Title: Encyclopaedia Britannica, 11th Edition, "Geodesy" to "Geometry"

Author: Various

Release date: September 17, 2011 [EBook #37461]

Language: English

Credits: Produced by Marius Masi, Don Kretz and the Online
Distributed Proofreading Team at http://www.pgdp.net

Transcriber's note:	A few typographical errors have been corrected. They appear in the text like this, and the explanation will appear when the mouse pointer is moved over the marked passage. Sections in Greek will yield a transliteration when the pointer is moved over them, and words using diacritic characters in the Latin Extended Additional block, which may not display in some fonts or browsers, will display an unaccented version.
	Links to other EB articles: Links to articles residing in other EB volumes will be made available when the respective volumes are introduced online.

THE ENCYCLOPÆDIA BRITANNICA

A DICTIONARY OF ARTS, SCIENCES, LITERATURE AND GENERAL INFORMATION

ELEVENTH EDITION

VOLUME XI SLICE VI

Geodesy to Geometry

Articles in This Slice

GEODESY (from the Gr. γῆ, the earth, and δαίειν, to divide), the science of surveying 607
(*q.v.*) extended to large tracts of country, having in view not only the production of a system of maps of very great accuracy, but the determination of the curvature of the surface of the earth, and eventually of the figure and dimensions of the earth. This last, indeed, may be the sole object in view, as was the case in the operations conducted in Peru and in Lapland by the celebrated French astronomers P. Bouguer, C.M. de la Condamine, P.L.M. de
Maupertuis, A.C. Clairault and others; and the measurement of the meridian arc of France 608
by P.F.A. Méchain and J.B.J. Delambre had for its end the determination of the true length of the "metre" which was to be the legal standard of length of France (see Earth, Figure of the).

The basis of every extensive survey is an accurate triangulation, and the operations of geodesy consist in the measurement, by theodolites, of the angles of the triangles; the measurement of one or more sides of these triangles on the ground; the determination by astronomical observations of the azimuth of the whole network of triangles; the determination of the actual position of the same on the surface of the earth by observations, first for latitude at some of the stations, and secondly for longitude; the determination of altitude for all stations.

For the computation, the points of the actual surface of the earth are imagined as projected along their plumb lines on the mathematical figure, which is given by the stationary sea-level, and the extension of the sea through the continents by a system of imaginary canals. For many purposes the mathematical surface is assumed to be a plane; in other cases a sphere of radius 6371 kilometres (20,900,000 ft.). In the case of extensive operations the surface must be considered as a compressed ellipsoid of rotation, whose minor axis coincides with the earth's axis, and whose compression, flattening, or ellipticity is about 1/298.

Measurement of Base Lines.

To determine by actual measurement on the ground the length of a side of one of the triangles ("base line"), wherefrom to infer the lengths of all the other sides in the triangulation, is not the least difficult operation of a trigonometrical survey. When the problem is stated thus—To determine the number of times that a certain standard or unit of length is contained between two finely marked points on the surface of the earth at a distance of some miles asunder, so that the error of the result may be pronounced to lie between certain very narrow limits,—then the question demands very serious consideration. The representation of the unit of length by means of the distance between two fine lines on the surface of a bar of metal at a certain temperature is never itself free from uncertainty and probable error, owing to the difficulty of knowing at any moment the precise temperature of the bar; and the transference of this unit, or a multiple of it, to a measuring bar will be affected not only with errors of observation, but with errors arising from uncertainty of temperature of both bars. If the measuring bar be not self-compensating for temperature, its expansion must be determined by very careful experiments. The thermometers required for this purpose must be very carefully studied, and their errors of division and index error determined.

In order to avoid the difficulty in exactly determining the temperature of a bar by the mercury thermometer, F.W. Bessel introduced in 1834 near Königsberg a compound bar which constituted a metallic thermometer.[1] A zinc bar is laid on an iron bar two toises long, both bars being perfectly planed and in free contact, the zinc bar being slightly shorter and the two bars rigidly united at one end. As the temperature varies, the difference of the lengths of the bars, as perceived by the other end, also varies, and affords a quantitative correction for temperature variations, which is applied to reduce the length to standard temperature. During the measurement of the base line the bars were not allowed to come into contact, the interval being measured by the insertion of glass wedges. The results of the comparisons of four measuring rods with one another and with the standards were elaborately computed by the method of least-squares. The probable error of the measured length of 935 toises (about 6000 ft.) has been estimated as 1/863500 or 1.2 μ (μ denoting a millionth). With this apparatus fourteen base lines were measured in Prussia and some neighbouring states; in these cases a somewhat higher degree of accuracy was obtained.

The principal triangulation of Great Britain and Ireland has seven base lines: five have been measured by steel chains, and two, more exactly, by the compensation bars of General T.F. Colby, an apparatus introduced in 1827-1828 at Lough Foyle in Ireland. Ten base lines were measured in India in 1831-1869 by the same apparatus. This is a system of six compound-bars self-correcting for temperature. The bars may be thus described: Two bars, one of brass and the other of iron, are laid in parallelism side by side, firmly united at their centres, from which they may freely expand or contract; at the standard temperature they are of the same length. Let AB be one bar, A′B′ the other; draw lines through the corresponding extremities AA′ (to P) and BB′ (to Q), and make A′P = B′Q, AA′ being equal to BB′. If the ratio A′P/AP equals the ratio of the coefficients of expansion of the bars A′B′ and AB, then, obviously, the distance PQ is constant (or nearly so). In the actual instrument P and Q are finely engraved dots 10 ft. apart. In practice the bars, when aligned, are not in contact, an interval of 6 in. being allowed between each bar and its neighbour. This distance is accurately measured by an ingenious micrometrical arrangement constructed on exactly the same principle as the bars themselves.

