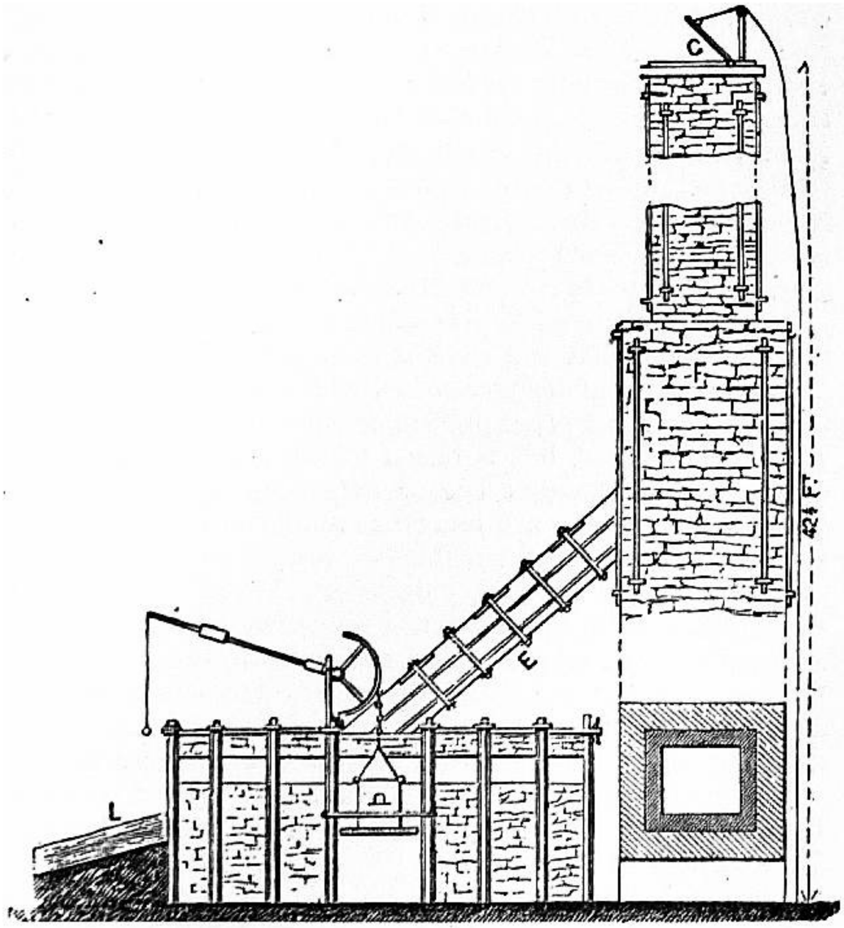


Fig. 4.—Reverberatory Furnace (elevation at flue end).

Figs. 2, 3 and 4 represent a reverberatory furnace such as is used for the fusion of copper ores for regulus, and may be taken as generally representing its class. The fire-place A is divided from the working bed B by a low wall C known as the fire bridge, and at the opposite end there is sometimes, though not invariably, a second bridge of less height called the flue bridge D. A short diagonal flue or up-take E conveys the current of spent flame to the chimney F, which is of square section, diminishing by steps at two or three different heights, and provided at the top with a covering plate or damper G, which may be raised or lowered by a chain reaching to the ground, and serves for regulating the speed of the exhaust gases, and thereby the draught of air through the fire. Where several furnaces are connected with the same chimney stack, the damper takes the form of a sliding plate in the mouth of the connecting flue, so that the draught in one may be modified without affecting the others. The fire bridge is partially protected against the intense heat of the 360 body of flame issuing through the fire arch by a passage to which the air has free access. The material to be melted is introduced into the furnace from the hoppers HH through the charging holes in the roof. When melted the products separate on the bed (which is made of closely packed sand or other infusible substances), according to their density; the lighter earthy matters forming an upper layer of slag are drawn out by the slag hole K at the flue end into an iron wagon or bogie, while the metal subsides to the bottom of the bed, and at the termination of the operation is run out by the tap hole L into moulds or granulated into water. The opposite opening M is the working door, through which the tool for stirring the charge is introduced. It is covered by a plate suspended to a lever, similar to that seen in the end elevation (fig. 4) in front of the slag hole.

According to the purposes to which they are applied, reverberatory furnaces may be classed into two groups, namely, fusion or melting furnaces, and calcining or wasting furnaces, also called calciners. The former have a very extended application in many branches of industry, being used by both founders and smelters in the fusion of metals; in the concentration of poor metallic compounds by fusion into regulus; in the reduction of lead and tin ores; for refining copper and silver; and for making malleable iron by the puddling processes and welding. Calcining furnaces have a less extended application, being chiefly employed in the conversion of metallic sulphides into oxides by continued exposure to the action of air at a temperature far below that of fusion, or into chlorides by roasting with common salt. As some of these substances (for example, lead sulphide and copper pyrites) are readily fusible when first heated, but become more refractory as part of the sulphur is dissipated and oxygen takes its place, it is important that the heat should be very carefully regulated at first, otherwise the mass may become clotted or fritted together, and the oxidizing effect of the air soon ceases unless the fritted masses be broken small again. This is generally done by making the bed of the furnace very long in proportion to its breadth and to the fire-grate area, which may be the more easily done as a not inconsiderable amount of heat is given out during the oxidation of the ore—such increased length being often obtained by placing two or even three working beds one above the other, and allowing the flame to pass over them in order from below upwards. Such calciners are used especially in roasting zinc blende into zinc oxide, and in the conversion of copper sulphides into chlorides in the wet extraction process. In some processes of lead-smelting, where the minerals treated contain sand, the long calciner is provided with a melting bottom close to the fire-place, so that the desulphurized ore leaves the furnace as a glassy slag or silicate, which is subsequently reduced to the metallic state by fusion with fluxes in blast furnaces. Reverberatory furnaces play an important part in the manufacture of sodium carbonate; descriptions and illustrations are given in the article Alkali Manufacture.

Muffle, Crucible and Retort Furnaces.—A third class of furnaces is so arranged that the work is done by indirect heating; that is, the material under treatment, whether subjected to calcination, fusion or any other process, is not brought in contact either with fuel or flame, but is raised to the proper temperature by exposure in a chamber heated externally by the products of combustion. These are known as muffle or chamber furnaces; and by supposing the crucibles or retorts to represent similar chambers of only temporary duration, the ordinary pot melting air furnaces, and those for the reduction of zinc ores or the manufacture of coal gas, may be included in the same category. These are almost invariably air furnaces, though sometimes air under pressure is used, as, for example, in the combustion of small anthracitic coal, where a current of air from a fan-blower is sometimes blown under the grate to promote combustion. Types of muffle furnaces are figured in the article Annealing, Hardening and Tempering.

Furnace Materials.—The materials used in the construction of furnaces are divisible into two classes, namely, ordinary and refractory or fire-resisting. The former are used principally as casing, walls, pillars or other supporting parts of the structure, and includes ordinary red or yellow bricks, clay-slate, granite and most building stones; the latter are reserved for the parts immediately in contact with the fuel and flame, such as the lining of the fire-place, the arches, roof and flues, the lower part if not the whole of the chimney lining in reverberatory furnaces, and the whole of the internal walls of blast furnaces. Among such substances are fireclay and firebricks, certain sandstones, silica in the form of ganister, and Dinas stone and bricks, ferric oxide and alumina, carbon (as coke and graphite), magnesia, lime and chromium oxide—their relative importance being indicated by their order, the last two or three indeed being only of limited use.

The most essential point in good fireclays, or in the bricks or other objects made from them, is the power of resisting fusion at the highest heat to which they may be exposed. This supposes them to be free from metallic oxides forming easily fusible compounds with silica, such as lime or iron, the presence of the former even in comparatively small proportion being very detrimental. As clays they must be sufficiently plastic to be readily moulded, but at the same time possess sufficient stiffness not to contract too strongly in drying, whereby the objects produced would be liable to be warped or cracked before firing. In most cases, however, the latter tendency is guarded against, in making up the paste for moulding, by adding to the fresh clay a certain proportion of burnt material of the same kind, such as old bricks or potsherds, ground to a coarse powder. Coke dust or graphite is used for the same purpose in crucible making (see Firebrick).

The most highly valued fireclays are derived from the Coal Measures. Among the chief localities are the neighbourhood of Stourbridge in Worcestershire and Stannington near Sheffield, which supply most of the materials for crucibles used in steel and brass melting, and the pots for glass houses; Newcastle-on-Tyne and Glenboig near Glasgow, where heavy blast furnace and other firebricks, gas retorts, &c., are made in large quantities. Coarse-grained but very strong firebricks are also made of the waste of china clay works.

In Belgium the clay raised at Andenne is very largely used for making retorts for zinc furnaces. The principal French fireclays are derived from the Tertiary strata in the south, and more nearly resemble porcelain clays than those of the Coal Measures. They give wares of remarkably fine texture and surface, combined with high refractory character.

In Germany, Ips and Passau on the Danube, and Gross Almerode in Hesse, are the best known localities producing fireclay goods, the crucibles from the last-mentioned place, known as Hessian crucibles, going all over the world. These, though not showing a great resistance to extreme heat, are very slightly affected by sudden alternations in heating, as they may be plunged cold into a strongly heated furnace without cracking, a treatment to which French and Stourbridge pots cannot be subjected with safety.

Plumbago or graphite is largely used in the production of crucibles, not in the pure state but in admixture with fireclay; the proportion of the former varies with the quality from 25 to nearly 50%. These are the most enduring of all crucibles, the best lasting out 70 or 80 meltings in brass foundries, about 50 with bronze, and 8 to 10 in steel-melting.

Silica is used in furnace-building in the forms of sand, ganister, a finely ground sandstone from the Coal Measures of Yorkshire, and the analogous substance known as Dinas clay, which is really nearly pure silica, containing at most about 2½% of bases. Dinas clay is found at various places in the Vale of Neath in South Wales, in the form of a loose disintegrated sandstone, which is crushed between rollers, mixed with about 1% of lime, and moulded into bricks that are fired in kilns at a very high temperature. These bricks are specially used for the roof, fire arches, and other parts subjected to intense heat in reverberatory steel-melting furnaces, and, although infusible under ordinary conditions, are often fairly melted by the heat without fluxing or corrosion after a certain amount of exposure. Ganister, a slightly plastic siliceous sand, is similarly used for the lining of Bessemer steel converters; it is found in the neighbourhood of Sheffield.

Alumina as a refractory material is chiefly used in the form of bauxite, but its applications are somewhat special. It has been found to stand well for the linings of rotatory puddling furnaces, where, under long-continued heating, it changes into a substance as hard and infusible as natural emery. In the Paris Exhibition 361 of 1878 bricks very hard and dense in character, said to be of pure alumina, were exhibited by Muller & Co. of Paris, as well as bricks of magnesia, the latter being specially remarkable for their great weight. They are intended for use at the extreme temperatures obtainable in steel furnaces, or for the melting of platinum before the oxy-hydrogen blowpipe. For the latter purpose, however, lime is generally used; but as this substance has only small stability, it is usually bedded in a casing of firebrick. Oxide of chromium and chrome iron ore have been proposed as refractory crucible materials. The former may be used as a bed for melting platinum in the same way as lime or magnesia, without affecting the quality of the metal.

Ferric oxide, though not strictly infusible, is largely used as a protecting lining for furnaces in which malleable iron is made, a portion of the ore being reduced and recovered in the process. In an oxidizing atmosphere it is indifferent to silica, and therefore siliceous bricks containing a considerable proportion of ferric oxide, when used in flues of boilers, brewers’ coppers, &c. and similar situations, are perfectly fire-resisting so long as the heated gas contains a large proportion of unconsumed air. The red firebricks known as Windsor bricks, which are practically similar in composition to soft red sandstone, are of this character.

The electric furnace has led to the discovery of several important materials, which have been employed as furnace linings. Carborundum (*q.v.*) was applied by Engels in 1899, firebricks being washed with carborundum paste and then baked. Siloxicon, a compound of carbon, silicon and oxygen, formed from carbon and silica in the electric furnace, was patented by E.G. Acheson in 1903. It is very refractory, and is applied by mixing with water and some bond, such as sodium silicate or gas-tar. An amorphous, soft silicon carbide, also formed in the electric furnace, was patented by B. Talbot in 1899. For basic linings, magnesia crystallized in the electric furnace is being extensively used, replacing dolomite to some extent (see E. Kilburn Scott, “Refractory Materials for Furnace Linings,” *Faraday Soc.*, 1906, p. 289).

Furnace Construction.—In the construction of furnaces provision has to be made for the unequal expansion of the different parts under the effect of heat. This is especially necessary in the case of reverberatory furnaces, which are essentially weak structures, and therefore require to be bound together by complicated systems of tie rods and uprights or buck staves. The latter are very commonly made of old flat bottom rails, laid with the flat of the flange against the wall. Puddling furnaces are usually entirely cased with iron plates, and blast furnaces with hoops round each course of the stack, or in those of thinner constructions the firebrick work is entirely enclosed in a wrought iron casing or jacket. Such parts as may be subjected to extreme heat and the fretting action of molten material, as the tuyere and slag breasts of blast furnaces, and the fire bridges and bed plates of reverberatory furnaces, are often made in cast iron with double walls, a current of water or air being kept circulating through the intermediate space. In this way the metal, owing to its high conductivity and low specific heat as compared to that of water, is kept at a temperature far below its melting point if the water is renewed quickly enough. It is of course necessary in such cases that the circulation shall be perfectly free, in order to prevent the accumulation of steam under pressure in the interior of the casting. This method has received considerable extension, notably in furnace-smelting of iron ores containing manganese, where the entire hearth is often completely water-cased, and in some lead furnaces where no firebrick lining is used, the lower part of the furnace stack being a mere double iron box cooled by water sufficiently to keep a coating of slag adhering to the inner shell which prevents the metal from being acted upon.

Mechanical Furnaces.—The introduction and withdrawal of the charges in fusion furnaces is effected by gravitation, the solid masses of raw ore, fuel and flux being thrown in at the top, and flowing out of the furnace at the taphole or slag run at the bottom. Vertical kilns, such as those used for burning limestone, are worked in a similar manner—the raw stone going in at the top, and the burnt product falling through holes in the bottom when allowed to do so. With reverberatory calciners, however, where the work is done upon a horizontal bed, a considerable amount of hand labour is expended in raking out the charge when finished, and in drawing slags from fusion furnaces; and more particularly in the puddling process of refining iron the amount of manual exertion required is very much greater. To diminish the item of expenditure on this head, various kinds of mechanical furnaces have been adopted, all of which can be classified under three heads of gravitating furnaces, mechanical stirrers and revolving furnaces.

1. In *gravitating furnaces* the bed is laid at a slope just within the angle of repose of the charge, which is introduced at the upper end, and is pushed down the slope by fresh material, when necessary, in the contrary direction to the flame which enters at the lower end. Gerstenhofer’s pyrites burner is a furnace of this class. It has a tall vertical chamber heated from below, and traversed by numerous narrow horizontal cross bars at different heights. The ore in fine powder is fed in at the top, through a hopper, in a regular thin stream, by a pair of rollers, and in falling lodges on the flats of the bars, forming a talus upon each of the height corresponding to the angle of rest of the material, which is, however, at short intervals removed to lower levels by the arrival of fresh ore from above. In this way a very large surface is exposed to the heat, and the ore, if containing sufficient sulphur to maintain the combustion, is perfectly burned when it arrives at the bottom; if, however, it is imperfectly sized or damp, or if it contains much earthy matter, the result is not very satisfactory. There are many other furnaces in which the same principle is utilized.

2. *Mechanical stirrers* constitute a second division of mechanical furnaces, in which the labour of rabbling or stirring the charges is performed by combinations of levers and wheel-work taking motion from a rotating shaft, and more or less perfectly imitating the action of hand labour. They are almost entirely confined to puddling furnaces.

3. *Revolving furnaces*, the third and most important division of mechanical furnaces, are of two kinds. The first of these resemble an ordinary reverberatory furnace by having a flat bed which, however, has the form of a circular disk mounted on a central shaft, and receives a slow movement of rotation from a water-wheel or other motor, so that every part of the surface is brought successively under the action of the fire, the charge being stirred and ultimately removed by passing under a series of fixed scraper arms placed above the surface at various points. Brunton’s calciner, used in the “burning” of the pyritic minerals associated with tin ore, is a

familiar example of this type. The hearth may either rotate on an inclined axis, so that the path of its surface is oblique to that of the flame, or the working part may be a hollow cylinder, between the fireplace and flue, with its axis horizontal or nearly so, whose inner surface represents the working bed, mounted upon friction rollers, and receiving motion from a special steam-engine by means of a central belt of spur gearing. Furnaces of the second kind were first used in alkali works for the conversion of sulphate into carbonate of sodium in the process known as black ash fusion, but have since been applied to other processes. As calciners they are used in tin mines and for the chlorination of silver ores. Mechanical furnaces are figured in the article ALKALI MANUFACTURE.

Use of Heated Air.—The calorific intensity of fuel is found to be very considerably enhanced, if the combustion be effected with air previously heated to any temperature between that of boiling water and a dull red heat, the same effect being observed both with solid and gaseous fuel. The latter, especially when brought to the burning point at a high temperature, produces a heat that can be resisted by the most refractory substances only, such as silica, alumina and magnesia. This is attained in the regenerative furnace of Siemens, detailed consideration of which belongs more properly to the subject of iron.

Economy of Waste Heat.—In every system of artificial heating, the amount of heat usefully applied is but a small proportion of that developed by combustion. Even under the most advantageous application, that of evaporation of water in a steam boiler where the gases of the fire have to travel through a great length of flues bounded by thin iron surfaces of great heat-absorbing capacity, the temperature of the current at the chimney is generally much above that required to maintain an active draught in the fireplace; and other tubes containing water, often in considerable numbers, forming the so-called fuel economizers, may often be interposed between the boiler and the chimney with marked advantage as regards saving of fuel. In reverberatory and air furnaces used in the different operations of iron manufacture, where an extremely high temperature has to be maintained in spaces of comparatively small extent, such as the beds of puddling, welding and steel-melting furnaces, the temperature of the exhaust gases is exceedingly high, and if allowed to pass directly into the chimney they appear as a great body of flame at the top. It is now general to save a portion of this heat by passing the flame through flues of steam boilers, air-heating apparatus, or both—so that the steam required for the necessary operations of the forge and heated blast for the furnace itself may be obtained without further expenditure of fuel. The most perfect method of utilizing the waste heat hitherto applied is that of the Siemens regenerator, in which the spent gases are made to travel through chambers, known as regenerators or recuperators of heat, containing a quantity of thin firebricks piled into a cellular mass so as to offer a very large heat-absorbing surface, whereby their temperature is very considerably reduced, and they arrive at the chimney at a heat not exceeding 300 or 400 degrees. As soon as the bricks have become red hot, the current is diverted to an adjacent chamber or pair of chambers, and the acquired heat is removed by a current of cool gas or air passing towards the furnace, where it arrives at a temperature sufficiently high to ensure the greatest possible heating effect in combustion.