The last base line measured in India had a length of 8913 ft. In consequence of some suspicion as to the accuracy of the compensation apparatus, the measurement was repeated four times, the operations being conducted so as to determine the actual values of the probable errors of the apparatus. The direction of the line (which is at Cape Comorin) is north and south. In two of the measurements the brass component was to the west, in the others to the east; the differences between the individual measurements and the mean of the four were +0.0017, −0.0049, −0.0015, +0.0045 ft. These differences are very small; an elaborate investigation of all sources of error shows that the probable error of a base line in India is on the average ±2.8 μ. These compensation bars were also used by Sir Thomas Maclear in the measurement of the base line in his extension of Lacaille’s arc at the Cape. The account of this operation will be found in a volume entitled *Verification and Extension of Lacaille’s Arc of Meridian at the Cape of Good Hope*, by Sir Thomas Maclear, published in 1866. A rediscussion has been given by Sir David Gill in his *Report on the Geodetic Survey of South Africa, &c., 1896*.

A very simple base apparatus was employed by W. Struve in his triangulations in Russia from 1817 to 1855. This consisted of four wrought-iron bars, each two toises (rather more than 13 ft.) long; one end of each bar is terminated in a small steel cylinder presenting a slightly convex surface for contact, the other end carries a contact lever rigidly connected with the bar. The shorter arm of the lever terminates below in a polished hemisphere, the upper and longer arm traversing a vertical divided arc. In measuring, the plane end of one bar is brought into contact with the short arm of the contact lever (pushed forward by a weak spring) of the next bar. Each bar has two thermometers, and a level for determining the inclination of the bar in measuring. The manner of transferring the end of a bar to the ground is simply this: under the end of the bar a stake is driven very firmly into the ground, carrying on its upper surface a disk, capable of movement in the direction of the measured line by means of slow-motion screws. A fine mark on this disk is brought vertically under the end of the bar by means of a theodolite which is planted at a distance of 25 ft. from the stake in a direction perpendicular to the base. Struve investigated for each base the probable errors of the measurement arising from each of these seven causes: Alignment, inclination, comparisons with standards, readings of index, personal errors, uncertainties of temperature, and the probable errors of adopted rates of expansion. He found that ±0.8 μ was the mean of the probable errors of the seven bases measured by him. The Austro-Hungarian apparatus is similar; the distance of the rods is measured by a slider, which rests on one of the ends of each rod. Twenty-two base lines were measured in 1840-1899.

General Carlos Ibañez employed in 1858-1879, for the measurement of nine base lines in Spain, two apparatus similar to the apparatus previously employed by Porro in Italy; one is complicated, the other simplified. The first, an apparatus of the brothers Brunner of Paris, was a thermometric combination of two bars, one of platinum and one of brass, in length 4 metres, furnished with three levels and four thermometers. Suppose A, B, C three micrometer microscopes very firmly supported at intervals of 4 metres with their axes vertical, and aligned in the plane of the base line by means of a transit instrument, their micrometer screws being in the line of measurement. The measuring bar is brought under say A and B, and those micrometers read; the bar is then shifted and brought under B and C. By repetition of this process, the reading of a micrometer indicating the end of each position of the bar, the measurement is made.

Quite similar apparatus (among others) has been employed by the French and Germans. Since, however, it only permitted a distance of about 300 m. to be measured daily, Ibañez introduced a simplification; the measuring rod being made simply of steel, and provided with inlaid mercury thermometers. This apparatus was used in Switzerland for the measurement of three base lines. The accuracy is shown by the estimated probable errors: ±0.2 μ to ±0.8 μ. The distance measured daily amounts at least to 800 m.

A greater daily distance can be measured with the same accuracy by means of Bessel’s apparatus; this permits the ready measurement of 2000 m. daily. For this, however, it is important to notice that a large staff and favourable ground are necessary. An important improvement was introduced by Edward Jäderin of Stockholm, who measures with stretched wires of about 24 metres long; these wires are about 1.65 mm. in diameter, and when in use are stretched by an accurate spring balance with a tension of 10 kg.[2] The nature of the ground has a very trifling effect on this method. The difficulty of temperature determinations is removed by employing wires made of invar, an alloy of steel (64%) and nickel (36%) which has practically no linear expansion for small thermal changes at ordinary temperatures; this 609
alloy was discovered in 1896 by Benôit and Guillaume of the International Bureau of Weights and Measures at Breteuil. Apparently the future of base-line measurements rests with the invar wires of the Jäderin apparatus; next comes Porro’s apparatus with invar bars 4 to 5 metres long.