In iron-smelting blast furnaces the waste gases are of considerable fuel value, and may render important 362
services if properly applied. Owing to the conditions of the work, which require the maintenance of a sensibly reducing atmosphere, they contain a very notable proportion of carbonic oxide, and are drawn off by large wrought iron tubes near the top of the furnace and conveyed by branch pipes to the different boilers and air-heating apparatus, which are now entirely heated by the combustion of such gases, or mixed with air and exploded in gas engines. Formerly they were allowed to burn to waste at the mouth of a short chimney place above the furnace top, forming a huge body of flame, which was one of the most striking features of the Black Country landscape at night.

Laboratory and Portable Furnaces.—Small air-furnaces with hot plates or sand bath flues were formerly much employed in chemical laboratories, as well as small blast furnaces for crucibles heated with charcoal or coke. The use of such furnaces has very considerably diminished, owing to the general introduction of coal-gas for heating purposes in laboratories, which has been rendered possible by the invention of the Bunsen burner, in which the mixture of air and gas giving the least luminous but most powerfully heating flame is effected automatically by the effluent gas. These burners, or modifications of them, have also been applied to muffle furnaces, which are convenient when only a few assays have to be made—the furnace being a mere clay shell and soon brought to a working temperature; but the fuel is too expensive to allow of their being used habitually or on a large scale. Petroleum, or rather the heavy oils obtained in tar refineries, having an equal or superior heating power to coal-gas, may also be used in laboratories for producing high temperatures. The oil is introduced in a thin stream upon a series of inclined and channelled bars, where it is almost immediately volatilized and burnt by air flowing in through parallel orifices. Furnaces of this kind may be used for melting cast iron or bronze in small quantities, and were employed by H. Sainte Claire Deville in experiments in the metallurgy of the platinum group of metals.

Sefstrom’s blast furnace, used in Sweden for the assay of iron ores, is a convenient form of portable furnace applied to melting in crucibles. It consists of a sheet-iron cylinder about 8 or 9 in. in diameter, within which is fixed one of smaller size lined with fireclay. The space between the two cylinders serves as a heater and distributor for the blast, which is introduced through the nozzle at the bottom, and enters the furnace through a series of several small tuyeres arranged round the inner lining. Charcoal is the fuel used, and the crucibles stand upon the bottom of the clay lining. When a large body of fuel is required, the cylinder can be lengthened by an iron hoop which fits over the top ring. Deville’s portable blast furnace is very similar in principle to the above, but the body of the furnace is formed of a single cast iron cylinder lined with fireclay, closed below by a cast iron plate perforated by a ring of small holes—a hemispherical basin below forming the air-heating chamber.

FURNEAUX, TOBIAS (1735-1781), English navigator, was born at Swilly near Plymouth on the 21st of August 1735. He entered the royal navy, and was employed on the French and African coasts and in the West Indies during the latter part of the Seven Years’ War (1760-1763). He served as second lieutenant of the “Dolphin” under Captain Samuel Wallis on the latter’s voyage round the globe (August 1766-May 1768); was made a commander in November 1771; and commanded the “Adventure” which accompanied Captain Cook (in the “Resolution”) in Cook’s second voyage. On this expedition Furneaux was twice separated from his leader (February 8-May 19, 1773; October 22, 1773-July 14, 1774, the date of his return to England). On the former occasion he explored a great part of the south and east coasts of Tasmania, and made the earliest British chart of the same. Most of his names here survive; Cook, visiting this shore-line on his third voyage, confirmed Furneaux’s account and delineation of it (with certain minor criticisms and emendations), and named after him the islands in Banks Straits, opening into Bass’s Straits, and the group now known as the Low Archipelago. After the “Adventure” was finally separated from the “Resolution” off New Zealand in October 1773, Furneaux returned home alone, bringing with him Omai of Ulaietea. This first South Sea Islander seen in the British Isles returned to his home with Cook in 1776-1777. Furneaux was made a captain in 1775, and commanded the “Syren” in the British attack of the 28th of June 1776 upon Charleston, South Carolina. His successful efforts to introduce domestic animals and potatoes into the South Sea Islands are worthy of note. He died at Swilly on the 19th of September 1781.

See Hawkesworth’s *Narrative of Wallis’ Voyage*; Captain Cook’s *Narrative of his Second Voyage*; also T. Furneaux’s life by Rev. Henry Furneaux in the *Dictionary of National Biography*.

FURNES (Flem. *Veurne*), an old-fashioned little town amid the dunes near the coast in West Flanders, Belgium, about 26 m. S.W. of Bruges. Pop. (1904) 6099. It is the centre of a considerable area extending to the French frontier, and its market is an important one for the disposal of corn, stock, hops and dairy produce. During the Norman raids Furnes was destroyed, and the present town was built by Baldwin Bras de Fer, first count of Flanders, about the year 870. At the height of the prosperity of the Flemish communes in the 14th century there were dependent on the barony of Furnes not fewer than fifty-two rich villages, but these have all disappeared, partly no doubt as the consequence of repeated French invasions down to the end of the 18th century, but chiefly through the encroachment of the sea followed by the accumulation of sand along the whole of this portion of the coast. Furnes contains many curious old houses and the church of St Walburga, which is a fine survival of the 13th century with some older portions. The old church and buildings, grouped round the Grand Place, which is the scene of the weekly market, present a quaint picture which is perhaps not to be equalled in the country. Near Furnes on the seashore is the fashionable bathing place called La Panne.

Furnes one day a year becomes a centre of attraction to all the people of Flanders. This is the last Sunday in July, when the fête of Calvary and the Crucifixion is celebrated. Of all popular festivities in Belgium this is the nearest approach to the old Passion Play. The whole story of Christ is told with great precision by means of succeeding groups which typify the different phases of the subject. The people of Furnes pose as Roman soldiers or Jewish priests, as the apostles or mere spectators, while the women put on long black veils so that they may figure in the procession as the just women.

FURNESS, HORACE HOWARD (1833-   ), American Shakespearian scholar, was born in Philadelphia on the 2nd of November 1833, being the son of William Henry Furness (1802-1896) minister of the First Unitarian church in that city, a powerful preacher and writer. He graduated at Harvard in 1854, and was admitted to the bar in 1859, but soon devoted himself to the study of Shakespeare. He accumulated a collection of illustrative material of great richness and extent, and brought out in 1871 the first volume of a new Variorum edition, designed to represent and summarize the conclusions of the best authorities in all languages—textual, critical and annotative. The volumes appeared as follows: *Romeo and Juliet* (1871); *Macbeth* (1873) (revised edition, 1903); *Hamlet* (2 vols., 1877); *King Lear* (1880); *Othello* (1886); *The Merchant of Venice* (1888); *As You Like It* (1890); *The Tempest* (1892); *A Midsummer Night’s Dream* (1895); *The Winter’s Tale* (1898); *Much Ado about Nothing* (1899); *Twelfth Night* (1901); *Love’s Labour’s Lost* (1904). The edition has been generally accepted as a thorough and scholarly piece of work; its chief fault is that, beginning with *Othello* (1858), the editor used the First Folio text as his basis, while in others he makes the text of the Cambridge (Globe) editors his foundation. His wife, Helen Kate Furness (1837-1883), compiled *A Concordance to the Poems of Shakespeare* (1872).

FURNESS, a district of Lancashire, England, separated from the major portion of the county by Morecambe Bay. It is bounded S.E. by this inlet of the Irish Sea, S.W. by the sea, W. by the Duddon estuary and Cumberland, and N. and E. by Westmorland. Its area is about 250 sq. m. It forms the greater part of the North Lonsdale parliamentary division of Lancashire, and contains the parliamentary borough of Barrow-in-Furness. The surface is almost entirely hilly. The northern half is included in the celebrated Lake District, and contains such eminences as the Old Man of Coniston and Wetherlam. Apart from the Duddon, which forms part of the western boundary, the principal rivers are the Leven and Crake, flowing southward into a common estuary in Morecambe Bay. The Leven drains Windermere and the Crake Coniston Lake. The usage of the term “Lake District,” however, tends to limit the name of Furness in common thought to the district south of the Lakes, where several of the place-names are suffixed with that of the district, as Barrow-in-Farness, Dalton-in-Furness, Broughton-in-Furness. Between the Duddon and Morecambe Bay lies Walney Island, 8 m. in length, and in the shallow strait between it and the mainland are several smaller islands. That part of Furness which forms a peninsula between the Leven estuary and Morecambe Bay, and the Duddon estuary, is rich in hematite iron ore, which has been worked from very early times. It was known and smelted by British and Romans, and by the monks of Furness Abbey and Conishead Priory, both in the district. It was owing to the existence of this ore that the town of Barrow grew up in the 19th century; at first as a port from which the ore was exported to South Wales, while later furnaces were established on the spot, and acquired additional importance on the introduction of the Bessemer process, which requires a non-phosphoric ore such as is found here. The hematite is also worked at Ulverston, Askam, Dalton and elsewhere, but the furnaces now depend in part upon ore imported from Spain. The supposed extension of the ore under the sands of the Duddon estuary led to the construction of a sea wall to facilitate the working. The district is served by the main line of the Furness railway, from Carnforth (junction with the London & North-Western railway), passing the pleasant watering-place of Grange, and approximately following the coast by Ulverston, Dalton and Barrow, with branches to Lake Side, Windermere, and to Coniston.

Furness Abbey.

Apart from its industrial importance and scenic attractions, Furness has an especial interest on account of its famous abbey. The ruins of this, beautifully situated in a wooded valley, are extensive, and mainly of fine transitional Norman and Early English date, acquiring additional picturesqueness from the warm colour of the red sandstone of which they are built. The abbey of Furness, otherwise Furdenesia or the further *nese* (promontory), which was dedicated to St Mary, was founded in 1127 by a small body of monks belonging to the Benedictine order of Savigny. In 1124 they had settled at Tulketh, near Preston, but migrated in 1127 to Furness under the auspices of Stephen, count of Boulogne, afterwards king, at that time lord of the liberty of Furness. In 1148 the brotherhood joined the Cistercian order. Stephen granted to the monks the lordship of Furness, and his charter was confirmed by Henry I., Henry II. and subsequent kings. The abbot’s power throughout the lordship was almost absolute; he had a market and fair at Dalton, was free from service to the county and wapentake, and held a sheriff’s tourn. By a succession of gifts the abbey became one of the richest in England and was the largest Cistercian foundation in the kingdom. At the Dissolution its revenues amounted to between £750 and £800 a year, exclusive of meadows, pastures, fisheries, mines, mills and salt works, and the wealth of the monks enabled them to practise a regal hospitality. The abbot was one of the twenty Cistercian abbots summoned to the parliament of 1264, but was not cited after 1330, as he did not hold of the king *in capite per baroniam*. The abbey founded several offshoot houses, one of the most important being Rushen Abbey in the Isle of Man. In 1535 the royal commissioners visited the abbey and reported four of its inmates, including the abbot, for incontinence. In 1536 the abbot was charged with complicity in the Pilgrimage of Grace, and on the 7th of April 1537, under compulsion, surrendered the abbey to the king. A few monks were granted pensions, and the abbot was endowed with the profits of the rectory of Dalton, valued at £33, 6s. 8d. per annum. In 1540 the estates and revenues were annexed by act of parliament to the Duchy of Lancaster. About James I.’s reign the site and territories were alienated to the Prestons of Preston-Patrick, from whom they descended to the dukes of Devonshire.

Conishead Priory, near Ulverston, an Augustinian foundation of the reign of Henry II., has left no remains, but of the priory of Cartmel (1188) the fine church is still in use. It is a cruciform structure of transitional Norman and later dates, its central tower having the upper storey set diagonally upon the lower. The chancel contains some superb Jacobean carved oak screens, with stalls of earlier date.

FURNISS, HARRY (1854-   ), British caricaturist and illustrator, was born at Wexford, Ireland, of English and Scottish parents. He was educated in Dublin, and in his schooldays edited a *Schoolboy’s Punch* in close imitation of the original. He came to London when he was nineteen, and began to draw for the illustrated papers, being for some years a regular contributor to the *Illustrated London News*. His first drawing in *Punch* appeared in 1880, and he joined its staff in 1884. He illustrated Lucy’s “Diary of Toby, M.P.,” in *Punch*, where his political caricatures became a popular feature. Among his other successes were a series of “Puzzle Heads,” and his annual “Royal Academy guy’d.” In *Royal Academy Antics* (1890) he published a volume of caricatures of the work of leading artists. He resigned from the staff of *Punch* in 1894, produced for a short time a weekly comic paper *Lika Joko*, and in 1898 began a humorous monthly, *Fair Game*; but these were short-lived. Among the numerous books he illustrated were James Payn’s *Talk of the Town*, Lewis Carroll’s *Sylvie and Bruno*, Gilbert à Beckett’s *Comic Blackstone*, G.E. Farrow’s *Wallypug Book*, and his own novel, *Poverty Bay* (1905). *Our Joe, his great Fight* (1903), was a collection of original cartoons. His volume of reminiscences, *Confessions of a Caricaturist* (1901), was followed by *Harry Furniss at Home* (1904). In 1905 he published *How to draw in Pen and Ink*, and produced the first number of *Harry Furniss’s Christmas Annual*.

FURNITURE (from “furnish,” Fr. *fournir*), a general term of obscure origin, used to describe the chattels and fittings required to adapt houses and other buildings for use. Wood, ivory, precious stones, bronze, silver and gold have been used from the most ancient times in the construction or for the decoration of furniture. The kinds of objects required for furniture have varied according to the changes of manners and customs, as well as with reference to the materials at the command of the workman, in different climates and countries.

Of really ancient furniture there are very few surviving examples, partly by reason of the perishable materials of which it was usually constructed; and partly because, however great may have been the splendour of Egypt, however consummate the taste of Greece, however luxurious the life of Rome, the number of household appliances was very limited. The chair, the couch, the table, the bed, were virtually the entire furniture of early peoples, whatever the degree of their civilization, and so they remained until the close of what are known in European history as the middle ages. During the long empire-strewn centuries which intervened between the lapse of Egypt and the obliteration of Babylon, the extinction of Greece and the dismemberment of Rome and the great awakening of the Renaissance, household comfort developed but little. The Ptolemies were as well lodged as the Plantagenets, and peoples who spent their lives in the open air, going to bed in the early hours of darkness, and rising as soon as it was light, needed but little household furniture.

Indoor life and the growth of sedentary habits exercised a powerful influence upon the development of furniture. From being splendid, or at least massive, and exceedingly sparse and costly, it gradually became light, plentiful and cheap. In the ancient civilizations, as in the periods when our own was slowly growing, household plenishings, save in the rudest and most elementary forms, were the privilege of the great—no person of mean degree could have obtained, or would have dared to use if he could, what is now the commonest object in every house, the chair (*q.v.*). Sparse examples of the furniture of Egypt, Nineveh, Greece and Rome are to be found in museums; but our chief sources of information are mural and sepulchral paintings and sculptures. The Egyptians used wooden furniture carved and gilded, covered with splendid textiles, and supported upon the legs of wild animals; they employed chests and coffers as receptacles for clothes, valuables and small objects generally. Wild animals and beasts of the chase were carved upon the furniture of Nineveh also; the lion, the bull and the ram were especially characteristic. The Assyrians were magnificent in their household appointments; their tables and couches were inlaid with ivory and precious metals. Cedar and ebony were much used by these great Eastern peoples, and it is probable that they were familiar with rosewood, walnut and teak. Solomon’s bed was of cedar of Lebanon. Greek furniture was 364
essentially Oriental in form; the more sumptuous varieties were of bronze, damascened with gold and silver. The Romans employed Greek artists and workmen and absorbed or adapted many of their mobiliary fashions, especially in chairs and couches. The Roman tables were of splendid marbles or rare woods. In the later ages of the empire, in Rome and afterwards in Constantinople, gold and silver were plentifully used in furniture; such indeed was the abundance of these precious metals that even cooking utensils and common domestic vessels were made of them.

The architectural features so prominent in much of the medieval furniture begin in these Byzantine and late Roman thrones and other seats. These features became paramount as Pointed architecture became general in Europe, and scarcely less so during the Renaissance. Most of the medieval furniture, chests, seats, trays, &c., of Italian make were richly gilt and painted. In northern Europe carved oak was more generally used. State seats in feudal halls were benches with ends carved in tracery, backs panelled or hung with cloths (called cloths of estate), and canopies projecting above. Bedsteads were square frames, the testers of panelled wood, resting on carved posts. Chests of oak carved with panels of tracery, or of Italian cypress (when they could be imported), were used to hold and to carry clothes, tapestries, &c., to distant castles and manor houses; for house furniture, owing to its scarcity and cost, had to be moved from place to place. Copes and other ecclesiastical vestments were kept in chests with ornamental lock plates and iron hinges. The splendour of most feudal houses depended on pictorial tapestries which could be packed and carried from place to place. Wardrobes were rooms fitted for the reception of dresses, as well as for spices and other valuable stores. Excellent carving in relief was executed on caskets, which were of wood or of ivory, with painting and gilding, and decorated with delicate hinge and lock metal-work. The general subjects of sculpture were taken from legends of the saints or from metrical romances. Renaissance art made a great change in architecture, and this change was exemplified in furniture. Cabinets (*q.v.*) and panelling took the outlines of palaces and temples. In Florence, Rome, Venice, Milan and other capitals of Italy, sumptuous cabinets, tables, chairs, chests, &c., were made to the orders of the native princes. Vasari (*Lives of Painters*) speaks of scientific diagrams and mathematical problems illustrated in costly materials, by the best artists of the day, on furniture made for the Medici family. The great extent of the rule of Charles V. helped to give a uniform training to artists from various countries resorting to Italy, so that cabinets, &c., which were made in vast numbers in Spain, Flanders and Germany, can hardly be distinguished from those executed in Italy. Francis I. and Henry VIII. encouraged the revived arts in their respective dominions. *Pietra dura*, or inlay of hard pebbles, agate, lapis lazuli, and other stones, ivory carved and inlaid, carved and gilt wood, marquetry or veneering with thin woods, tortoise-shell, brass, &c., were used in making sumptuous furniture during the first period of the Renaissance. Subjects of carving or relief were generally drawn from the theological and cardinal virtues, from classical mythology, from the seasons, months, &c. Carved altarpieces and woodwork in churches partook of the change in style.

The great period of furniture in almost every country was, however, unquestionably the 18th century. That century saw many extravagances in this, as in other forms of art, but on the whole it saw the richest *floraison* of taste, and the widest sense of invention. This is the more remarkable since the furniture of the 17th century has often been criticized as heavy and coarse. The criticism is only partly justified. Throughout the first three-quarters of the period between the accession of James I. and that of Queen Anne, massiveness and solidity were the distinguishing characteristics of all work. Towards the reign of James II., however, there came in one of the most pleasing and elegant styles ever known in England. Nearly a generation before then Boulle was developing in France the splendid and palatial method of inlay which, although he did not invent it, is inseparably associated with his name. We owe it perhaps to the fact that France, as the neighbour of Italy, was touched more immediately by the Renaissance than England that the reign of heaviness came earlier to an end in that country than on the other side of the Channel. But there is a heaviness which is pleasing as well as one which is forbidding, and much of the furniture made in England any time after the middle of the 17th century was highly attractive. If English furniture of the Stuart period be not sought after to the same extent as that of a hundred years later, it is yet highly prized and exceedingly decorative. Angularity it often still possessed, but generally speaking its elegance of form and richness of upholstering lent it an attraction which not long before had been entirely lacking. Alike in France and in England, the most attractive achievements of the cabinetmaker belong to the 18th century—English Queen Anne and early Georgian work is universally charming; the regency and the reigns of Louis XV. and XVI. formed a period of the greatest artistic splendour. The inspiration of much of the work of the great English school was derived from France, although the gropings after the Chinese taste and the earlier Gothic manner were mainly indigenous. The French styles of the century, which began with excessive flamboyance, closed before the Revolution with a chaste perfection of detail which is perhaps more delightful than anything that has ever been done in furniture. In the achievements of Riesener, David Röntgen, Gouthière, Oeben and Rousseau de la Rottière we have the high-water mark of craftsmanship. The marquetry of the period, although not always beautiful in itself, was executed with extraordinary smoothness and finish; the mounts of gilded bronze, which were the leading characteristic of most of the work of the century, were finished with a minute delicacy of touch which was until then unknown, and has never been rivalled since. If the periods of Francis I. and Henry II., of Louis XIV. and the regency produced much that was sumptuous and even elegant, that of Louis XVI., while men's minds were as yet undisturbed by violent political convulsions, stands out as, on the whole, the one consummate era in the annals of furniture. Times of great achievement are almost invariably followed directly by those in which no tall thistles grow and in which every little shrub is magnified to the dimensions of a forest tree; and the so-called "empire style" which had begun even while the last monarch of the *ancien régime* still reigned, lacked alike the graceful conception and the superb execution of the preceding style. Heavy and usually uninspired, it was nurtured in tragedy and perished amid disaster. Yet it is a profoundly interesting style, both by reason of the classical roots from which it sprang and the attempt, which it finally reflected, to establish new ideas in every department of life. Founded upon the wreck of a lingering feudalism it reached back to Rome and Greece, and even to Egypt. If it is rarely charming, it is often impressive by its severity. Mahogany, satinwood and other rich timbers were characteristic of the style of the end of the 18th century; rosewood was most commonly employed for the choicer work of the beginning of the 19th. Bronze mounts were in high favour, although their artistic character varied materially.

Previously to the middle of the 18th century the only cabinetmaker who gained sufficient personal distinction to have had his name preserved was André Charles Boulle; beginning with that period France and England produced many men whose renown is hardly less than that of artists in other media. With Chippendale there arose a marvellously brilliant school of English cabinetmakers, in which the most outstanding names are those of Sheraton, Heppelwhite, Shearer and the Adams. But if the school was splendid it was lamentably short-lived, and the 19th century produced no single name in the least worthy to be placed beside these giants. Whether, in an age of machinery, much room is left for fine individual execution may be doubted, and the manufacture of furniture now, to a great extent, takes place in large factories both in England and on the continent. Owing to the necessary subdivision of labour in these establishments, each piece of furniture passes through numerous distinct workshops. The master and a few artificers formerly superintended each piece of work, which, therefore, was never far removed from the 365
designer's eye. Though accomplished artists are retained by the manufacturers of London, Paris and other capitals, there can no longer be the same relation between the designer and his work. Many operations in these modern factories are carried on by machinery. This, though an economy of labour, entails loss of artistic effect. The chisel and the knife are no longer in such cases guided and controlled by the sensitive touch of the human hand.

Plate I.

Fig. 1.—Venetian Folding Chair of carved and gilt walnut, leather back and seat; about 1530.

Fig. 2.—Oak Arm-chair. English, 17th century.

Fig. 3.—Arm-chair, solid seat, cane back; about 1660.

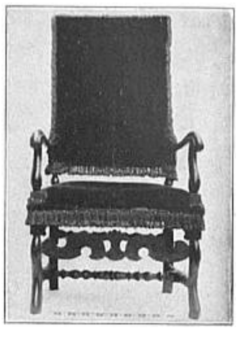


Fig. 4.—Arm-chair, stuffed back and seat; about 1650.

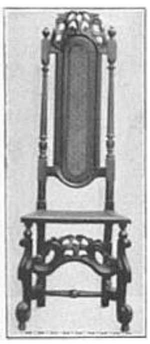


Fig. 5.—Painted and carved High-Back Chair; about 1660.

Fig. 6.—Carved Walnut Chairs. English, early 18th century. The arm-chair is inlaid.

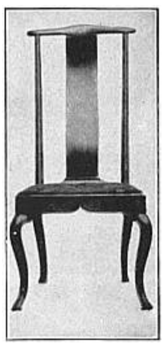


Fig. 7.—Walnut Chair; about 1710.

Fig. 8.—Carved Mahogany Chair in the style of Chippendale; 2nd half of 18th century.

Fig. 9.—Carved Mahogany Arm-chair, in the style of Chippendale, with ribbon pattern.

Fig. 10.—Carved and Inlaid Mahogany Chair, in the style of Hepplewhite; late 18th century.

Fig. 11.—Mahogany Chair in the style of Sheraton; about 1780.

Fig. 12.—Painted and gilt Arm-chair with cane seat, in the style of Adam; about 1790.

Fig. 13.—Arm-chair of carved and gilt wood with stuffed back, seat and arms. French, Louis XV. style.

Fig. 14.—Mahogany Arm-chair. Empire style, early 19th century, said to have belonged to the Bonaparte family.

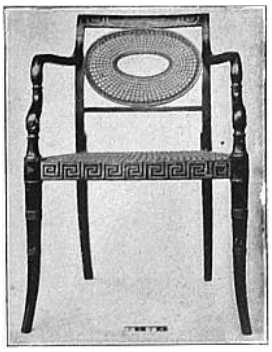


Fig. 15.—Painted and gilt Beech Chair. English, about 1800.

Plate II.

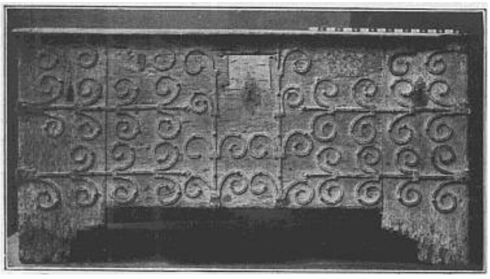


Fig. 1.—Front of Oak Coffer with wrought iron bands. French, 2nd half of 13th century.

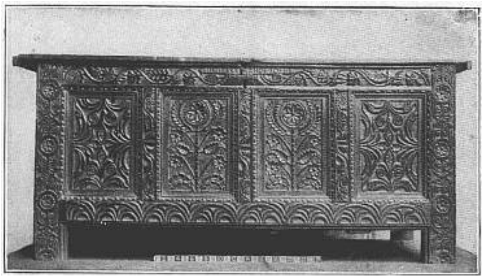


Fig. 2.—English Oak Chest, dated 1637.

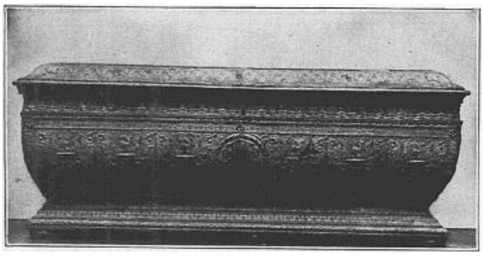


Fig. 3.—Italian (Florentine) Coffer of Wood with gilt arabesque stucco ornament, about 1480.

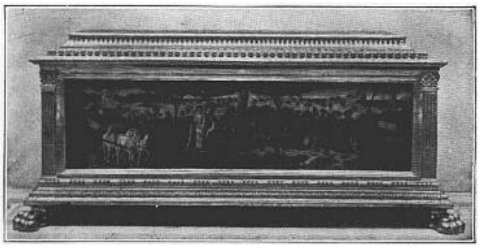


Fig. 4.—Italian “Cassone” or Marriage Coffer, 13th century. Carved and gilt wood with painted front and ends.

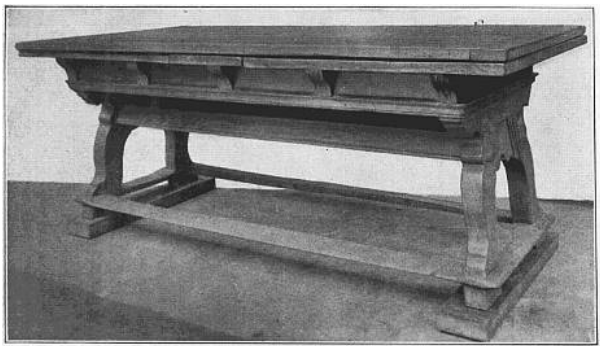


Fig. 5.—Walnut Table with expanding leaves. Swiss, 17th century.

Fig. 6.—Oak Gate-Legged Table. English, 17th century.

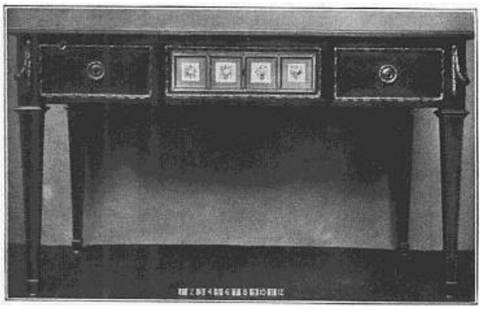


Fig. 7.—Writing Table. French, end of Louis XV. period. Riesener marquetry, ormolu mounts and Sèvres plaques.

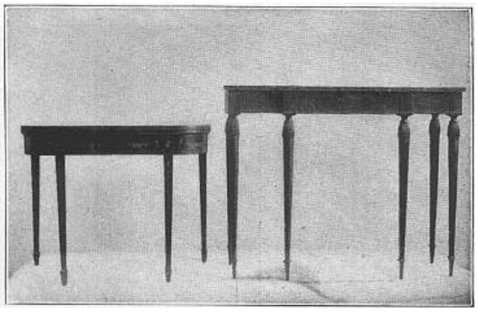


Fig. 8.—Painted Satin-Wood Tables, in the style of Sheraton, about 1790.

(The above are in the Victoria and Albert Museum, except Fig. 8, which were in the Bethnal Green Exhibition, 1892.)

Plate III.

1. CARVED OAK SIDEBOARD. English, 17th century. Victoria and Albert Museum.

2. CARVED OAK COURT CUPBOARD. English, early 17th century. Victoria and Albert Museum.

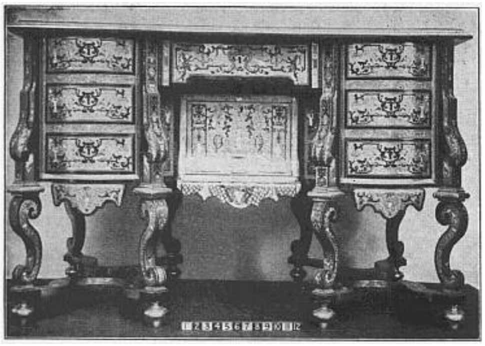
3. EBONY CARVED CABINET. The interior decorated with inlaid ivory and coloured woods; French or Dutch, middle of 17th century. Victoria and Albert Museum.

4. VENEERED CHEST OF DRAWERS. About 1690. Lent to Bethnal Green Exhibition by Sir Spencer Ponsonby-Fane, G.C.B.

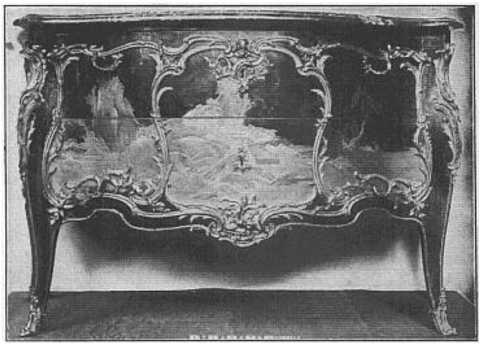
5. EBONY ARMOIRE. With tortoise-shell panels inlaid with brass and other metals, and ormolu mountings. Designed by Bérain, and executed by André Boulle. French, Louis XIV. period. Victoria and Albert Museum.

6. GLASS-FRONTED BOOKCASE AND CABINET. Of mahogany. In the style of Sheraton, about 1790. Lent to the Bethnal Green Exhibition by the late Vincent J. Robinson, C.I.E.

Plate IV.



1. COMMODE OF PINE. With marquetry of brass, ebony, tortoise-shell, mother-of-pearl, ivory, and green-stained bone. “Boulle” work with designs in the style of Bérain. French, late period of Louis XIV.

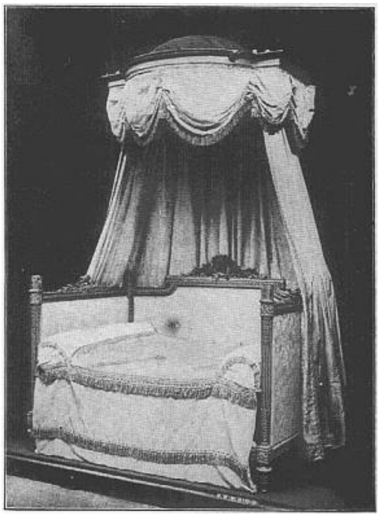


2. COMMODE. With panels of Japanese lacquer and ormolu mountings, in the style of Caffieri. French, Louis XV. period.

3. TABLE OF KING AND TULIP WOODS. With ormolu mountings. Louis XV. period.

4. ESCRITOIRE À TOILETTE. Formerly belonging to Marie Antoinette. Of tulip and sycamore woods inlaid with other coloured woods, ormolu mounts. Louis XV. period.

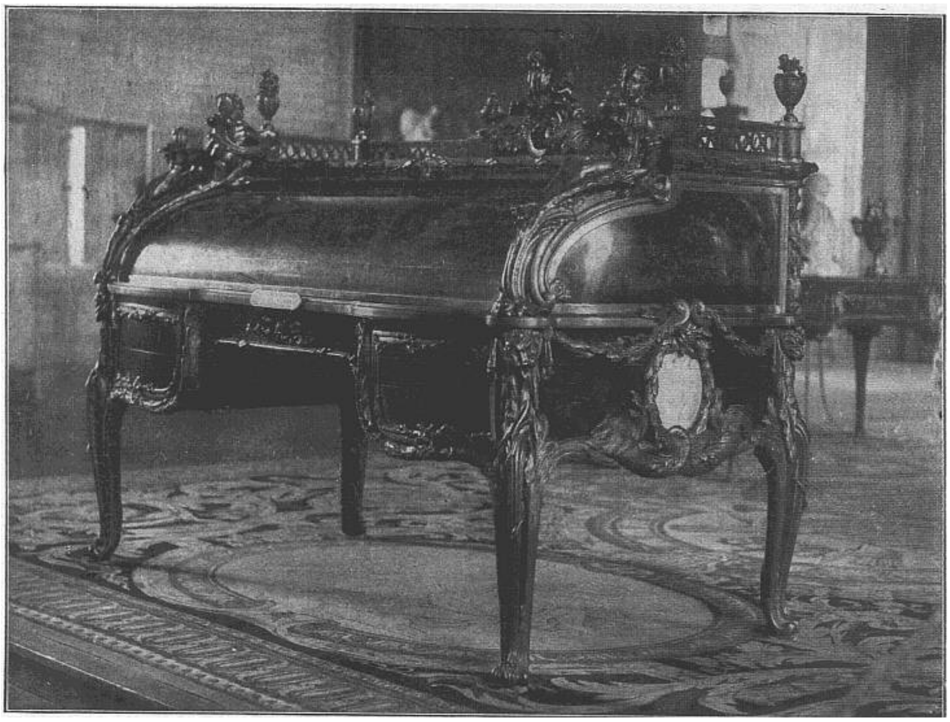
5. FOUR-POST BEDSTEAD. Of oak inlaid with bog-oak and holly, from the “Inlaid Room” at Sizergh Castle, Westmorland. Latter half of sixteenth century.