Results have been obtained in the United States, of great importance in view of their accuracy, rapidity of determination and economy. For the measurement of the arc of meridian in longitude 98° E., in 1900, nine base lines of a total length of 69.2 km. were measured in six months. The total cost of one base was $1231. At the beginning and at the end of the field-season a distance of exactly 100 m. was measured with R.S. Woodward’s “5-m. ice-bar” (invented in 1891); by means of the remeasurement of this length the standardization of the apparatus was done under the same conditions as existed in the case of the base measurements. For the measurements there were employed two steel tapes of 100 m. long, provided with supports at distances of 25 m., two of 50 m., and the duplex apparatus of Eimbeck, consisting of four 5-m. rods. Each base was divided into sections of about 1000 m.; one of these, the “test kilometre,” was measured with all the five apparatus, the others only with two apparatus, mostly tapes. The probable error was about ±0.8 μ, and the day’s work a distance of about 2000 m. Each of the four rods of the duplex apparatus consists of two bars of brass and steel. Mercury thermometers are inserted in both bars; these serve for the measurement of the length of the base lines by each of the bars, as they are brought into their consecutive positions, the contact being made by an elastic-sliding contact. The length of the base lines may be calculated for each bar only, and also by the supposition that both bars have the same temperature. The apparatus thus affords three sets of results, which mutually control themselves, and the contact adjustments permit rapid work. The same device has been applied to the older bimetallic-compensating apparatus of Bache-Würdemann (six bases, 1847-1857) and of Schott. There was also employed a single rod bimetallic apparatus on F. Porro’s principle, constructed by the brothers Repsold for some base lines. Excellent results have been more recently obtained with invar tapes.

The following results show the lengths of the same German base lines as measured by different apparatus:

				metres.
Base at Berlin	1864	Apparatus of	Bessel	2336·3920
” ”	1880	”	Brunner	·3924
Base at Strehlen	1854	”	Bessel	2762·5824
” ”	1879	”	Brunner	·5852
Old base at Bonn	1847	”	Bessel	2133·9095
” ”	1892	”	”	·9097
New base at Bonn	1892	”	”	2512·9612
” ”	1892	”	Brunner	·9696

It is necessary that the altitude above the level of the sea of every part of a base line be ascertained by spirit levelling, in order that the measured length may be reduced to what it would have been had the measurement been made on the surface of the sea, produced in imagination. Thus if l be the length of a measuring bar, h its height at any given position in the measurement, r the radius of the earth, then the length radially projected on to the level of the sea is $l(1 - h/r)$. In the Salisbury Plain base line the reduction to the level of the sea is −0.6294 ft.

The total number of base lines measured in Europe up to the present time is about one hundred and ten, nineteen of which do not exceed in length 2500 metres, or about 1½ miles, and three—one in France, the others in Bavaria—exceed 19,000 metres. The question has been frequently discussed whether or not the advantage of a long base is sufficiently great to warrant the expenditure of time that it requires, or whether as much precision is not obtainable in the end by careful triangulation from a short base. But the answer cannot be given generally; it must depend on the circumstances of each particular case. With Jäderin's apparatus, provided with invar wires, bases of 20 to 30 km. long are obtained without difficulty.

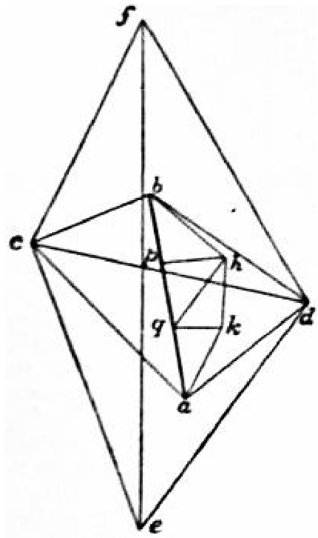


Fig. 1.

In working away from a base line ab, stations c, d, e, f are carefully selected so as to obtain from well-shaped triangles gradually increasing sides. Before, however, finally leaving the base line, it is usual to verify it by triangulation thus: during the measurement two or more points, as p, q (fig. 1), are marked in the base in positions such that the lengths of the different segments of the line are known; then, taking suitable external stations, as h, k, the angles of the triangles bhp, phq, hqk, kqa are measured. From these angles can be computed the ratios of the segments, which must agree, if all operations are correctly performed, with the ratios resulting from the measures. Leaving the base line, the sides increase up to 10, 30 or 50 miles occasionally, but seldom reaching 100 miles. The triangulation points may either be natural objects presenting themselves in suitable positions, such as church towers; or they may be objects specially constructed in stone or wood on mountain tops or other prominent ground. In every case it is necessary that the precise centre of the station be marked by some permanent mark. In India no expense is spared in making permanent the principal trigonometrical stations—costly towers in masonry being erected. It is essential that every trigonometrical station shall present a fine object for observation from surrounding stations.

Horizontal Angles.