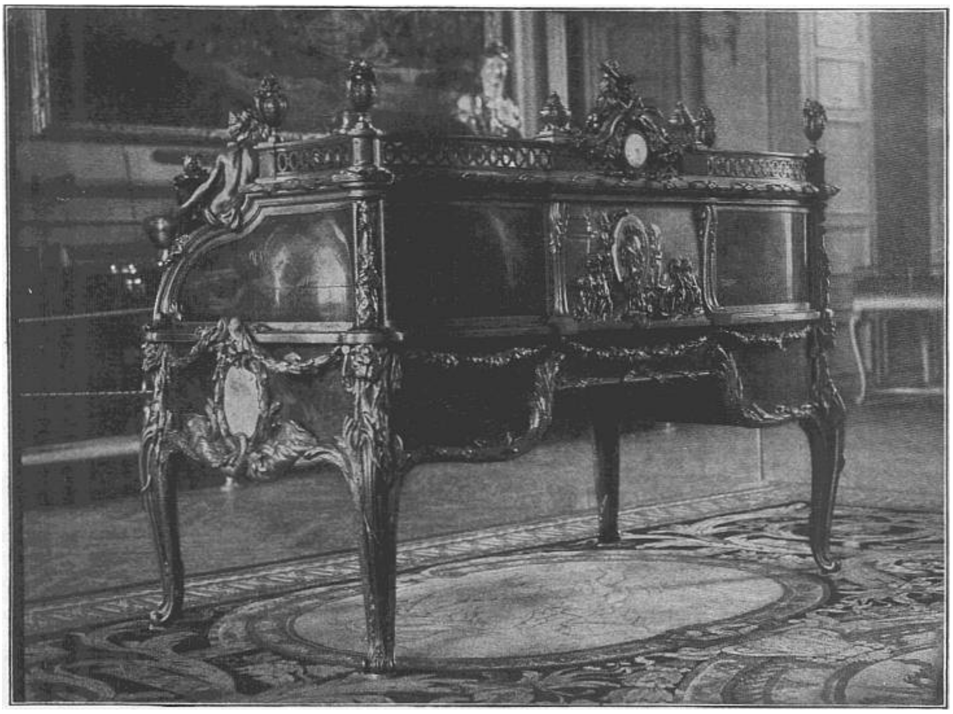


6. CARVED AND GILT BEDSTEAD. With blue silk damask coverings and hangings. French, late 18th century. Louis XVI. period.

From the Victoria and Albert Museum, S. Kensington.

Plate V.





Photo, Mansell & Co.
THE “BUREAU DU ROI,” MADE FOR LOUIS XV., NOW IN THE LOUVRE. For description, see Desk.

A decided, if not always intelligent, effort to devise a new style in furniture began during the last few years of the 19th century, which gained the name of “*l’art nouveau*.” Its pioneers professed to be free from all old traditions and to seek inspiration from nature alone. Happily nature is less forbidding than many of these interpretations of it, and much of the “new art” is a remarkable exemplification of the impossibility of altogether ignoring traditional forms. The style was not long in degenerating into extreme extravagance. Perhaps the most striking consequence of this effort has been, especially in England, the revival of the use of oak. Lightly polished, or waxed, the cheap foreign oaks often produce very agreeable results, especially when there is applied to them a simple inlay of boxwood and stained holly, or a modern form of pewter. The simplicity of these English forms is in remarkable contrast to the tortured and ungainly outlines of continental seekers after a conscious and unpleasing “originality.”

Until a very recent period the most famous collections of historic furniture were to be found in such French museums as the Louvre, Cluny and the Garde Meuble. Now, however, they are rivalled, if not surpassed, by the magnificent collections of the Victoria and Albert Museum at South Kensington, and the Wallace collection at Hertford House, London. The latter, in conjunction with the Jones bequest at South Kensington, forms the finest of all gatherings of French furniture of the great periods, notwithstanding that in the Bureau du Roi the Louvre possesses the most magnificent individual example in existence. In America there are a number of admirable collections representative of the graceful and homely “colonial furniture” made in England and the United States during the Queen Anne and Georgian periods.

See also the separate articles in this work on particular forms of furniture. The literature of the subject has become very extensive, and it is needless to multiply here the references to books. Perrot and Chipiez, in their great *Histoire de l’art dans l’antiquité* (1882 et seq.) deal with ancient times, and A. de Champeaux, in *Le Meuble* (1885), with the middle ages and later period; English furniture is admirably treated by Percy Macquoid in his *History of English Furniture* (1905); and Lady Dilke’s *French Furniture in the 18th Century* (1901), and Luke Vincent Lockwood’s *Colonial Furniture in America* (1901), should also be consulted.

(J. P. B.)

FURNIVALL, FREDERICK JAMES (1825-1910), English philologist and editor, was born at Egham, Surrey, on the 4th of February 1825, the son of a surgeon. He was called to the bar in 1849, but his attention was soon diverted to philological studies and social problems. He gave Frederick Denison Maurice valuable assistance in the Christian Socialist movement, and was one of the founders of the Working Men’s College. For half a century he indefatigably promoted the study of early English literature, partly by his own work as editor, and still more efficaciously by the agency of the numerous learned societies of which he was both founder and director, especially the Early English Text Society (1864), which has been of inestimable service in promoting the study of early and middle English. He also established and conducted the Chaucer, Ballad, New Shakespeare and Wyclif Societies, and at a later period societies for the special study of Browning and Shelley. He edited texts for the Early English Text Society, for the Roxburghe Club and the Rolls Series; but his most important labours were devoted to Chaucer, whose study he as an editor greatly assisted by his “Six-Text” edition of the *Canterbury Tales*, and other publications of the Chaucer Society. He was the honorary secretary of the Philological Society, and was one of the original promoters of the Oxford *New English Dictionary*. He co-operated with its first editor, Herbert Coleridge, and after his death was for some time principal editor during the preliminary period of the collection of material. The completion of his half-century of labour was acknowledged in 1900 by a handsome testimonial, including the preparation by his friends of a volume of philological essays specially dedicated to him, *An English Miscellany* (Oxford, 1901), and a considerable donation to the Early English Text Society. Dr Furnivall was always an enthusiastic oarsman, and till the end kept up his interest in rowing; with John Beesley in 1845 he introduced the new type of narrow sculling boat, and in 1886 started races on the Thames for sculling fours and sculling eights. He died on the 2nd of July 1910.

FURSE, CHARLES WELLINGTON (1868-1904), English painter, born at Staines, the son of the Rev. C.W. Furse, archdeacon of Westminster, was descended collaterally from Sir Joshua Reynolds, and in his short span of life achieved such rare excellence as a portrait and figure painter that he forms an important link in the chain of British portraiture which extends from the time when Van Dyck was called to the court of Charles I. to our own day. His talent was precocious; at the age of seven he gave indications of it in a number of drawings illustrating Scott’s novels. He entered the Slade school in 1884, winning the Slade scholarship in the following year, and completed his education at Julian’s *atelier* in Paris. Hard worker as he was, his activity was frequently interrupted by spells of illness, for he had developed signs of consumption when he was still attending the Slade school. An important canvas called “Cain” was his first contribution (1888) to the Royal Academy, to the associateship of which he was elected in the year of his death. For some years before he had been a staunch supporter of the New English Art Club, to the exhibitions of which he was a regular contributor. He was married in October 1900 to Katherine, daughter of John Addington Symonds. His fondness for sport and of an open-air life found expression in his art and introduced a new, fresh and vigorous note into portraiture. There is never a suggestion of the studio or of the fatiguing pose in his portraits. The sitters appear unconscious of being painted, and are generally seen in the pursuit of their favourite outdoor sport or pastime, in the full enjoyment of life. Such are the “Diana of the Uplands,” the “Lord Roberts” and “The Return from the Ride” at the Tate Gallery; the four children in the “Cubbing with the York and Ainsty,” “The Lilac Gown,” “Mr and Mrs Oliver Fishing” and the portrait of Lord Charles Beresford. Most of these pictures, and indeed nearly all the work completed in the few years of Furse’s activity, show a pronounced decorative tendency. His sense of space, composition and decorative design can best be judged by his admirable mural decorations for Liverpool town hall, executed between 1899 and 1902. A memorial exhibition of Furse’s paintings and sketches was held at the Burlington Fine Arts Club in 1906.

FÜRST, JULIUS (1805-1873), German Orientalist, was born of Jewish parents at Zerkowo in Posen, on the 12th of May 1805. He studied philosophy and philology at Berlin, and oriental literature at Posen, Breslau and Halle. In 1857 he was appointed to a lectureship at the university of Leipzig, and he was promoted to a professorship in 1864, which he held until his death at Leipzig on the 9th of February 1873. Among his writings may be mentioned *Lehrgebäude der aramäischen Idiome* (Leipzig, 1835); *Librorum sacrorum Veteris Testamenti concordantiae Hebraicae atque Chaldaicae* (Leipzig, 1837-1840); *Hebräisches und chaldäisches Wörterbuch* (1851, English translation by S. Davidson 1867); *Kultur und Literaturgeschichte der Juden in Asien* (1849). Fürst also edited a valuable *Bibliotheca Judaica* (Leipzig, 1849-1863), and was the author of some other works of minor importance. From 1840 to 1851 he was editor of *Der Orient*, a journal devoted to the language, literature, history and antiquities of the Jews.

FÜRSTENBERG, the name of two noble houses of Germany.

1. The more important is in possession of a mediatized principality in the district of the Black Forest and the Upper Danube, which comprises the countship of Heiligenberg, about 7 m. to the N. of the Lake of Constance, the landgraviates of Stühlingen and Baar, and the lordships of Jungnau, Trochtelfingen, Hausen 366
and Möskirch or Messkirch. The territory is discontinuous; and as it lies partly in Baden, partly in Württemberg, and partly in the Prussian province of Sigmaringen, the head of the family is an hereditary member of the first chamber of Baden and of the chamber of peers in Württemberg and in Prussia. The relations of the principality with Baden are defined by the treaty of May 1825, and its relations with Württemberg by the royal declaration of 1839. The *Stammort* or ancestral seat of the family is Fürstenberg in the Black Forest, about 13 m. N. of Schaffhausen, but the principal residence of the present representatives of the main line is at Donaueschingen.

The family of Fürstenberg claims descent from a certain Count Unruoch, a contemporary of Charlemagne, but their authentic pedigree is only traceable to Egino II., count of Urach, who died before 1136. In 1218 his successors inherited the possessions of the house of Zähringen in the Baar district of the Black Forest, where they built the town and castle of Fürstenberg. Of the two sons of Egino V. of Urach, Conrad, the elder, inherited the Breisgau and founded the line of the counts of Freiburg, while the younger, Heinrich (1215-1284), received the territories lying in the Kinzigthal and Baar, and from 1250 onward styled himself first lord, then count, of Fürstenberg. His territories were subsequently divided among several branches of his descendants, though temporarily reunited under Count Friedrich III., whose wife, Anna, heiress of the last count of Wardenberg, brought him the countship of Heiligenberg and lordships of Jungnau and Trochtelfingen in 1534. On Friedrich’s death (1559) his territories were divided between his two sons, Joachim and Christof I. Of these the former founded the line of Heiligenberg, the latter that of Kinzigthal. The Kinzigthal branch was again subdivided in the 17th century between the two sons of Christof II. (d. 1614), the elder, Wratislaw II. (d. 1642), founding the line of Mösskirch, the younger, Friedrich Rudolf (d. 1655), that of Stühlingen. The Heiligenberg branch received an accession of dignity by the elevation of Count Hermann Egon (d. 1674) to the rank of prince of the Empire in 1664, but his line became extinct with the death of his son Prince Anton Egon, favourite of King Augustus the Strong and regent of Saxony, in 1716. The heads of both the Mösskirch and Stühlingen lines were now raised to the dignity of princes of the Empire (1716). The Mösskirch branch died out with Prince Karl Friedrich (d. 1744); the territories of the Stühlingen branch had been divided on the death of Count Prosper Ferdinand (1662-1704) between his two sons, Joseph Wilhelm Ernst (1699-1762) and Ludwig August Egon (1705-1759). The first of these was created prince of the Empire on the 10th of December 1716, and founded the princely line of the Swabian Fürstenbergs; in 1772 he obtained from the emperor Francis I. for all his legitimate sons and their descendants the right to bear, instead of the style of landgrave, that of prince, which had so far been confined to the reigning head of the family. Ludwig, on the other hand, founded the family of the landgraves of Fürstenberg, who, since their territories lay in Austria and Moravia, were known as the "cadet line in Austria." The princely line became extinct with the death of Karl Joachim in 1804, and the inheritance passed to the Bohemian branch of the Austrian cadet line in the person of Karl Egon II. (see below). Two years later the principality was mediatized.

In 1909 there were two branches of the princely house of Fürstenberg: (1) the main branch, that of Fürstenberg-Donaueschingen, the head of which was Prince Maximilian Egon (b. 1863), who succeeded his cousin Karl Egon III. in 1896; (2) that of Fürstenberg-Königshof, in Bohemia, the head of which was Prince Emil Egon (b. 1876), chamberlain and secretary of legation to the Austro-Hungarian embassy in London (1907). The cadet line of the landgraves of Fürstenberg is now extinct, its last representative having been the landgrave Joseph Friedrich Ernst of Fürstenberg-Weitra (1860-1896), son of the landgrave Ernst (1816-1889) by a morganatic marriage. He was not recognized as *ebenbürtig* by the family. The landgraves of Fürstenberg were in 1909 represented only by the landgravines Theresa (b. 1839) and Gabrielle (b. 1844), daughters of the landgrave Johann Egon (1802-1879).

From the days of Heinrich of Urach, a relative and notable supporter of Rudolph of Habsburg, the Fürstenbergs have played a stirring part in German history as statesmen, ecclesiastics and notably soldiers. There was a popular saying that "the emperor fights no great battle but a Fürstenberg falls." In the Heiligenberg line the following may be more particularly noticed.

Franz Egon (1625-1682), bishop of Strassburg, was the elder son of Egon VII., count of Fürstenberg (1588-1635), who served with distinction as a Bavarian general in the Thirty Years' War. He began life as a soldier in the imperial service, but on the elevation of his friend Maximilian Henry of Bavaria to the electorate of Cologne in 1650, he went to his court and embraced the ecclesiastical career. He soon gained a complete ascendancy over the weak-minded elector, and, with his brother William Egon (see below), was mainly instrumental in making him the tool of the aggressive policy of Louis XIV. of France. Ecclesiastical preferments were heaped upon him. As a child he had been appointed to a canonry of Cologne; to these he added others at Strassburg, Liége, Hildesheim and Spires; he became also suffragan bishop and dean of Cologne and provost of Hildesheim, and in 1663 bishop of Strassburg. Later he was also prince-abbot of Lüders and Murbach and abbot of Stablo and Malmedy. On the conclusion of a treaty between the emperor and the elector of Cologne, on the 11th of May 1674, Franz was deprived of all his preferments in Germany, and was compelled to take refuge in France. He was, however, amnestied with his brother William by a special article of the treaty of Nijmwegen (1679), whereupon he returned to Cologne. After the French occupation of Strassburg (1681) he took up his residence there and died on the 1st of April 1682.

His brother William Egon (1629-1704), bishop of Strassburg, began his career as a soldier in the French service. He went to the court of the elector of Cologne at the same time as Franz Egon, whose zeal for the cause of Louis XIV. of France he shared. In 1672 the intrigues of the two Fürstenbergs had resulted in a treaty of offensive alliance between the French monarchy and the electorate of Cologne, and, the brothers being regarded by the Imperialists as the main cause of this disaster, William was seized by imperial soldiers in the monastery of St Pantaleon at Cologne, hurried off to Vienna and there tried for his life. He was saved by the intervention of the papal nuncio, but was kept in prison till the signature of the treaty of Nijmwegen (1679). As a reward for his services Louis XIV. appointed him bishop of Strassburg in succession to his brother in 1682, in 1686 obtained for him from Pope Innocent XI. the cardinal's hat, and in 1688 succeeded in obtaining his election as coadjutor-archbishop of Cologne and successor to the elector Maximilian Henry. At the instance of the emperor, however, the pope interposed his veto; the canons followed the papal lead, and, the progress of the Allies against Louis XIV. depriving him of all prospect of success, William Egon retired to France. Here he took up his abode at his abbey of St Germain des Près near Paris, where he died on the 10th of April 1704.

In the Stühlingen line the most notable was Karl Egon (1796-1854), prince of Fürstenberg, the son of Prince Karl Alois of Fürstenberg, a general in the Austrian service, who was killed at the battle of Loptingen on the 25th of March 1799. In 1804 he inherited the Swabian principality of Fürstenberg and all the possessions of the family except the Moravian estates. He studied at Freiburg and Würzburg, and in 1815 accompanied Prince Schwarzenberg to Paris as staff-officer. In 1817 he came of age, and in the following year married the princess Amalie of Baden. By the mediatization of his principality in 1806 the greater part of his vast estates had fallen under the sovereignty of the grand-duke of Baden, and Prince Fürstenberg took a conspicuous part in the upper house of the grand-duchy. In politics he distinguished himself by a liberalism rare in a great German noble, carrying through by his personal influence with his peers the abolition of tithes and feudal dues and stanchly advocating the freedom of the press. He was not less distinguished by his large charities: among other foundations he established a hospital at Donaueschingen. For the industrial development of the country, too, he did much, and proved himself also a notable patron of the arts. His palace of Donaueschingen, with its collections of paintings, engravings and coins, was a centre of culture, where poets, painters and musicians met with princely entertainment. He died on the 14th of September 1869, and was succeeded by his son Karl Egon II. (1820-1892), with the death of whose son, Karl Egon III., in 1896, the title and estates passed to Prince Maximilian Egon, head of the cadet line of Fürstenberg-Pürglitz.

See Münch, *Gesch. des Hauses und des Landes Fürstenberg*, 4 vols. (Aix-la-Chapelle, 1829-1847); S. Riezler, *Gesch. des fürstlichen Hauses Fürstenberg bis 1507* (Tübingen, 1883); *Fürstenbergisches Urkundenbuch*, edited by S. Riezler and F.L. Baumann, vols. i.-vii. (Tübingen, 1877-1891), continued *s. tit. Mitteilungen aus dem fürstlich. Fürstenbergischem Archiv* by Baumann and G. Tumbült, 2 vols. (ib. 1899-1902); Stokvis, *Manuel d'histoire* (Leiden, 1890-1893); *Almanach de Gotha; Allgemeine deutsche Biographie*.

2. The second Fürstenberg family has its possessions in Westphalia and the country of the Rhine, and takes its name from the castle of Fürstenberg on the Ruhr. The two most remarkable men whom it has produced are Franz Friedrich Wilhelm, freiherr von Fürstenberg, and Franz Egon, count von Fürstenberg-Stammheim. The former (1728-1810) became ultimately vicar-general of the prince-bishop of Münster, and effected a great number of important reforms in the administration of the country, besides doing much for its educational and industrial development. The latter (1797-1859) was an enthusiastic patron of art, who zealously advocated the completion of the Cologne cathedral, and erected the beautiful church of St Apollinaris near Remagen on the Rhine. He was a member of the Prussian Upper House in 1849, collaborated in founding the *Preussisches Wochenblatt*, and was an ardent defender of Catholic interests. His son, Count Gisbert von Fürstenberg-Stammheim (b. 1836), was in 1909 head of the Rhenish line of the house of Fürstenberg.