In placing the theodolite over a station to be observed from, the first point to be attended to is that it shall rest upon a perfectly solid foundation. The method of obtaining this desideratum must depend entirely on the nature of the ground; the instrument must if possible be supported on rock, or if that be impossible a solid foundation must be obtained by digging. When the theodolite is required to be raised above the surface of the ground in order to command particular points, it is necessary to build two scaffolds,—the outer one to carry the observatory, the inner one to carry the instrument,—and these two edifices must have no point of contact. Many cases of high scaffolding have occurred on the English Ordnance Survey, as for instance at Thaxted church, where the tower, 80 ft. high, is surmounted by a spire of 90 ft. The scaffold for the observatory was carried from the base to the top of the spire; that for the instrument was raised from a point of the spire 140 ft. above the ground, having its bearing upon timbers passing through the spire at that height. Thus the instrument, at a height of 178 ft. above the ground, was insulated, and not affected by the action of the wind on the observatory.

At every station it is necessary to examine and correct the adjustments of the theodolite, which are these: the line of collimation of the telescope must be perpendicular to its axis of rotation; this axis perpendicular to the vertical axis of the instrument; and the latter perpendicular to the plane of the horizon. The micrometer microscopes must also measure correct quantities on the divided circle or circles. The method of observing is this. Let A, B, C ... be the stations to be observed taken in order of azimuth; the telescope is first directed to A and the cross-hairs of the telescope made to bisect the object presented by A, then the microscopes or verniers of the horizontal circle (also of the vertical circle if necessary) are read and recorded. The telescope is then turned to B, which is observed in the same manner; then C and the other stations. Coming round by continuous motion to A, it is again observed, and the agreement of this second reading with the first is some test of the stability of the instrument. In taking this round of angles—or "arc," as it is called on the Ordnance Survey—it is desirable that the interval of time between the first and second observations of A should be as small as may be consistent with due care. Before taking the next arc the horizontal circle is moved through 20° or 30°; thus a different set of divisions of the circle is used in each arc, which tends to eliminate the errors of division.

It is very desirable that all arcs at a station should contain one point in common, to which all angular measurements are thus referred,—the observations on each arc commencing and ending with this point, which is on the Ordnance Survey called the “referring object.” It is usual for this purpose to select, from among the points which have to be observed, that one which affords the best object for precise observation. For mountain tops a “referring object” is constructed of two rectangular plates of metal in the same vertical plane, their edges parallel and placed at such a distance apart that the light of the sky seen through appears as a vertical line about 10″ in width. The best distance for this object is from 1 to 2 miles.

This method seems at first sight very advantageous; but if, however, it be desired to attain the highest accuracy, it is better, as shown by General Schreiber of Berlin in 1878, to measure only single angles, and as many of these as possible between the directions to be determined. Division-errors are thus more perfectly eliminated, and errors due to the variation in the stability, &c., of the instruments are diminished. This method is rapidly gaining precedence.

The theodolites used in geodesy vary in pattern and in size—the horizontal circles ranging from 10 in. to 36 in. in diameter. In Ramsden’s 36-in. theodolite the telescope has a focal length of 36 in. and an aperture of 2.5 in., the ordinarily used magnifying power being 54; this last, however, can of course be changed at the requirements of the observer or of the weather. The probable error of a single observation of a fine object with this theodolite is about 0″.2. Fig. 2 represents an altazimuth theodolite of an improved pattern used on the Ordnance Survey. The horizontal circle of 14-in. diameter is read by three micrometer microscopes; the vertical circle has a diameter of 12 in., and is read by two microscopes. In the great trigonometrical survey of India the theodolites used in the more important parts of the work have been of 2 and 3 ft. diameter—the circle read by five equidistant microscopes. Every angle is measured twice in each position of the zero of the horizontal circle, of which there are generally ten; the entire number of measures of an angle is never less than 20. An 610
examination of 1407 angles showed that the probable error of an observed angle is on the average ±0″.28.

For the observations of very distant stations it is usual to employ a heliotrope (from the Gr. ἥλιος, sun; τρόπος, a turn), invented by Gauss at Göttingen in 1821. In its simplest form this is a plane mirror, 4, 6, or 8 in. in diameter, capable of rotation round a horizontal and a vertical axis. This mirror is placed at the station to be observed, and in fine weather it is kept so directed that the rays of the sun reflected by it strike the distant observing telescope. To the observer the heliotrope presents the appearance of a star of the first or second magnitude, and is generally a pleasant object for observing.

Observations at night, with the aid of light-signals, have been repeatedly made, and with good results, particularly in France by General François Perrier, and more recently in the United States by the Coast and Geodetic Survey; the signal employed being an acetylene bicycle-lamp, with a lens 5 in. in diameter. Particularly noteworthy are the trigonometrical connexions of Spain and Algeria, which were carried out in 1879 by Generals Ibañez and Perrier (over a distance of 270 km.), of Sicily and Malta in 1900, and of the islands of Elba and Sardinia in 1902 by Dr Guarducci (over distances up to 230 km.); in these cases artificial light was employed: in the first case electric light and in the two others acetylene lamps.