FÜRSTENWALDE, a town of Germany, in the Prussian province of Brandenburg, on the right bank of the Spree, and on the railway from Berlin to Frankfort-on-Oder, 28 m. E. of the former city. Pop. (1905) 20,498. Its beautiful cathedral church contains several old monuments. The industries are important, including, besides brewing and malting, manufactures of starch, vinegar, electric lamps and gas-fittings, stoves, &c., iron-founding and wool-weaving. Fürstenwalde is one of the oldest towns of Brandenburg. From 1385 it was the seat of the bishop of Lebus, whose bishopric was incorporated with the electorate of Brunswick in 1595.

FÜRTH, a manufacturing town of Germany, in the kingdom of Bavaria, at the confluence of the Pegnitz with the Regnitz, 5 m. N.W. from Nuremberg by rail, at the junction of lines to Hof and Würzburg. Pop. (1885) 35,455; (1905) 60,638. It is a modern town in appearance, with broad streets and palatial business houses. Of its four Evangelical churches, the old St Michaeliskirche is a handsome structure; but its chief edifices are the new town hall, with a tower 175 ft. high and the magnificent synagogue. The Jews have also a high school, which enjoys a great reputation. There are besides a classical, a wood-carving and an agricultural school and a library. Fürth is the seat of several important industries; particularly, the production of chromolithographs and picture-books, the manufacture of mirrors and mirror-frames, bronze and gold-leaf wares, pencils, toys, haberdashery, optical instruments, silver work, turnery, chicory, machinery, fancy boxes and cases, and an extensive trade is carried on in these goods as also in hops, metals, wool, groceries and coal. A large annual fair is held at Michaelmas and lasts for eleven days. The earliest railway in Germany was that between Nuremberg and Fürth (opened on the 7th of December 1835).

Fürth was founded, according to tradition, by Charlemagne, who erected a chapel there. It was for a time a *Vogtei* (advocateship) under the burgraves of Nuremberg, but about 1314 it was bequeathed to the see of Bamberg, and in 1806 it came into the possession of Bavaria. In 1632 Gustavus Adolphus besieged it in vain, and in 1634 it was pillaged and burnt by the Croats. It owes its rise to prosperity to the tolerance it meted out to the Jews, who found here an asylum from the oppression under which they suffered in Nuremberg.

See Fronmüller, *Chronik der Stadt Fürth* (1887).

FURTWÄNGLER, ADOLF (1853-1907), German archaeologist, was born at Freiburg im Breisgau, and was educated there, at Leipzig and at Munich, where he was a pupil of H. Brunn, whose comparative method in art-criticism he much developed. He took part in the excavations at Olympia in 1878, became an assistant in the Berlin Museum in 1880, and professor at Berlin (1884) and later at Munich. His latest excavation work was at Aegina. He was a prolific writer, with a prodigious knowledge and memory, and a most ingenious and confident critic; and his work not only dominated the field of archaeological criticism but also raised its standing both at home and abroad. Among his numerous publications the most important were a volume on the bronzes found at Olympia, vast works on ancient gems and Greek vases, and the invaluable *Masterpieces of Greek Sculpture* (English translation by Eugénie Strong). He died at Athens on the 10th of October 1907.

FURZE, Gorse or Whin; botanical name *Ulex* (Ger. *Stechginster*, Fr. *ajonc*), a genus of thorny papilionaceous shrubs, of few species, confined to west and central Europe and north-west Africa. Common furze, *U. europaeus*, is found on heaths and commons in western Europe from Denmark to Italy and Greece, and in the Canaries and Azores, and is abundant in nearly all parts of the British Isles. It grows to a height of 2-6 ft.; it has hairy stems, and the smaller branches end each in a spine; the leaves, sometimes lanceolate on the lowermost branches, are mostly represented by spines from 2 to 6 lines long, and branching at their base; and the flowers, about three-quarters of an inch in length, have a shaggy, yellowish-olive calyx, with two small ovate bracts at its base, and appear in early spring and late autumn. They are yellow and sweet-scented and visited by bees. The pods are few-seeded; their crackling as they burst may often be heard in hot weather. This species comprises the varieties *vulgaris*, or *U. europaeus* proper, which has spreading branches, and strong, many-ridged spines, and *strictus* (Irish furze), with erect branches, and slender 4-edged spines. The other British species of furze is *U. nanus*, dwarf furze, a native of Belgium, Spain and the west of France; it is a procumbent plant, less hairy than *U. europaeus*, with smaller and more orange-coloured flowers, which spring from the primary spines, and have a nearly smooth calyx, with minute basal bracts. Furze, or gorse, is sometimes employed for fences.

Notwithstanding its formidable spines, the young shoots yield a palatable and nutritious winter forage for horses and cattle. To fit it for this purpose it must be chopped and bruised to destroy the spines. This is sometimes done in a primitive and laborious way by laying the gorse upon a block of wood and beating it with a mallet, flat at one end and armed with crossed knife-edges at the other, by the alternate use of which it is bruised and chopped. There are now a variety of machines by which this is done rapidly and efficiently, and which are in use where this kind of forage is used to any extent. The agricultural value of this plant has often been over-rated by theoretical writers. In the case of very poor, dry soils it does, however, yield much valuable food at a season when green forage is not otherwise to be had. It is on this account of importance to dairymen; and to them it has this further recommendation, that cows fed upon it give much rich milk, which is free from any unpleasant flavour. To turn it to good account, it must be sown in drills, kept clean by hoeing, and treated as a regular green crop. If sown in March, on land fitly prepared and afterwards duly cared for, it is ready for use in the autumn of the following year. A succession of cuttings of proper age is obtained for several years from the same field. It is cut by a short stout scythe, and must be brought from the field daily; for when put in a heap after being chopped and bruised it heats rapidly. It is given to horses and cows in 368
combination with chopped hay or straw. An acre will produce about 2000 faggots of green two-year-old gorse, weighing 20 ℔ each.

This plant is invaluable in mountain sheep-walks. The rounded form of the furze bushes that are met with in such situations shows how diligently the annual growth, as far as it is accessible, is nibbled by the sheep. The food and shelter afforded to them in snowstorms by clusters of such bushes is of such importance that the wonder is our sheep farmers do not bestow more pains to have it in adequate quantity. Young plants of whin are so kept down by the sheep that they can seldom attain to a profitable size unless protected by a fence for a few years. In various parts of England it is cut for fuel. The ashes contain a large proportion of alkali, and are a good manure, especially for peaty land.

FUSARO, LAGO, a lake of Campania, Italy, ½ m. W. of Baia, and 1 m. S. of the acropolis of Cumae. It is the ancient *Acherusia palus*, separated from the sea on the W. by a line of sandhills. It may have been the harbour of Cumae in early antiquity. In the 1st century A.D. an artificial outlet was dug for it at its S. end, with a tunnel, lined with *opus reticulatum* and brick, under the hill of Torregaveta. This hill is covered with the remains of a large villa, which is almost certainly that of Servilius Vatia, described by Seneca (*Epist.* 55). There are remains of other villas on the shores of the lake. Oyster cultivation is carried on there.

See J. Beloch, *Campanien* (2nd ed., Breslau, 1890), 188.

(T. As.)

FUSELI, HENRY (1741-1825), English painter and writer on art, of German-Swiss family, was born at Zürich in Switzerland on the 7th of February 1741; he himself asserted in 1745, but this appears to have been a mere whim. He was the second child in a family of eighteen. His father was John Caspar Füssli, of some note as a painter of portraits and landscapes, and author of Lives of the *Helvetic Painters*. This parent destined his son for the church, and with this view sent him to the Caroline college of his native town, where he received an excellent classical education. One of his schoolmates there was Lavater, with whom he formed an intimate friendship.

After taking orders in 1761 Fuseli was obliged to leave his country for a while in consequence of having aided Lavater to expose an unjust magistrate, whose family was still powerful enough to make its vengeance felt. He first travelled through Germany, and then, in 1765, visited England, where he supported himself for some time by miscellaneous writing: there was a sort of project of promoting through his means a regular literary communication between England and Germany. He became in course of time acquainted with Sir Joshua Reynolds, to whom he showed his drawings. By Sir Joshua’s advice he then devoted himself wholly to art. In 1770 he made an art-pilgrimage to Italy, where he remained till 1778, changing his name from Füssli to Fuseli, as more Italian-sounding. Early in 1779 he returned to England, taking Zürich on his way. He found a commission awaiting him from Alderman Boydell, who was then organizing his celebrated Shakespeare gallery. Fuseli painted a number of pieces for this patron, and about this time published an English edition of Lavater’s work on physiognomy. He likewise gave Cowper some valuable assistance in preparing the translation of Homer. In 1788 Fuseli married Miss Sophia Rawlins (who it appears was originally one of his models, and who proved an affectionate wife), and he soon after became an associate of the Royal Academy. Two years later he was promoted to the grade of Academician. In 1799 he exhibited a series of paintings from subjects furnished by the works of Milton, with a view to forming a Milton gallery corresponding to Boydell’s Shakespeare gallery. The number of the Milton paintings was forty-seven, many of them very large; they were executed at intervals within nine years. This exhibition, which closed in 1800, proved a failure as regards profit. In 1799 also he was appointed professor of painting to the Academy. Four years afterwards he was chosen keeper, and resigned his professorship; but he resumed it in 1810, and continued to hold both offices till his death. In 1805 he brought out an edition of Pilkington’s *Lives of the Painters*, which, however, did not add much to his reputation. Canova, when on his visit to England, was much taken with Fuseli’s works, and on returning to Rome in 1817 caused him to be elected a member of the first class in the Academy of St Luke. Fuseli, after a life of uninterrupted good health, died at Putney Hill on the 16th of April 1825, at the advanced age of eighty-four, and was buried in the crypt of St Paul’s cathedral. He was comparatively rich at his death, though his professional gains had always appeared to be meagre.

As a painter, Fuseli had a daring invention, was original, fertile in resource, and ever aspiring after the highest forms of excellence. His mind was capable of grasping and realizing the loftiest conceptions, which, however, he often spoiled on the canvas by exaggerating the due proportions of the parts, and throwing his figures into attitudes of fantastic and over-strained contortion. He delighted to select from the region of the supernatural, and pitched everything upon an ideal scale, believing a certain amount of exaggeration necessary in the higher branches of historical painting. “Damn Nature! she always puts me out,” was his characteristic exclamation. In this theory he was confirmed by the study of Michelangelo’s works and the marble statues of the Monte Cavallo, which, when at Rome, he used often to contemplate in the evening, relieved against a murky sky or illuminated by lightning. But this idea was by him carried out to an excess, not only in the forms, but also in the attitudes of his figures; and the violent and intemperate action which he often displays destroys the grand effect which many of his pieces would otherwise produce. A striking illustration of this occurs in his famous picture of “Hamlet breaking from his Attendants to follow the Ghost”: Hamlet, it has been said, looks as though he would burst his clothes with convulsive cramps in all his muscles. This intemperance is the grand defect of nearly all Fuseli’s compositions. On the other hand, his paintings are never either languid or cold. His figures are full of life and earnestness, and seem to have an object in view which they follow with rigid intensity. Like Rubens he excelled in the art of setting his figures in motion. Though the lofty and terrible was his proper sphere, Fuseli had a fine perception of the ludicrous. The grotesque humour of his fairy scenes, especially those taken from *A Midsummer-Night’s Dream*, is in its way not less remarkable than the poetic power of his more ambitious works. As a colourist Fuseli has but small claims to distinction. He scorned to set a palette as most artists do; he merely dashed his tints recklessly over it. Not unfrequently he used his paints in the form of a dry powder, which he rubbed up with his pencil with oil, or turpentine, or gold size, regardless of the quantity, and depending for accident on the general effect. This recklessness may perhaps be explained by the fact that he did not paint in oil till he was twenty-five years of age. Despite these drawbacks he possessed the elements of a great painter.

Fuseli painted more than 200 pictures, but he exhibited only a minority of them. His earliest painting represented “Joseph interpreting the Dreams of the Baker and Butler”; the first to excite particular attention was the “Nightmare,” exhibited in 1782. He produced only two portraits. His sketches or designs numbered about 800; they have admirable qualities of invention and design, and are frequently superior to his paintings.

His general powers of mind were large. He was a thorough master of French, Italian, English and German, and could write in all these tongues with equal facility and vigour, though he preferred German as the vehicle of his thoughts. His writings contain passages of the best art-criticism that English literature can show. The principal work is his series of *Lectures* in the Royal Academy, twelve in number, commenced in 1801.

Many interesting anecdotes of Fuseli, and his relations to contemporary artists, are given in his *Life* by John Knowles, who also edited his works in 3 vols. 8vo, London, 1831.

(W. M. R.)

FUSEL OIL (from the Ger. *Fusel*, bad spirits), the name applied to the volatile oily liquids, of a nauseous fiery taste and smell, which are obtained in the rectification of spirituous liquors made by the fermentation of grain, potatoes, the marc of grapes, and other material, and which, as they are of higher boiling point than 369
ethyl alcohol, occur in largest quantity in the last portions of the distillate. Besides ethyl or ordinary alcohol, and amyl alcohol, which are present in them all, there have been found in fusel oil several other bodies of the $C_nH_{2n+1}\cdot OH$ series, also certain ethers, and members of the $C_nH_{2n+1}\cdot CO_2H$ series of fatty acids. Normal propyl alcohol is contained in the fusel oil of the marc brandy of the south of France, and isoprimary butyl alcohol in that of beet-root molasses. The chief constituent of the fusel oil procured in the manufacture of alcohol from potatoes and grain, usually known as fusel oil and potato-spirit, is isoprimary amyl alcohol, or isobutylcarbinol. Ordinary fusel oil yields also an isomeric amyl alcohol (active amyl alcohol) boiling at about 128°. Variable quantities of fusel oil, less or greater according to the stage of ripening, exist in commercial spirits (see Spirits).

Fusel oil and its chief constituent, amyl alcohol, are direct nerve poisons. In small doses it causes only thirst and headache, with furred tongue and some excitement. In large doses it is a convulsent poison. Impure beverages induce all the graver neurotic and visceral disorders in alcoholism; and, like fusel oil, furfurol and the essence of absinthe, are convulsent poisons. Pure ethyl alcohol intoxication, indeed, is rarely seen, being modified in the case of spirits by the higher alcohols contained in fusel oil. According to Rabuteau the toxic properties of the higher alcohols increase with their molecular weight and boiling point. Richet considers that the fusel oil contained in spirits constitutes the chief danger in the consumption of alcoholic beverages. The expert can immediately detect the peculiarly virulent characters of the mixed intoxication due to the consumption of spirits containing a large percentage of fusel oil.

FUSIBLE METAL, a term applied to certain alloys, generally composed of bismuth, lead and tin, which possess the property of melting at comparatively low temperatures. Newton's fusible metal (named after Sir Isaac Newton) contains 50 parts of bismuth, 31.25 of lead and 18.75 of tin; that of Jean Darcet (1725-1801), 50 parts of bismuth with 25 each of lead and tin; and that of Valentin Rose the elder, 50 of bismuth with 28.1 of lead and 24.1 of tin. These melt between 91° and 95° C. The addition of cadmium gives still greater fusibility; in Wood's metal, for instance, which is Darcet's metal with half the tin replaced by cadmium, the melting point is lowered to 66°-71° C.; while another described by Lipowitz and containing 15 parts of bismuth, 8 of lead, 4 of tin and 3 of cadmium, softens at about 55° and is completely liquid a little above 60°. By the addition of mercury to Darcet's metal the melting point may be reduced so low as 45°. These fusible metals have the peculiarity of expanding as they cool; Rose's metal, for instance, remains pasty for a considerable range of temperature below its fusing point, contracts somewhat rapidly from 80° to 55°, expands from 55° to 35°, and contracts again from 35° to 0°. For this reason they may be used for taking casts of anatomical specimens or making *clichés* from wood-blocks, the expansion on cooling securing sharp impressions. By suitable modification in the proportions of the components, a series of alloys can be made which melt at various temperatures above the boiling point of water; for example, with 8 parts of bismuth, 8 of lead and 3 of tin the melting point is 123°, and with 8 of bismuth, 30 of lead and 24 of tin it is 172°. With tin and lead only in equal proportions it is 241°. Such alloys are used for making the fusible plugs inserted in the furnace-crowns of steam boilers, as a safeguard in the event of the water-level being allowed to fall too low. When this happens the plug being no longer covered with water is heated to such a temperature that it melts and allows the contents of the boiler to escape into the furnace. In automatic fire-sprinklers the orifices of the pipes are closed with fusible metal, which melts and liberates the water when, owing to an outbreak of fire in the room, the temperature rises above a predetermined limit.

FUSILIER, originally (in French about 1670, in English about 1680) the name of a soldier armed with a light flintlock musket called the fusil; now a regimental designation. Various forms of flintlock small arms had been used in warfare since the middle of the 16th century. At the time of the English civil war (1642-1652) the term "firelock" was usually employed to distinguish these weapons from the more common matchlock musket. The special value of the firelock in armies of the 17th century lay in the fact that the artillery of the time used open powder barrels for the service of the guns, making it unsafe to allow lighted matches in the muskets of the escort. Further, a military escort was required, not only for the protection, but also for the surveillance of the artillerymen of those days. Companies of "firelocks" were therefore organized for these duties, and out of these companies grew the "fusiliers" who were employed in the same way in the wars of Louis XIV. In the latter part of the Thirty Years' War (1643) fusiliers were simply mounted troops armed with the fusil, as carabiniers were with the carbine. But the escort companies of artillery came to be known by the name shortly afterwards, and the regiment of French Royal Fusiliers, organized in 1671 by Vauban, was considered the model for Europe. The general adoption of the flintlock musket and the suppression of the pike in the armies of Europe put an end to the original special duties of fusiliers, and they were subsequently employed to a large extent in light infantry work, perhaps on account of the greater individual aptitude for detached duties naturally shown by soldiers who had never been restricted to a fixed and unchangeable place in the line of battle. The senior fusilier regiment in the British service, the (7th) Royal Fusiliers (City of London Regiment), was formed on the French model in 1685; the 5th foot (now Northumberland Fusiliers), senior to the 7th in the army, was not at that time a fusilier regiment. The distinctive head-dress of fusiliers in the British service is a fur cap, generally resembling, but smaller than and different in details from, that of the Foot Guards.

In Germany the name "fusilier" is borne by certain infantry regiments and by one battalion in each grenadier regiment.

FUSION, the term generally applied to the melting of a solid substance, or the change of state of aggregation from the solid to the liquid. The term "liquefaction" is frequently employed in the same sense, but is often restricted to the condensation of a gas or vapour. The converse process of freezing or solidification, the change from the liquid to the solid state, is subject to the same laws, and must be considered together with fusion. The solution of a solid in a foreign liquid, and the deposition or crystallization of a solid from a solution, are so closely related to the fusion of a pure substance, that it will also be necessary to consider some of the analogies which they present.

1. *General Phenomena.*—There are two chief varieties of the process of fusion, namely, crystalline and amorphous, which are in many ways distinct, although it is possible to find intermediate cases which partake of the characteristics of both. The melting of ice may be taken as a typical case of crystalline fusion. The

passage from rigid solid to mobile liquid occurs at a definite surface without any intermediate stage or plastic condition. The change takes place at a definite temperature, the fusing or freezing point (abbreviated F.P.), and requires the addition of a definite quantity of heat to the solid, which is called the latent heat of fusion. There is also in general a considerable change of volume during fusion, which amounts in the case of ice to a contraction of 9%. Typical cases of amorphous solidification are those of silica, glass, plastic sulphur, pitch, alcohol and many organic liquids. In this type the liquid gradually becomes more and more viscous as the temperature falls, and ultimately attains the rigidity characteristic of a solid, without any definite freezing point or latent heat. The condition of the substance remains uniform throughout, if its temperature is uniform; there is no separation into the two distinct phases of solid and liquid, and there is no sudden change of volume at any temperature.

A change or transition from one crystalline form to another may occur in the solid state with evolution or absorption of heat at a definite temperature, and is analogous to the change from solid to liquid, but usually takes place more slowly owing to the small molecular mobility of the solid state. Thus rhombic sulphur when heated passes slowly at 95.6° C. into the monosymmetric form which melts at 120°, but if heated rapidly the 370
rhombic form melts at 114.5. The two forms, rhombic and monosymmetric, can exist in equilibrium at 95.6°, the transition point at which they have the same vapour pressure. Similarly a solid solution of carbon in iron, when cooled slowly, passes at about 700° C., with considerable evolution of heat, into the form of “pearlite,” which is soft when cold, but if rapidly chilled the carbon remains in solution and the steel is very hard (see also Alloys).

In the case of crystalline fusion it is necessary to distinguish two cases, the homogeneous and the heterogeneous. In the first case the composition of the solid and liquid phases are the same, and the temperature remains constant during the whole process of fusion. In the second case the solid and liquid phases differ in composition; that of the liquid phase changes continuously, and the temperature does not remain constant during the fusion. The first case comprises the fusion of pure substances, and that of eutectics, or cryohydrates; the second is the general case of an alloy or a solution. These have been very fully studied and their phenomena greatly elucidated in recent years.

There is also a sub-variety of amorphous fusion, which may be styled colloid or gelatinous, and may be illustrated by the behaviour of solutions of water in gelatin. Many of these jellies melt at a fairly definite temperature on heating, and coagulate or set at a definite temperature on cooling. But in some cases the process is not reversible, and there is generally marked hysteresis, the temperature of setting and other phenomena depending on the rate of cooling. This case has not yet been fully worked out; but it appears probable that in many cases the jelly possesses a spongy framework of solid, holding liquid in its meshes or interstices. It might be regarded as a case of “heterogeneous” amorphous fusion, in which the liquid separates into two phases of different composition, one of which solidifies before the other. The two phases cannot, as a rule, be distinguished optically, but it is generally possible to squeeze out some of the liquid phase when the jelly has set, which proves that the substance is not really homogeneous. In very complicated mixtures, such as acid lavas or slags containing a large proportion of silica, amorphous and crystalline solidification may occur together. In this case the crystals separate first during the process of cooling, the mother liquor increases gradually in viscosity, and finally sets as an amorphous ground-mass or matrix, in which crystals of different kinds and sizes, formed at different stages of the cooling, remain embedded. The formation of crystals in an amorphous solid after it has set is also of frequent occurrence. It is termed devitrification, but is a very slow process unless the solid is in a plastic state.

2. *Homogeneous Crystalline Fusion.*—The fusion of a solid of this type is characterized most clearly by the perfect constancy of temperature during the process. In fact, the law of constant temperature, which is generally stated as the first of the so-called “laws of fusion,” does not strictly apply except to this case. The constancy of the F.P. of a pure substance is so characteristic that change of the F.P. is often one of the most convenient tests of the presence of foreign material. In the case of substances like ice, which melt at a low temperature and are easily obtained in large quantities in a state of purity, the point of fusion may be very accurately determined by observing the temperature of an intimate mixture of the solid and liquid while slowly melting as it absorbs heat from surrounding bodies. But in the majority of cases it is more convenient to observe the freezing point as the liquid is cooled. By this method it is possible to ensure perfect uniformity of temperature throughout the mass by stirring the liquid continuously during the process of freezing, whereas it is difficult to ensure uniformity of temperature in melting a solid, however gradually the heat is supplied, unless the solid can be mixed with the liquid. It is also possible to observe the F.P. in other ways, as by noting the temperature at the moment of the breaking of a wire, of the stoppage of a stirrer, or of the maximum rate of change of volume, but these methods are generally less certain in their indications than the point of greatest constancy of temperature in the case of homogeneous crystalline solids.

Fusing Points of Common Metals

Mercury	−38.8°	Antimony	630°
Potassium	62.5°	Aluminium	655°
Sodium	95.6°	Silver	962°
Tin	231.9°	Gold	1064°
Bismuth	269.2°	Copper	1082°
Cadmium	320.7°	Nickel	1427°
Lead	327.7°	Palladium	1535°
Zinc	419.0°	Platinum	1710°

The above table contains some of the most recent values of fusing points of metals determined (except the first three and the last three) with platinum thermometers. The last three values are those obtained by extrapolation with platinum-rhodium and platinum-iridium couples. (See Harker, *Proc. Roy. Soc.* A 76, p. 235, 1905.) Some doubt has recently been raised with regard to the value for platinum, which is much lower than that previously accepted, namely 1775°.

3. *Superfusion, Supersaturation.*—It is generally possible to cool a liquid several degrees below its normal freezing point without a separation of crystals, especially if it is protected from agitation, which would assist the molecules to rearrange themselves. A liquid in this state is said to be "undercooled" or "superfused." The phenomenon is even more familiar in the case of solutions (*e.g.* sodium sulphate or acetate) which may remain in the "metastable" condition for an indefinite time if protected from dust, &c. The introduction into the liquid under this condition of the smallest fragment of the crystal, with respect to which the solution is supersaturated, will produce immediate crystallization, which will continue until the temperature is raised to the saturation point by the liberation of the latent heat of fusion. The constancy of temperature at the normal freezing point is due to the equilibrium of exchange existing between the liquid and solid. Unless both solid and liquid are present, there is no condition of equilibrium, and the temperature is indeterminate.

It has been shown by H.A. Miers (*Jour. Chem. Soc.*, 1906, 89, p. 413) that for a supersaturated solution in metastable equilibrium there is an inferior limit of temperature, at which it passes into the "labile" state, *i.e.* spontaneous crystallization occurs throughout the mass in a fine shower. This seems to be analogous to the fine misty condensation which occurs in a supersaturated vapour in the absence of nuclei (see Vaporization) when the supersaturation exceeds a certain limit.

4. *Effect of Pressure on the F.P.*—The effect of pressure on the fusing-point depends on the change of volume during fusion. Substances which expand on *freezing*, like ice, have their freezing points lowered by increase of pressure; substances which expand on *fusing*, like wax, have their melting points raised by pressure. In each case the effect of pressure is to retard increase of volume. This effect was first predicted by James Thomson on the analogy of the effect of pressure on the boiling point, and was numerically verified by Lord Kelvin in the case of ice, and later by Bunsen in the case of paraffin and spermaceti. The equation by which the change of the F.P. is calculated may be proved by a simple application of the Carnot cycle, exactly as in the case of vapour and liquid. (See Thermodynamics.) If L be the latent heat of fusion in mechanical units, v' the volume of unit mass of the solid, and v'' that of the liquid, the work done in an elementary Carnot cycle of range $d\theta$ will be $dp(v'' - v')$, if dp is the increase of pressure required to produce a change $d\theta$ in the F.P. Since the ratio of the work-difference or cycle-area to the heat-transferred L must be equal to $d\theta/\theta$, we have the relation

$$d\theta/dp = \theta\,(v'' - v')/L. \qquad (1)$$

The sign of $d\theta$, the change of the F.P., is the same as that of the change of volume $(v'' - v')$. Since the change of volume seldom exceeds 0.1 c.c. per gramme, the change of the F.P. per atmosphere is so small that it is not as a rule necessary to take account of variations of atmospheric pressure in observing a freezing point. A variation of 1 cm. in the height of the barometer would correspond to a change of .0001° C. only in the F.P. of ice. This is far beyond the limits of accuracy of most observations. Although the effect of pressure is so small, it produces, as is well known, remarkable results in the motion of glaciers, the moulding and regelation of ice, and many other phenomena. It has also been employed to explain the apparent inversion of the order of crystallization in rocks like granite, in which the arrangement of the crystals indicates that the quartz matrix solidified subsequently to the crystals of felspar, mica or hornblende embedded in it, although the quartz has 371
a higher melting point. It is contended that under enormous pressure the freezing points of the more fusible constituents might be raised above that of the quartz, if the latter is less affected by pressure. Thus Bunsen found the F.P. of paraffin wax 1.4° C. below that of spermaceti at atmospheric pressure. At 100 atmospheres the two melted at the same temperature. At higher pressures the paraffin would solidify first. The effect of pressure on the silicates, however, is much smaller, and it is not so easy to explain a change of several hundred degrees in the F.P. It seems more likely in this particular case that the order of crystallization depends on the action of superheated water or steam at high temperatures and pressures, which is well known to exert a highly solvent and metamorphic action on silicates.

5. *Variation of Latent Heat.*—C.C. Person in 1847 endeavoured to show by the application of the first law of thermodynamics that the increase of the latent heat per degree should be equal to the difference $(s'' - s')$ between the specific heats of the liquid and solid. If, for instance, water at 0° C. were first frozen and then cooled to $-t$° C., the heat abstracted per gramme would be $(L' + s't)$ calories. But if the water were first cooled to $-t$° C., and then frozen at $-t$°C., by abstracting heat L'', the heat abstracted would be $L'' + s''t$. Assuming that the heat abstracted should be the same in the two cases, we evidently obtain $L' - L'' = (s'' - s')t$. This theory has been approximately verified by Petterson, by observing the freezing of a liquid cooled below its normal F.P. (*Jour. Chem. Soc.* 24, p. 151). But his method does not represent the true variation of the latent heat with temperature, since the freezing, in the case of a superfused liquid, really takes place at the normal freezing point. A quantity of heat $s''t$ is abstracted in cooling to $-t$, $(L'' - s''t)$ in raising to 0° and freezing at 0°, and $s't$ in cooling the ice to $-t$. The latent heat L'' at $-t$ does not really enter into the experiment. In order to make the liquid freeze at a different temperature, it is necessary to subject it to pressure, and the effect of the pressure on the latent heat cannot be neglected. The entropy of a liquid φ'' at its F.P. reckoned from any convenient zero φ_0 in the solid state may be represented by the expression

$$\varphi'' - \varphi_0 = \int s'd\theta/\theta + L/\theta. \qquad (2)$$

Since $\theta d\varphi''/d\theta = s''$, we obtain by differentiation the relation

$$dL/d\theta = s'' - s' + L/\theta, \qquad (3)$$

which is exactly similar to the equation for the specific heat of a vapour maintained in the saturated condition. If we suppose that the specific heats s' and s'' of the solid and liquid at equilibrium pressure are nearly the same as those ordinarily observed at constant pressure, the relation (3) differs from that of Person only by the addition of the term L/θ. Since s'' is greater than s' in all cases hitherto investigated, and L/θ is necessarily positive, it is clear that the latent heat of fusion must increase with rise of temperature, or diminish with fall of temperature. It is possible to imagine the F.P. so lowered by pressure (positive or negative) that the latent heat should vanish, in which case we should probably obtain a continuous passage from the liquid to the solid state similar to that which occurs in the case of amorphous substances. According to equation (3), the rate of change of the latent heat of water is approximately 0.80 calorie per degree at 0° C. (as compared with 0.50, Person), if we assume s″ = 1, and s′ = 0.5. Putting (s″ − s′) = 0.5 in equation (2), we find L = 0 at −160° C. approximately, but no stress can be laid on this estimate, as the variation of (s″ − s′) is so uncertain.

6. *Freezing of Solutions and Alloys.*—The phenomena of freezing of heterogeneous crystalline mixtures may be illustrated by the case of aqueous solutions and of metallic solutions or alloys, which have been most widely studied. The usual effect of an impurity, such as salt or sugar in solution in water, is to lower the freezing point, so that no crystallization occurs until the temperature has fallen below the normal F.P. of the pure solvent, the depression of F.P. being nearly proportional to the concentration of the solution. When freezing begins, the solvent generally separates out from the solution in the pure state. This separation of the solvent involves an increase in the strength of the remaining solution, so that the temperature does not remain constant during the freezing, but continues to fall as more of the solvent is separated. There is a perfectly definite relation between temperature and concentration at each stage of the process, which may be represented in the form of a curve as AC in fig. 1, called the freezing point curve. The equilibrium temperature, at the surface of contact between the solid and liquid, depends only on the composition of the liquid phase and not at all on the quantity of solid present. The abscissa of the F.P. curve represents the composition of that portion of the original solution which remains liquid at any temperature. If instead of starting with a dilute solution we start with a strong solution represented by a point N, and cool it as shown by the vertical line ND, a point D is generally reached at which the solution becomes "saturated." The dissolved substance or "solute" then separates out as the solution is further cooled, and the concentration diminishes with fall of temperature in a definite relation, as indicated by the curve CB, which is called the solubility curve. Though often called by different names, the two curves AC and CB are essentially of a similar nature. To take the case of an aqueous solution of salt as an example, along CB the solution is saturated with respect to salt, along AC the solution is saturated with respect to ice. When the point C is reached along either curve, the solution is saturated with respect to both salt and ice. The concentration cannot vary further, and the temperature remains constant, while the salt and ice crystallize out together, maintaining the exact proportions in which they exist in the solution. The resulting solid was termed a cryohydrate by F. Guthrie, but it is really an intimate mixture of two kinds of crystals, and not a chemical compound or hydrate containing the constituents in chemically equivalent proportions. The lowest temperature attainable by means of a freezing mixture is the temperature of the F.P. of the corresponding cryohydrate. In a mixture of salt and ice with the least trace of water a saturated brine is quickly formed, which dissolves the ice and falls rapidly in temperature, owing to the absorption of the latent heat of fusion. So long as both ice and salt are present, if the mixture is well stirred, the solution must necessarily become saturated with respect to both ice and salt, and this can only occur at the cryohydric temperature, at which the two curves of solubility intersect.

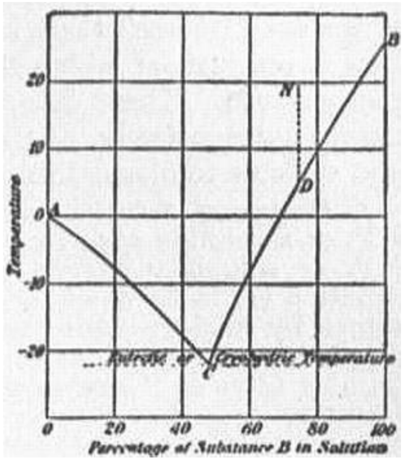


Fig. 1.—F.P. or Solubility Curve: simple case.

The curves in fig. 1 also illustrate the simplest type of freezing point curve in the case of alloys of two metals A and B which do not form mixed crystals or chemical compounds. The alloy corresponding to the cryohydrate, possessing the lowest melting point, is called the eutectic alloy, as it is most easily cast and worked. It generally possesses a very fine-grained structure, and is not a chemical compound. (See Alloys.)

To obtain a complete F.P. curve even for a binary alloy is a laborious and complicated process, but the information contained in such a curve is often very valuable. It is necessary to operate with a number of different alloys of suitably chosen composition, and to observe the freezing points of each separately. Each alloy should also be analysed after the process if there is any risk of its composition having been altered by oxidation or otherwise. The freezing points are generally best determined by observing the gradual cooling of a considerable mass, which is well stirred so long as it remains liquid. The curve of cooling may most conveniently be recorded, either photographically, using a thermocouple and galvanometer, as in the method of Sir W. Roberts-Austen, or with pen and ink, if a platinum thermometer is available, according to the method put in practice by C.T. Heycock and F.H. Neville. A typical set of curves obtained in this manner is shown in fig. 2. When the pure metal A in cooling reaches its F.P. the temperature suddenly becomes stationary, and remains accurately constant for a considerable period. Often it falls slightly below the F.P. owing to super-fusion, but rises to the F.P. and remains constant as soon as freezing begins. The second curve shows the cooling of A with 10% of another metal B added. The freezing begins at a lower temperature with the separation of pure A. The temperature no longer remains constant during freezing, but 372 falls more and more rapidly as the proportion of B in the liquid increases. When the eutectic temperature is reached there is a second F.P. or arrest at which the whole of the remaining liquid solidifies. With 20% of B the first F.P. is further lowered, and the temperature falls faster. The eutectic F.P. is of longer duration, but still at the same temperature. For an alloy of the composition of the eutectic itself there is no arrest until the eutectic temperature is reached, at which the whole solidifies without change of temperature. There is a great advantage in recording these curves automatically, as the primary arrest is often very slight, and difficult to observe in any other way.

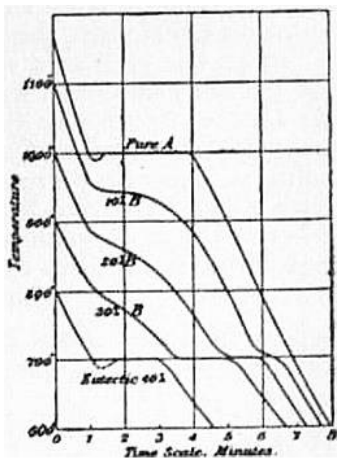


Fig. 2.—Cooling Curves of Alloys: typical case.