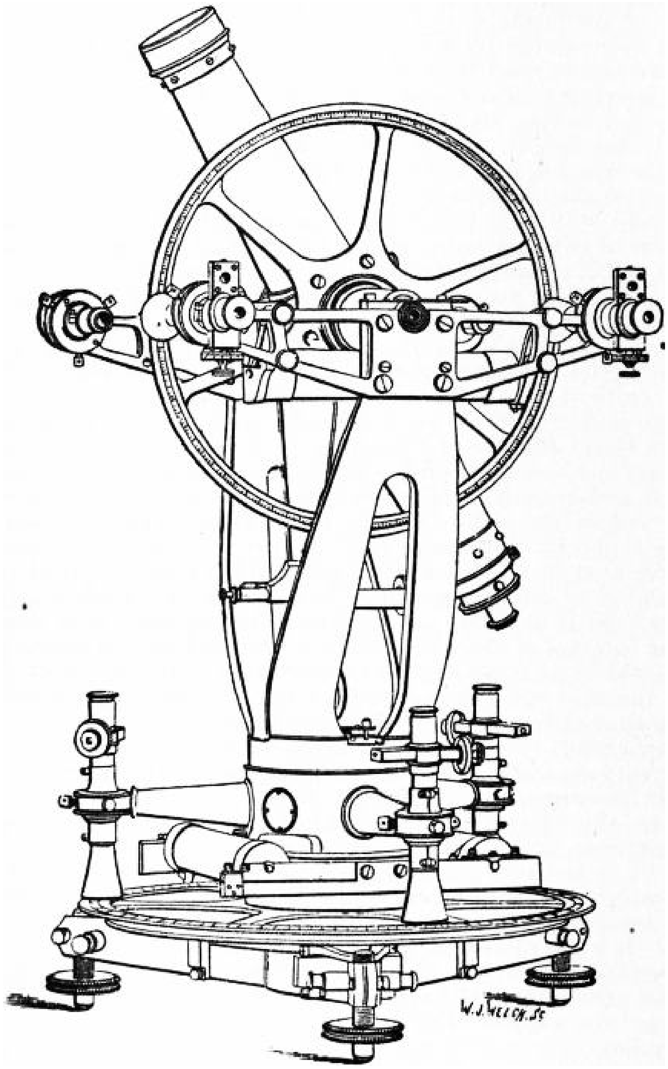


FIG. 2.—Altazimuth Theodolite.

Astronomical Observations.

The direction of the meridian is determined either by a theodolite or a portable transit instrument. In the former case the operation consists in observing the angle between a terrestrial object—generally a mark specially erected and capable of illumination at night—and a close circumpolar star at its greatest eastern or western azimuth, or, at any rate, when very near that position. If the observation be made t minutes of time before or after the time of greatest azimuth, the azimuth then will differ from its maximum value by $(450t)^2 \sin 1'' \sin 2\delta / \sin z$, in seconds of angle, omitting smaller terms, δ being the star's declination and z its zenith distance. The collimation and level errors are very carefully determined before and after these observations, and it is usual to arrange the observations by the reversal of the telescope so that collimation error shall disappear. If b, c be the level and collimation errors, the correction to the circle reading is $b \cot z \pm c \operatorname{cosec} z$, b being positive when the west end of the axis is high. It is clear that any uncertainty as to the real state of the level will produce a corresponding uncertainty in the resulting value of the azimuth,—an uncertainty which increases with the latitude and is very large in high latitudes. This may be partly remedied by observing in connexion with the star its reflection in mercury. In determining the value of "one division" of a level tube, it is necessary to bear in mind that in some the value varies considerably with the temperature. By experiments on the level of Ramsden's 3-foot theodolite, it was found that though at the ordinary temperature of 66° the value of a division was about one second, yet at 32° it was about five seconds.

In a very excellent portable transit used on the Ordnance Survey, the uprights carrying the telescope are constructed of mahogany, each upright being built of several pieces glued and screwed together; the base, which is a solid and heavy plate of iron, carries a reversing apparatus for lifting the telescope out of its bearings, reversing it and letting it down again. Thus is avoided the change of temperature which the telescope would incur by being lifted by the hands of the observer. Another form of transit is the German diagonal form, in which the rays of light after passing through the object-glass are turned by a total reflection prism through one of the transverse arms of the telescope, at the extremity of which arm is the eye-piece. The unused half of the ordinary telescope being cut away is replaced by a counterpoise. In this instrument there is the advantage that the observer without moving the position of his eye commands the whole meridian, and that the level may remain on the pivots whatever be the elevation of the telescope. But there is the disadvantage that the flexure of the transverse axis causes a variable collimation error depending on the zenith distance of the star to which it is directed; and moreover it has been found that in some cases the personal error of an observer is not the same in the two positions of the telescope.

To determine the direction of the meridian, it is well to erect two marks at nearly equal angular distances on either side of the north meridian line, so that the pole star crosses the vertical of each mark a short time before and after attaining its greatest eastern and western azimuths.

If now the instrument, perfectly levelled, is adjusted to have its centre wire on one of the marks, then when elevated to the star, the star will traverse the wire, and its exact position in the field at any moment can be measured by the micrometer wire. Alternate observations of the star and the terrestrial mark, combined with careful level readings and reversals of the instrument, will enable one, even with only one mark, to determine the direction of the meridian in the course of an hour with a probable error of less than a second. The second mark enables one to complete the station more rapidly and gives a check upon the work. As an instance, at Findlay Seat, in latitude 57° 35′, the resulting azimuths of the two marks were 177° 45′ 37″.29 ± 0″.20 and 182° 17′ 15″.61 ± 0″.13, while the angle between the two marks directly measured by a theodolite was found to be 4° 31′ 37″.43 ± 0″.23.

We now come to the consideration of the determination of time with the transit instrument. Let fig. 3 represent the sphere stereographically projected on the plane of the horizon,—ns being the meridian, we the prime vertical, Z, P the zenith and the pole. Let p be the point in which the production of the axis of the instrument meets the celestial sphere, S the position of a star when observed on a wire whose distance from the collimation centre is c. Let a be the azimuthal deviation, namely, the angle wZp, b the level error so that Zp = 90° − b. Let also the hour angle corresponding to p be 90° − n, and the declination of the same = m, the star's declination being δ, and the latitude φ. Then to find the hour angle ZPS = τ of the star when observed, in the triangles pPS, pPZ we have, since pPS = 90 + τ − n,

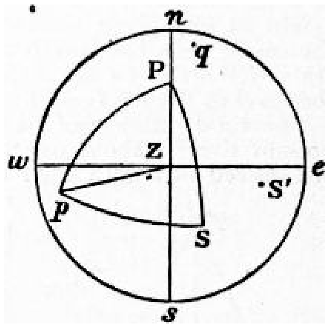


FIG. 3.

$$-\operatorname{Sin} c = \sin m \sin \delta + \cos m \cos \delta \sin (n - \tau),$$
$$\operatorname{Sin} m = \sin b \sin \varphi - \cos b \cos \varphi \sin a,$$
$$\operatorname{Cos} m \sin n = \sin b \cos \varphi + \cos b \sin \varphi \sin a.$$

And these equations solve the problem, however large be the errors of the instrument. Supposing, as usual, a, b, m, n to be small, we have at once $\tau = n + c \sec \delta + m \tan \delta$, which is the correction to the observed time of transit. Or, eliminating m and n by means of the second and third equations, and putting z for the zenith distance of the star, t for the observed time of transit, the corrected time is $t + (a \sin z + b \cos z + c) / \cos \delta$. Another very convenient form for stars near the zenith is $\tau = b \sec \varphi + c \sec \delta + m (\tan \delta - \tan \varphi)$.

Suppose that in commencing to observe at a station the error of the chronometer is not known; then having secured for the instrument a very solid foundation, removed as far as possible level and collimation errors, and placed it by estimation nearly in the meridian, let two stars differing considerably in declination be observed—the instrument not being reversed between them. From these two stars, neither of which should be a close circumpolar star, a good approximation to the chronometer error can be obtained; thus let ε_1, ε_2, be the apparent clock errors given by these stars if δ_1, δ_2 be their declinations the real error is 611

$$\varepsilon = \varepsilon_1 + (\varepsilon_1 - \varepsilon_2) (\tan \varphi - \tan \delta_1) / (\tan \delta_1 - \tan \delta_2).$$

Of course this is still only approximate, but it will enable the observer (who by the help of a table of natural tangents can compute ε in a few minutes) to find the meridian by placing at the proper time, which he now knows approximately, the centre wire of his instrument on the first star that passes—not near the zenith.

The transit instrument is always reversed at least once in the course of an evening's observing, the level being frequently read and recorded. It is necessary in most instruments to add a correction for the difference in size of the pivots.

The transit instrument is also used in the prime vertical for the determination of latitudes. In the preceding figure let q be the point in which the northern extremity of the axis of the instrument produced meets the celestial sphere. Let nZq be the azimuthal deviation = a, and b being the level error, Zq = 90° − b; let also nPq = τ and Pq = ψ. Let S′ be the position of a star when observed on a wire whose distance from the collimation centre is c, positive when to the south, and let h be the observed hour angle of the star, viz. ZPS′. Then the triangles qPS′, gPZ give

$$-\text{Sin } c = \sin\delta \cos\psi - \cos\delta \sin\psi \cos(h + \tau),$$
$$\text{Cos } \psi = \sin b \sin\varphi + \cos b \cos\varphi \cos a,$$
$$\text{Sin } \psi \sin\tau = \cos b \sin a.$$

Now when a and b are very small, we see from the last two equations that $\psi = \varphi - b$, $a = \tau \sin\psi$, and if we calculate φ' by the formula $\cot\varphi' = \cot\delta \cos h$, the first equation leads us to this result—

$$\varphi = \varphi' + (a \sin z + b \cos z + c) / \cos z,$$

the correction for instrumental error being very similar to that applied to the observed time of transit in the case of meridian observations. When a is not very small and z is small, the formulae required are more complicated.

The method of determining latitude by transits in the prime vertical has the disadvantage of being a somewhat slow process, and of requiring a very precise knowledge of the time, a disadvantage from which the zenith telescope is free. In principle this instrument is based on the proposition that when the meridian zenith distances of two stars at their upper culminations—one being to the north and the other to the south of the zenith—are equal, the latitude is the mean of their declinations; or, if the zenith distance of a star culminating to the south of the zenith be Z, its declination being δ, and that of another culminating to the north with zenith distance Z′ and declination δ′, then clearly the latitude is ½(δ + δ′) + ½(Z − Z′). Now the zenith telescope does away with the divided circle, and substitutes the measurement micrometrically of the quantity Z′ − Z.