7. *Change of Solubility with Temperature.*—The lowering of the F.P. of a solution with increase of concentration, as shown by the F.P. or solubility curves, may be explained and calculated by equation (1) in terms of the osmotic pressure of the dissolved substance by analogy with the effect of mechanical pressure. It is possible in salt solutions to strain out the salt mechanically by a suitable filter or "semi-permeable membrane," which permits the water to pass, but retains the salt. To separate 1 gramme of salt requires the performance of work PV against the osmotic pressure P, where V is the corresponding diminution in the volume of the solution. In dilute solutions, to which alone the following calculation can be applied, the volume V is the reciprocal of the concentration C of the solution in grammes per unit volume, and the osmotic pressure P is equal to that of an equal number of molecules of gas in the same space, and may be deduced from the usual equation of a gas,

$$P = R\theta / VM = R\theta C / M, \tag{4}$$

where M is the molecular weight of the salt in solution, θ the absolute temperature, and R a constant which has the value 8.32 joules, or nearly 2 calories, per degree C. It is necessary to consider two cases, corresponding to the curves CB and AB in fig. 1, in which the solution is saturated with respect to salt and water respectively. To facilitate description we take the case of a salt dissolved in water, but similar results apply to solutions in other liquids and alloys of metals.

(*a*) If unit mass of salt is separated in the solid state from a saturated solution of salt (curve CB) by forcing out through a semi-permeable membrane against the osmotic pressure P the corresponding volume of water V in which it is dissolved, the heat evolved is the latent heat of saturated solution of the salt Q together with the work done PV. Writing (Q + PV) for L, and V for (v″ − v′) in equation (1), and substituting P for p, we obtain

$$Q + PV = V\theta dP / d\theta, \tag{5}$$

which is equivalent to equation (1), and may be established by similar reasoning. Substituting for P and V in terms of C from equation (4), if Q is measured in calories, R = 2, and we obtain

$$QC = 2\theta^2 dC / d\theta, \tag{6}$$

which may be integrated, assuming Q constant, with the result

$$2\log_e C'' / C' = Q / \theta' - Q / \theta'', \tag{7}$$

where C′, C″ are the concentrations of the saturated solution corresponding to the temperatures θ′ and θ″. This equation may be employed to calculate the latent heat of solution Q from two observations of the solubility. It follows from these equations that Q is of the same sign as dC/dθ, that is to say, the solubility increases with rise of temperature if heat is absorbed in the formation of the saturated solution, which is the usual case. If, on the other hand, heat is liberated on solution, as in the case of caustic potash or sulphate of calcium, the solubility diminishes with rise of temperature.

(*b*) In the case of a solution saturated with respect to ice (curve AC), if one gramme of water having a volume v is separated by freezing, we obtain a precisely similar equation to (5), but with L the latent heat of fusion of water instead of Q, and v instead of V. If the solution is dilute, we may neglect the external work Pv in comparison with L, and also the heat of dilution, and may write P/t for dP/dθ, where t is the depression of the F.P. below that of the pure solvent. Substituting for P in terms of V from equation (4), we obtain

$$t = 2\theta^2 v / LVM = 2\theta^2 w / LWM, \tag{8}$$

where W is the weight of water and w that of salt in a given volume of solution. If M grammes of salt are dissolved in 100 of water, w = M and W = 100. The depression of the F.P. in this case is called by van 't Hoff the "Molecular Depression of the F.P." and is given by the simple formula

$$t = .02\theta^2 / L. \tag{9}$$

Equation (8) may be used to calculate L or M, if either is known, from observations of t, θ and w/W. The results obtained are sufficiently approximate to be of use in many cases in spite of the rather liberal assumptions and approximations effected in the course of the reasoning. In any case the equations give a simple theoretical basis with which to compare experimental data in order to estimate the order of error involved in the assumptions. We may thus estimate the variation of the osmotic pressure from the value given by the gaseous equation, as the concentration of the solution or the molecular dissociation changes. The most uncertain factor in the formula is the molecular weight M, since the molecule in solution may be quite different from that denoted by the chemical formula of the solid. In many cases the molecule of a metal in dilute solution in another metal is either monatomic, or forms a compound molecule with the solvent containing one atom of the dissolved metal, in which case the molecular depression is given by putting the atomic weight for M. In other cases, as Cu, Hg, Zn, in solution in cadmium, the depression of the F.P. per atom, according to Heycock and Neville, is only half as great, which would imply a diatomic molecule. Similarly As and Au in Cd appear to be triatomic, and Sn in Pb tetratomic. Intermediate cases may occur in which different molecules exist together in equilibrium in proportions which vary according to the temperature and concentration. The most familiar case is that of an electrolyte, in which the molecule of the dissolved substance is partly dissociated into ions. In such cases the degree of dissociation may be estimated by observing the depression of the F.P., but the results obtained cannot always be reconciled with those deduced by other methods, such as measurement of electrical conductivity, and there are many difficulties which await satisfactory interpretation.

Exactly similar relations to (8) and (9) apply to changes of boiling point or vapour pressure produced by substances in solution (see Vaporization), the laws of which are very closely connected with the corresponding phenomena of fusion; but the consideration of the vapour phase may generally be omitted in dealing with the fusion of mixtures where the vapour pressure of either constituent is small.

8. *Hydrates.*—The simple case of a freezing point curve, illustrated in fig. 1, is generally modified by the

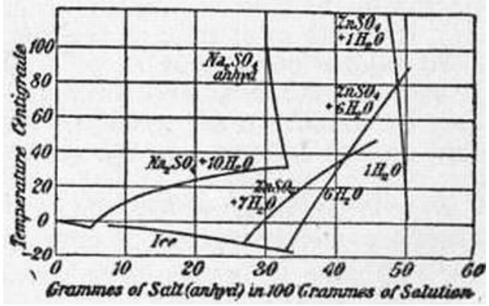


FIG. 3.—Solubility Curves of Hydrates.

occurrence of compounds of a character analogous to hydrates of soluble salts, in which the dissolved substance combines with one or more molecules of the solvent. These hydrates may exist as compound molecules in the solution, but their composition cannot be demonstrated unless they can be separated in the solid state. Corresponding to each crystalline hydrate there is generally a separate branch of the solubility curve along which the crystals of the hydrate are in equilibrium with the saturated solution. At any given temperature the hydrate possessing the least solubility is the most stable. If two are present in contact with the same solution, the more soluble will dissolve, and the less soluble will be formed at its expense until the conversion is complete. The two hydrates cannot be in equilibrium with the same solution except at the temperature at which their solubilities are equal, *i.e.* at the point where the corresponding curves of solubility intersect. This temperature is called the “Transition Point.” In the case of $ZnSO_4$, as shown in fig. 3, the heptahydrate, with seven molecules of water, is the least soluble hydrate at ordinary temperatures, and is generally deposited from saturated solutions. Above 39° C., however, the hexahydrate, with six molecules, is less soluble, and a rapid conversion of the hepta- into the hexahydrate occurs if the former is heated above the transition point. The solubility of the hexahydrate is greater than that of the heptahydrate below 39°, but increases more slowly with rise of temperature. At about 80° C. the hexahydrate gives place to the monohydrate, which dissolves in water with evolution of heat, and diminishes in solubility with rise of temperature. Intermediate hydrates exist, but they are more soluble, and cannot be readily isolated. Both the mono- and hexahydrates are capable of existing in equilibrium with saturated solutions at temperatures far below their transition points, provided that the less soluble hydrate is not present in the crystalline form. The solubility curves can therefore be traced, as in fig. 3, over an extended range of temperature. The equilibrium of each hydrate with the solvent, considered separately, would present a diagram of two branches similar to fig. 1, but as a rule only a small portion of each curve can be realized, and the complete solubility curve, as experimentally determined, is composed of a number of separate pieces corresponding to the ranges of minimum solubility of different hydrates. Failure to recognize this, coupled with the fact that in strong and viscous solutions the state of equilibrium is but slowly attained, 373
is the probable explanation of the remarkable discrepancies existing in many recorded data of solubility.

Transition Points of Hydrates.

$Na_2CrO_4 \cdot 10H_2O$	19.9°	$NaBr \cdot 2H_20$	50.7°
$Na_2SO_4 \cdot 10H_2O$	32.4°	$MnCl_2 \cdot 4H_2O$	57.8°
$Na_2CO_3 \cdot 10H_2O$	35.1°	$Na_3PO_4 \cdot 12H_2O$	73.4°
$Na_2S_2O_3 \cdot 5H_2O$	48.0°	$Ba(OH)_2 \cdot 8H_2O$	77.9°

The transition points of the hydrates given in the above list (Richards, *Proc. Amer. Acad.*, 1899, 34, p. 277) afford well-marked constant temperatures which can be utilized as fixed points for experimental purposes.

9. *Formation of Mixed Crystals.*—An important exception to the general type already described, in which the addition of a dissolved substance lowers the F.P. of the solvent, is presented by the formation of mixed crystals, or “solid solutions,” in which the solvent and solute occur mixed in varying proportions. This isomorphous replacement of one substance by another, in the same crystal with little or no change of form, has long been known and studied in the case of minerals and salts, but the relations between composition and melting-point have seldom been investigated, and much still remains obscure. In this case the process of freezing does not necessitate the performance of work of separation of the constituents of the solution, the F.P. is not necessarily depressed, and the effect cannot be calculated by the usual formula for dilute solutions. One of the simplest types of F.P. curve which may result from the occurrence of mixed crystals is illustrated by the case of alloys of gold and silver, or gold and platinum, in which the F.P. curve is nearly a straight line joining the freezing-points of the constituents. The equilibrium between the solid and liquid, in both of which the two metals are capable of mixing in all proportions, bears in this case an obvious and close analogy to the equilibrium between a mixed liquid (*e.g.* alcohol and water) and its vapour. In the latter case, as is well known, the vapour will contain a larger proportion of the more volatile constituent. Similarly in the case of the formation of mixed crystals, the liquid should contain a larger proportion of the more fusible constituent than the solid with which it is in equilibrium. The composition of the crystals which are being deposited at any moment will, therefore, necessarily change as solidification proceeds, following the change in the composition of the liquid, and the temperature will fall until the last portions of the liquid to solidify will consist chiefly of the more fusible constituent, at the F.P. of which the solidification will be complete. If, however, as seems to be frequently the case, the composition of the solid and liquid phases do not greatly differ from each other, the greater part of the solidification will occur within a comparatively small range of temperature, and the initial F.P. of the alloy will be well marked. It is possible in this case to draw a second curve representing the composition of the *solid* phase which is in equilibrium with the liquid at any temperature. This curve will not represent the average composition of the crystals, but that of the outer coating only which is in equilibrium with the liquid at the moment. H.W.B. Roozeboom (*Zeit. Phys. Chem.* xxx. p. 385) has attempted to classify some of the possible cases which may occur in the formation of mixed crystals on the basis of J.W. Gibbs’s thermodynamic potential, the general properties of which may be qualitatively deduced from a consideration of observed phenomena. But although this method may enable us to classify different types, and even to predict results in a qualitative manner, it does not admit of numerical calculation similar to equation (8), as the Gibbs’s function itself is of a purely abstract nature and its form is unknown. There is no doubt that the formation of mixed crystals may explain many apparent anomalies in the study of F.P. curves. The whole subject has been most fruitful of results in recent years, and appears full of promise for the future.

For further details in this particular branch the reader may consult a report by Neville (*Brit. Assoc. Rep.*, 1900), which contains numerous references to original papers by Roberts-Austen, Le Chatelier, Roozeboom and others. For the properties of solutions see Solution.

(H. L. C.)

FÜSSEN, a town of Germany, in the kingdom of Bavaria, at the foot of the Alps (Tirol), on the Lech, 2500 ft. above the sea, with a branch line to Oberdorf on the railway to Augsburg. Pop. 4000. It has six Roman Catholic churches, a Franciscan monastery and a castle. Rope-making is an important industry. The castle, lying on a rocky eminence, is remarkable for the peace signed here on the 22nd of April 1745 between the elector Maximilian III., Joseph of Bavaria and Maria Theresa. Two miles to the S.E., immediately on the Austrian frontier, romantically situated on a rock overlooking the Schwanensee, is the magnificent castle of Hohenschwangau, and a little to the north, on the site of an old castle, that of Neuschwanstein, built by Louis II. of Bavaria.

See H. Feistle, *Füssen und Umgebung* (1898).

FUST, JOHANN (?-1466), early German printer, belonged to a rich and respectable burgher family of Mainz, which is known to have flourished from 1423, and to have held many civil and religious offices. The name was always written Fust, but in 1506 Johann Schöffer, in dedicating the German translation of Livy to the emperor Maximilian, called his grandfather Faust, and thenceforward the family assumed this name, and the Fausts of Aschaffenburg, an old and quite distinct family, placed Johann Fust in their pedigree. Johann's brother Jacob, a goldsmith, was one of the burgomasters in 1462, when Mainz was stormed and sacked by the troops of Count Adolf of Nassau, on which occasion he seems to have perished (see a document, dated May 8, 1463, published by Wyss in *Quartalbl. des hist. Vereins für Hessen*, 1879, p. 24). There is no evidence that, as is commonly asserted, Johann Fust was a goldsmith, but he appears to have been a money-lender or banker. On account of his connexion with Gutenberg (*q.v.*), he has been represented by some as the inventor of printing, and the instructor as well as the partner of Gutenberg, by others as his patron and benefactor, who saw the value of his discovery and supplied him with means to carry it out, whereas others paint him as a greedy and crafty speculator, who took advantage of Gutenberg's necessity and robbed him of the fruits of his invention. However this may be, the Helmasperger document of November 6, 1455, shows that Fust advanced money to Gutenberg (apparently 800 guilders in 1450, and another 800 in 1452) for carrying on his work, and that Fust, in 1455, brought a suit against Gutenberg to recover the money he had lent, claiming 2020 (more correctly 2026) guilders for principal and interest. It appears that he had not paid in the 300 guilders a year which he had undertaken to furnish for expenses, wages, &c., and, according to Gutenberg, had said that he had no intention of claiming interest. The suit was apparently decided in Fust's favour, November 6, 1455, in the refectory of the Barefooted Friars of Mainz, when Fust made oath that he himself had borrowed 1550 guilders and given them to Gutenberg. There is no evidence that Fust, as is usually supposed, removed the portion of the printing materials covered by his mortgage to his own house, and carried on printing there with the aid of Peter Schöffer, of Gernsheim (who is known to have been a scriptor at Paris in 1449), to whom, probably about 1455,[1] he gave his only daughter Dyna or Christina in marriage. Their first publication was the Psalter, August 14, 1457, a folio of 350 pages, the first printed book with a complete date, and remarkable for the beauty of the large initials printed each in two colours, red and blue, from types made in two pieces.[2] The Psalter was reprinted with the same types, 1459 (August 29), 1490, 1502 (Schöffer's last publication) and 1516. Fust and Schöffer's other works are given below.[3] In 1464 Adolf of Nassau appointed 374
for the parish of St Quintin three *Baumeisters* (master-builders) who were to choose twelve chief parishioners as assistants for life. One of the first of these "Vervaren," who were named on May-day 1464, was Johannes Fust, and in 1467 Adam von Hochheim was chosen instead of "the late" (*selig*) Johannes Fust. Fust is said to have gone to Paris in 1466 and to have died of the plague, which raged there in August and September. He certainly was in Paris on the 4th of July, when he gave Louis de Lavernade of the province of Forez, then chancellor of the duke of Bourbon and first president of the parliament of Toulouse, a copy of his second edition of Cicero, as appears from a note in Lavernade's own hand at the end of the book, which is now in the library of Geneva. But nothing further is known than that on the 30th of October, probably in 1471, an annual mass was instituted for him by Peter Schöffer, Conrad Henlif (for Henekes, or Henckis, Schöffer's partner? who married Fust's widow about 1468[4]) and Johann Fust (the son), in the abbey-church of St Victor of Paris, where he was buried; and that Peter Schöffer founded a similar memorial service for Fust in 1473 in the church of the Dominicans at Mainz (Bockenheimer, *Gesch. der Stadt Mainz*, iv. 15).

Fust was formerly often confused with the famous magician Dr Johann Faust, who, though an historical figure, had nothing to do with him (see Faust).

See further the articles Gutenberg and Typography.

(J. H. H.)

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1 This date is uncertain; some place the marriage in 1453 or soon after, others about 1464. It is probable that Fust alluded to this relationship when he spoke of Schöffer as *pueri mei* in the colophons of Cicero's *De officiis* of 1465 and 1466.

2 This method was patented in England by Solomon Henry in 1780, and by Sir William Congreve in 1819.

3 (3) Durandus, *Rationale divinorum officiorum* (1459), folio, 160 leaves; (4) the *Clementine Constitutions*, with the gloss of Johannes Andreae (1460), 51 leaves; (5) *Biblia Sacra Latina* (1462), folio, 2 vols., 242 and 239 leaves, 48 lines to a full page; (6) the Sixth Book of Decretals, with Andreae's gloss, 17th December 1465, folio, 141 leaves; (7) Cicero, *De officiis* (1465). 4to, 88 leaves, the first edition of a Latin classic and the first book containing Greek characters, while in the colophon Fust for the first time calls Schöffer "puerum suum"; (8) the same, 4th February 1466; (9) *Grammatica rhytmica* (1466), folio, 11 leaves. They also printed in 1461-1462 several papal bulls, proclamations of Adolf of Nassau, &c. Nothing is known to have appeared for three years after the storming and capture of Mainz in 1462.

4 Some confusion in the history of the Fust family has arisen since the publication of Bernard's *Orig. de l'imprimerie* (1853). On p. 262, vol. i. he gave an extract from the correspondence between Oberlin and Bodmann (now preserved in the Paris Nat. Library), from which it would appear that Peter Schöffer was the son-in-law, not of Johann Fust, but of a brother of his, Conrad Fust. Of the latter, however, no other trace has been found, and he is no doubt a fiction of F.J. Bodmann, who, partly basing himself on the "Conrad" (Henlif, or Henckis) mentioned above, added the rest to gratify Oberlin (see Wyss in *Quartalblätter des hist. Vereins für Hessen*, 1879, p. 17).