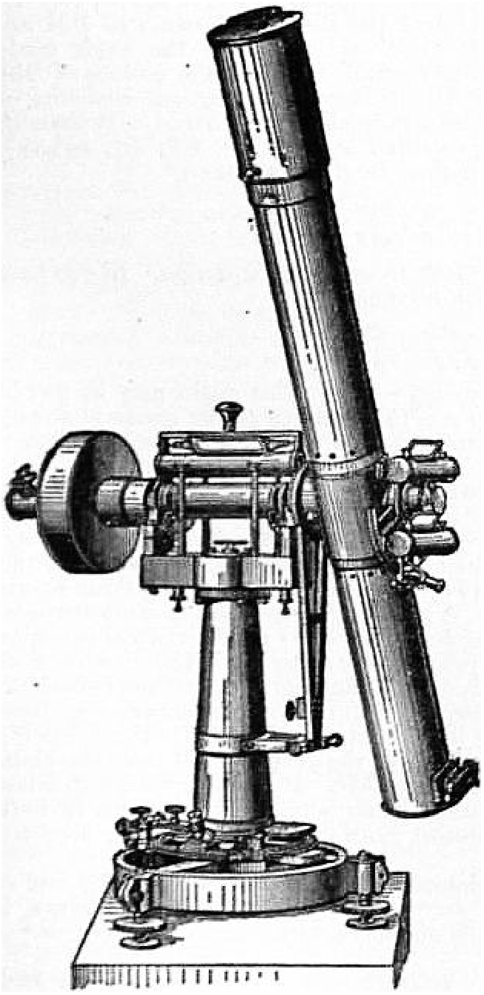


Fig. 4.—Zenith Telescope constructed for the International Stations at Mizusawa, Carloforte, Gaithersburg and Ukiah, by Hermann Wanschaff, Berlin.

In fig. 4 is shown a zenith telescope by H. Wanschaff of Berlin, which is the type used (according to the Central Bureau at Potsdam) since about 1890 for the determination of the variations of latitude due to different, but as yet imperfectly understood, influences. The instrument is supported on a strong tripod, fitted with levelling screws; to this tripod is fixed the azimuth circle and a long vertical steel axis. Fitting on this axis is a hollow axis which carries on its upper end a short transverse horizontal axis with a level. This latter carries the telescope, which, supported at the centre of its length, is free to rotate in a vertical plane. The telescope is thus mounted eccentrically with respect to the vertical axis around which it revolves. Two extremely sensitive levels are attached to the telescope, which latter carries a micrometer in its eye-piece, with a screw of long range for measuring differences of zenith distance. Two levels are employed for controlling and increasing the accuracy. For this instrument stars are selected in pairs, passing north and south of the zenith, culminating within a few minutes of time and within about twenty minutes (angular) of zenith distance of each other. When a pair of stars is to be observed, the telescope is set to the mean of the zenith distances and in the plane of the meridian. The first star on passing the central meridional wire is bisected by the micrometer; then the telescope is rotated very carefully through 180° round the vertical axis, and the second star on passing through the field is bisected by the micrometer on the centre wire. The micrometer has thus measured the difference of the zenith distances, and the calculation to get the latitude is most simple. Of course it is necessary to read the level, and the observations are not necessarily confined to the centre wire. In fact if n, s be the north and south readings of the level for the south star, n′, s′ the same for the north star, l the value of one division of the level, m the value of one division of the micrometer, r, r′ the refraction corrections, μ, μ′ the micrometer readings of the south and north star, the micrometer being supposed to read from the zenith, then, supposing the observation made on the centre wire,—

$$\varphi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(\mu - mu')m + \tfrac{1}{4}(n + n' - s - s')l + \tfrac{1}{2}(r - r').$$

It is of course of the highest importance that the value m of the screw be well determined. This is done most effectually by observing the vertical movement of a close circumpolar star when at its greatest azimuth.

In a single night with this instrument a very accurate result, say with a probable error of about 0″.2, could be obtained for latitude from, say, twenty pair of stars; but when the latitude is required to be obtained with the highest possible precision, two nights at least are necessary. The weak point of the zenith telescope lies in the circumstance that its requirements prevent the selection of stars whose positions are well fixed; very frequently it is necessary to have the declinations of the stars selected for this instrument specially observed at fixed observatories. The zenith telescope is made in various sizes from 30 to 54 in. in focal length; a 30-in. telescope is sufficient for the highest purposes and is very portable. The net observation probable-error for one pair of stars is only ±0″.1.

The zenith telescope is a particularly pleasant instrument to work with, and an observer has been known (a sergeant of Royal Engineers, on one occasion) to take every star in his list during eleven hours on a stretch, namely, from 6 o’clock P.M. until 5 A.M., and this on a very cold November night on one of the highest points of the Grampians. Observers accustomed to geodetic operations attain considerable powers of endurance. Shortly after the commencement of the observations on one of the hills in the Isle of Skye a storm carried away the wooden houses of the men and left the observatory roofless. Three observatory roofs were subsequently demolished, and for some time the observatory was used without a roof, being filled with snow every night and emptied every morning. Quite different, however, was the experience of the same party when on the top of Ben Nevis, 4406 ft. high. For about a fortnight the state of the atmosphere was unusually calm, so much so, that a lighted candle could often be carried between the tents of the men and the observatory, whilst at the foot of the hill the weather was wild and stormy.