FUSTEL DE COULANGES, NUMA DENIS (1830-1889), French historian, was born in Paris on the 18th of March 1830, of Breton descent. After studying at the École Normale Supérieure he was sent to the French school at Athens in 1853, directed some excavations in Chios, and wrote an historical account of the island. After his return he filled various educational offices, and took his doctor's degree with two theses, *Quid Vestae cultus in institutis veterum privatis publicisque valuerit* and *Polybe, ou la Grèce conquise par les Romains* (1858). In these works his distinctive qualities were already revealed. His minute knowledge of the language of the Greek and Roman institutions, coupled with his low estimate of the conclusions of contemporary scholars, led him to go direct to the original texts, which he read without political or religious bias. When, however, he had succeeded in extracting from the sources a general idea that seemed to him clear and simple, he attached himself to it as if to the truth itself, employing dialectic of the most penetrating, subtle and even paradoxical character in his deduction of the logical consequences. From 1860 to 1870 he was professor of history at the faculty of letters at Strassburg, where he had a brilliant career as a teacher, but never yielded to the influence exercised by the German universities in the field of classical and Germanic antiquities.

It was at Strassburg that he published his remarkable volume *La Cité antique* (1864), in which he showed forcibly the part played by religion in the political and social evolution of Greece and Rome. Although his making religion the sole factor of this evolution was a perversion of the historical facts, the book was so consistent throughout, so full of ingenious ideas, and written in so striking a style, that it ranks as one of the masterpieces of the French language in the 19th century. By this literary merit Fustel set little store, but he clung tenaciously to his theories. When he revised the book in 1875, his modifications were very slight, and it is conceivable that, had he recast it, as he often expressed the desire to do in the last years of his life, he would not have abandoned any part of his fundamental thesis. The work is now largely superseded.

Fustel de Coulanges was the most conscientious of men, the most systematic and uncompromising of historians. Appointed to a lectureship at the École Normale Supérieure in February 1870, to a professorship at the Paris faculty of letters in 1875, and to the chair of medieval history created for him at the Sorbonne in 1878, he applied himself to the study of the political institutions of ancient France. The invasion of France by the German armies during the war of 1870-71 attracted his attention to the Germanic invasions under the Roman Empire. Pursuing the theory of J.B. Dubos, but singularly transforming it, he maintained that those invasions were not marked by the violent and destructive character usually attributed to them; that the penetration of the German barbarians into Gaul was a slow process; that the Germans submitted to the imperial administration; that the political institutions of the Merovingians had their origins in the Roman laws at least as much as, if not more than, in German usages; and, consequently, that there was no conquest of Gaul by the Germans. This thesis he sustained brilliantly in his *Histoire des institutions politiques de l'ancienne France*, the first volume of which appeared in 1874. It was the author's original intention to complete this work in four volumes, but as the first volume was keenly attacked in Germany as well as in France, Fustel was forced in self-defence to recast the book entirely. With admirable conscientiousness he re-examined all the texts and wrote a number of dissertations, of which, though several (*e.g.* those on the Germanic mark and on the *allodium* and *beneficium*) were models of learning and sagacity, all were dominated by his general idea and characterized by a total disregard for the results of such historical disciplines as diplomatic. From this crucible issued an entirely new work, less well arranged than the original, but richer in facts and critical comments. The first volume was expanded into three volumes, *La Gaule romaine* (1891), *L'Invasion germanique et la fin de l'empire* (1891) and *La Monarchie franque* (1888), followed by three other volumes, *L'Alleu et le domaine rural pendant l'époque mérovingienne* (1889), *Les Origines du système féodal: le bénéfice et le patronat ...* (1890) and *Les Transformations de la royauté pendant l'époque carolingienne* (1892). Thus, in six volumes, he had carried the work no farther than the Carolingian period. The result of this enormous labour, albeit worthy of a great historian, clearly showed that the author lacked all sense of historical proportion. He was a diligent seeker after the truth, and was perfectly sincere when he informed a critic of the exact number of "truths" he had discovered, and when he remarked to one of his pupils a few days before his death, "Rest assured that what I have written in my book is the truth." Such superb self-confidence can accomplish much, and it undoubtedly helped to form Fustel's talent and to give to his style that admirable concision which subjugates even when it fails to convince; but a student instinctively distrusts an historian who settles the most controverted problems with such impassioned assurance. The dissertations not embodied in his great work were collected by himself and (after his death) by his pupil, Camille Jullian, and published as volumes of miscellanies: *Recherches sur quelques problèmes d'histoire* (1885), dealing with the Roman colonate, the land system in Normandy, the Germanic mark, and the judiciary organization in the kingdom of the Franks; *Nouvelles recherches sur quelques problèmes* *d’histoire* (1891); and *Questions historiques* (1893), which contains his paper on Chios and his thesis on Polybius.

His life was devoted almost entirely to his teaching and his books. In 1875 he was elected member of the Académie des Sciences Morales, and in 1880 reluctantly accepted the post of director of the École Normale. Without intervening personally in French politics, he took a keen interest in the questions of administration and social reorganization arising from the fall of the imperialist régime and the disasters of the war. He wished the institutions of the present to approximate more closely to those of the past, and devised for the 375
new French constitution a body of reforms which reflected the opinions he had formed upon the democracy at Rome and in ancient France. But these were dreams which did not hold him long, and he would have been scandalized had he known that his name was subsequently used as the emblem of a political and religious party. He died at Massy (Seine-et-Oise) on the 12th of September 1889. Throughout his historical career—at the École Normale and the Sorbonne and in his lectures delivered to the empress Eugénie—his sole aim was to ascertain the truth, and in the defence of truth his polemics against what he imagined to be the blindness and insincerity of his critics sometimes assumed a character of harshness and injustice. But, in France at least, these critics were the first to render justice to his learning, his talents and his disinterestedness.

See Paul Guiraud, *Fustel de Coulanges* (1896); H. d’Arbois de Jubainville, *Deux Manières d’écrire l’histoire: critique de Bossuet, d’Augustin Thierry et de Fustel de Coulanges* (1896); and Gabriel Monod, *Portraits et souvenirs* (1897).

(C. B.*)

FUSTIAN, a term which includes a variety of heavy woven cotton fabrics, chiefly prepared for men’s wear. It embraces plain twilled cloth called jean, and cut fabrics similar to velvet, known as velveteen, moleskin, corduroy, &c. The term was once applied to a coarse cloth made of cotton and flax; now, fustians are usually of cotton and dyed various colours. In the reign of Edward III. the name was given to a woollen fabric. The name is said to be derived from El-Fustat, a suburb of Cairo, where it was first made; and certainly a kind of cloth has long been known under that name. In a petition to parliament, *temp.* Philip and Mary, “fustian of Naples” is mentioned. In the 13th and 14th centuries priests’ robes and women’s dresses were made of fustian, but though dresses are still made from some kinds the chief use is for labourers’ clothes.

FUSTIC (Fr. *fustoc*, from Arab. *fustuq*, Gr. πιστάκη, pistachio) Yellow Wood or Old Fustic, a dye-stuff consisting of the wood of *Chlorophora tinctoria*, a large tree of the natural order Moraceae, growing in the West Indies and tropical America. Fustic occurs in commerce in blocks, which are brown without, and of a brownish-yellow within. It is sometimes employed for inlaid work. The dye-stuff termed young fustic or Zante fustic, and also Venetian sumach, is the wood of *Rhus cotinus* (fustet, or smoke tree), a southern European and Asiatic shrub of the natural order Anacardiaceae, called by Gerarde “red sumach,” and apparently the “coccygia” and “cotinus” of Pliny (*Nat. Hist.* xiii. 41, xvi. 30). Its colouring matter is fisetin, $C_{15}H_{10}O_6$, which was synthesized by S. von Kostanecki (*Ber.*, 1904, 37, p. 384). (See Dyeing.)

FUTURES, a term used in the produce markets for purchases or sales of commodities to be completed at a future date, as opposed to cash or “spot” transactions, which are settled immediately. See Market, and (for a detailed discussion of the question as affecting cotton) Cotton: *Marketing and Supply*.

FUX, JOHANN JOSEPH (1660-1741), Austrian musician, was born at Hirtenfeld (Styria) in 1660. Of his youth and early training nothing is known. In 1696 he was organist at one of the principal churches of Vienna, and in 1698 was appointed by the emperor Leopold I. as his “imperial court-composer,” with a salary of about £6 a month. At the court of Leopold and of his successors Joseph I. and Charles VI., Fux remained for the rest of his life. To his various court dignities that of organist at St Stephen’s cathedral was added in 1704. He married the daughter of the government secretary Schnitzbaum. As a proof of the high favour in which he was held by the art-loving Charles VI., it is told that at the coronation of that emperor as king of Bohemia in 1723 an opera, *La Constanza e la Fortezza*, especially composed by Fux for the occasion, was given at Prague in an open-air theatre. Fux at the time was suffering from gout, but the emperor had him carried in a litter all the way from Vienna, and gave him a seat in the imperial box. Fux died at Vienna on the 13th of February 1741. His life, although passed in the great world, was eventless, and his only troubles arose from the intrigues of his Italian rivals at court. Of the numerous operas which Fux wrote it is unnecessary to speak. They do not essentially differ from the style of the Italian *opera seria* of the time. Of greater importance are his sacred compositions, psalms, motets, oratorios and masses, the celebrated *Missa Canonica* amongst the latter. It is an all but unparalleled *tour de force* of learned musicianship, being written entirely in that most difficult of contrapuntal devices—the canon. As a contrapuntist and musical scholar generally, Fux was unsurpassed by any of his contemporaries, and his great theoretical work, the *Gradus ad Parnassum*, long remained by far the most thorough treatment of counterpoint and its various developments. The title of the original Latin edition is *Gradus ad Parnassum sive manuductio ad compositionem musicae regularem, methoda nova ac certa nondum ante tam exacta ordine in lucem edita, elaborata a Joanne Josepho Fux* (Vienna, 1715). It was translated into most European languages during the 18th century, and is still studied by musicians interested in the history of their art. The expenses of the publication were defrayed by the emperor Charles VI.

Fux's biography was published by Ludwig von Köchel (Vienna, 1871). It is based on minute original research and contains, amongst other valuable materials, a complete catalogue of the composer's numerous works.

FUZE or Fuse, an appliance for firing explosives in blasting operations, military shells, &c. (see Blasting and Ammunition, § *Shell*). The spelling is not governed by authority, but modern convenience has dictated the adoption of the "z" by military engineers as a general rule, in order to distinguish this sense from that of melting by heat (see below). The word, according to the *New English Dictionary*, is one of the forms in which the Lat. *fusus*, spindle, has been adapted through Romanic into English, the ordinary fuze taking the shape of a spindle-like tube. Similarly the term "fusee" (Fr. *fusée*, spindle full of tow, Late Lat. *fusata*) is applied to a coned spindle sometimes used in the wheel train of watches and spring clocks to equalize the action of the mainspring (see Watch); and the application of the same term to a special kind of match may also be due to its resemblance to a spindle. Again, in heraldry, another form, "fusil," derived through the French from a Late Lat. diminutive (*fusillus* or *fusellus*) of this same *fusus*, is used of a bearing, an elongated lozenge. According to other etymological authorities, however (see Skeat, *Etym. Dict.*, 1898), "fuze" or "fuse," and "fusee" in the sense of match, are all forms derived through the Fr. fusil, from Late Lat. *focile*, steel for striking fire from a flint, from Lat. *focus*, hearth. The Fr. *fusil* and English "fusil" were thus transferred to the "firelock," *i.e.* the light musket of the 17th century (see Fusilier).

In electrical engineering a "fuse" (always so spelled) is a safety device, commonly consisting of a strip or wire of easily fusible metal, which melts and thus interrupts the circuit of which it forms part, whenever that circuit, through some accident or derangement, is caused to carry a current larger than that for which it is intended. In this sense the word must be connected with *fusus*, the past participle of Lat. *fundere*, to pour, whence comes the verb "fuse," to melt by heat, often used figuratively in the sense of blend, mix.

FYNE, LOCH, an inlet of the sea, Argyllshire, Scotland. From the head, 6 m. above Inveraray, to the mouth on the Sound of Bute, it has a south-westerly and then southerly trend and is 44 m. long, its width varying from ¼ m. to 6 m. It receives the Fyne, Shira, Aray and many other streams, and, on the western side, gives off Lochs Shira, Gair, Gilp (with Ardrishaig, the Crinan Canal and Lochgilphead) and East Tarbert (with Tarbert village). The glens debouching on the lake are Fyne, Shira, Aray, Kinglas and Hell's Glen. The coast generally is picturesque and in many parts well wooded. All vessels using the Crinan Canal navigate the loch to and from Ardrishaig, and there are daily excursions during the season, as far up as Inveraray. There are ferries at St Catherine's and Otter, and piers at Tarbert, Ardrishaig, Kilmory, Crarae, Furnace, Inveraray, Strachur and elsewhere. The industries comprise granite quarrying at Furnace and Crarae, distilling at Ardrishaig, gunpowder-making at Furnace and Kilfinan, and, above all, fishing. Haddock, whiting and codling are taken, and the famous "Loch Fyne herrings" command the highest price in the market. 376

FYRD, the name given to the English army, or militia, during the Anglo-Saxon period (see Army, 60). It is first mentioned in the *Anglo-Saxon Chronicle* under the date 605. The ealdorman, or sheriff, of the shire was probably charged with the duty of calling out and leading the fyrd, which appears always to have retained a local character, as during the time of the Danish invasions we read of the fyrd of Kent, of Somerset and of Devon. As attendance at the fyrd was included in the *trinoda necessitas* it was compulsory on all holders of land; but that it was not confined to them is shown by the following extract from the laws of Ine, king of the West Saxons, dated about 690, which prescribes the penalty for the serious offence of neglecting the fyrd: "If a *gesithcund* man owning land neglect the fyrd, let him pay 120 shillings, and forfeit his land; one not owning land 60 shillings; a ceorlish man 30 shillings as *fyrdwite*." The fyrd was gradually superseded by the gathering of the thegns and their retainers, but it was occasionally called out for defensive purposes even after the Norman Conquest.

FYT, JOHANNES (1609-1661), Belgian animal painter, was born at Antwerp and christened on the 19th of August 1609. He was registered apprentice to Hans van den Berghe in 1621. Professionally van den Berghe was a restorer of old pictures rather than a painter of new ones. At twenty Johannes Fyt entered the gild of St Luke as a master, and from that time till his death in 1661 he produced a vast number of pictures in which the bold facility of Snyders is united to the powerful effects of Rembrandt, and harmonies of gorgeous tone are not less conspicuous than freedom of touch and a true semblance of nature. There never was such a master of technical processes as Fyt in the rendering of animal life in its most varied forms. He may have been less correct in outline, less bold in action than Snyders, but he was much more skilful and more true in the reproduction of the coat of deer, dogs, greyhounds, hares and monkeys, whilst in realizing the plumage of peacocks, woodcocks, ducks, hawks, and cocks and hens, he had not his equal, nor was any artist even of the Dutch school more effective in relieving his compositions with accessories of tinted cloth, porcelain ware, vases and fruit. He was not clever at figures, and he sometimes trusted for these to the co-operation of Cornelius Schut or Willeborts, whilst his architectural backgrounds were sometimes executed by Quellyn. “Silenus amongst Fruit and Flowers,” in the Harrach collection at Vienna, “Diana and her Nymphs with the Produce of the Chase,” in the Belvedere at Vienna, and “Dead Game and Fruit in front of a Triumphal Arch,” belonging to Baron von Rothschild at Vienna, are specimens of the co-operation respectively of Schut, Willeborts and Quellyn. They are also Fyt’s masterpieces. The earliest dated work of the master is a cat grabbing at a piece of dead poultry near a hare and birds, belonging to Baron Cetto at Munich, and executed in 1644. The latest is a “Dead Snipe with Ducks,” of 1660, sold with the Jäger collection at Cologne in 1871. Great power is shown in the bear and boar hunts at Munich and Ravensworth castle. A “Hunted Roedeer with Dogs in the Water,” in the Berlin Museum, has some of the life and more of the roughness of Snyders, but lacks variety of tint and finish. A splendid specimen is the Page and Parrot near a table covered with game, guarded by a dog staring at a monkey, in the Wallace collection. With the needle and the brush Fyt was equally clever. He etched 16 plates, and those representing dogs are of their kind unique.

FYZABAD, or Faizabad, a city, district and division of British India in the United Provinces. The city stands on the left bank of the river Gogra, 78 m. by rail E. of Lucknow. Pop. (1901) 75,085. To the E. of Fyzabad, and now forming a suburb, is the ancient site of Ajodhya(*q.v.*). Fyzabad was founded about 1730 by Sa’adat Ali Khan, the first nawab wazir of Oudh, who built a hunting-lodge here. It received its present name in the reign of his successor; and Shuja-ud-daula, the third nawab, laid out a large town and fortified it, and here he was buried. It was afterwards the residence of the Begums of Oudh, famous in connexion with the impeachment of Warren Hastings. When the court of Oudh was removed to Lucknow in 1775 all the leading merchants and bankers abandoned the place. At the census of 1869 Fyzabad contained only 37,804 inhabitants; but it is now again advancing in prosperity and population. On the outbreak of the Mutiny in 1857, the cantonment contained two regiments of infantry, a squadron of cavalry, and a light field battery of artillery—all natives. Owing to their threatening demeanour after the Meerut massacre, many of the European women and children were sheltered by one of the great landholders of Oudh, and others were sent to less disturbed parts of the country. The troops rose, as was anticipated, and although they at first permitted their officers to take boats and proceed towards Dinapur, a message was afterwards sent to a rebel force lower down the river to intercept the fugitives. Of four boats, one, having passed the rebels unnoticed, succeeded in reaching Dinapur safely. Of those in the other three boats, one alone escaped. Fyzabad is now a station for European as well as for native troops. It is the headquarters of a brigade in the 8th division of the northern army. There is a government college. Sugar-refining and trade in agricultural produce are important.

The District of Fyzabad, lying between the two great rivers Gogra and Gumti, has an area of 1740 sq. m. It is entirely alluvial and well wooded, and has a good climate. Pop. (1901) 1,225,374, an increase of .7% in the decade. The district is traversed throughout its length by the Oudh and Rohilkhand railway from Lucknow to Benares, with a branch to Allahabad. Tanda, with a population in 1901 of 19,853, has the largest production of cotton goods in Oudh.

The Division of Fyzabad has an area of 12,113 sq. m., and comprises the six districts of Fyzabad, Gonda, Bahraich, Sultanpur, Partabgarh and Bara Banki. Pop. (1901) 6,855,991, an increase of 2% in the decade.