The determination of the difference of longitude between two stations A and B resolves itself into the determination of the local time at each of the stations, and the comparison by signals of the clocks at A and B. Whenever telegraphic lines are available these comparisons are made by telegraphy. A small and delicately-made apparatus introduced into the mechanism of an astronomical clock or chronometer breaks or closes by the action of the clock an electric circuit every second. In order to record the minutes as well as seconds, one second in each minute, namely that numbered 0 or 60, is omitted. The seconds are recorded on a chronograph, which consists of a cylinder revolving uniformly at the rate of one revolution per minute covered with white paper, on which a pen having a slow movement in the direction of the axis of the cylinder describes a continuous spiral. This pen is deflected through the agency of an electromagnet every second, and thus the seconds of the clock are recorded on the chronograph by offsets from the spiral curve. An observer having his hand on a contact key in the same circuit can record in the same manner his observed times of transits of stars. The method of determination of difference of longitude is, therefore, virtually as follows. After the necessary observations for instrumental corrections, which are recorded only at the station of observation, the clock at A is put in connexion with the circuit so as to write on both chronographs, namely, that at A and that at B. Then the clock at B is made to write on both chronographs. It is clear that by this double operation one can eliminate the effect of the small interval of time consumed in the transmission of signals, for the difference of longitude obtained from the one chronograph will be in excess by as much as that obtained from the other will be in defect. The determination of the personal errors of the observers in this delicate operation is a matter of the greatest importance, as therein lies probably the chief source of residual error.

These errors can nevertheless be almost entirely avoided by using the impersonal 612
micrometer of Dr Repsold (Hamburg, 1889). In this device there is a movable micrometer wire which is brought by hand into coincidence with the star and moved along with it; at fixed points there are electrical contacts, which replace the fixed wires. Experiments at the Geodetic Institute and Central Bureau at Potsdam in 1891 gave the following personal equations in the case of four observers:—

	Older Procedure.	New Procedure.
A − B	$-0^s.108$	$-0^s.004$

A − G	$-0^s.314$	$-0^s.035$
A − S	$-0^s.184$	$-0^s.027$
B − G	$-0^s.225$	$+0^s.013$
B − S	$-0^s.086$	$-0^s.023$
G − S	$+0^s.109$	$-0^s.006$

These results show that in the later method the personal equation is small and not so variable; and consequently the repetition of longitude determinations with exchanged observers and apparatus entirely eliminates the constant errors, the probable error of such determinations on ten nights being scarcely $\pm 0^s.01$.

Calculation of Triangulation.

The surface of Great Britain and Ireland is uniformly covered by triangulation, of which the sides are of various lengths from 10 to 111 miles. The largest triangle has one angle at Snowdon in Wales, another on Slieve Donard in Ireland, and a third at Scaw Fell in Cumberland; each side is over a hundred miles and the spherical excess is 64″. The more ordinary method of triangulation is, however, that of chains of triangles, in the direction of the meridian and perpendicular thereto. The principal triangulations of France, Spain, Austria and India are so arranged. Oblique chains of triangles are formed in Italy, Sweden and Norway, also in Germany and Russia, and in the United States. Chains are composed sometimes merely of consecutive plain triangles; sometimes, and more frequently in India, of combinations of triangles forming consecutive polygonal figures. In this method of triangulating, the sides of the triangles are generally from 20 to 30 miles in length—seldom exceeding 40.

The inevitable errors of observation, which are inseparable from all angular as well as other measurements, introduce a great difficulty into the calculation of the sides of a triangulation. Starting from a given base in order to get a required distance, it may generally be obtained in several different ways—that is, by using different sets of triangles. The results will certainly differ one from another, and probably no two will agree. The experience of the computer will then come to his aid, and enable him to say which is the most trustworthy result; but no experience or ability will carry him through a large network of triangles with anything like assurance. The only way to obtain trustworthy results is to employ the method of least squares. We cannot here give any illustration of this method as applied to general triangulation, for it is most laborious, even for the simplest cases.

Three stations, projected on the surface of the sea, give a spherical or spheroidal triangle according to the adoption of the sphere or the ellipsoid as the form of the surface. A spheroidal triangle differs from a spherical triangle, not only in that the curvatures of the sides are different one from another, but more especially in this that, while in the spherical triangle the normals to the surface at the angular points meet at the centre of the sphere, in the spheroidal triangle the normals at the angles A, B, C meet the axis of revolution of the spheroid in three different points, which we may designate α, β, γ respectively. Now the angle A of the triangle as measured by a theodolite is the inclination of the planes BAα and CAα, and the angle at B is that contained by the planes ABβ and CBβ. But the planes ABα and ABβ containing the line AB in common cut the surface in two distinct plane curves. In order, therefore, that a spheroidal triangle may be exactly defined, it is necessary that the nature of the lines joining the three vertices be stated. In a mathematical point of view the most natural definition is that the sides be geodetic or shortest lines. C.C.G. Andrae, of Copenhagen, has also shown that other lines give a less convenient computation.

K.F. Gauss, in his treatise, *Disquisitiones generales circa superficies curvas*, entered fully into the subject of geodetic (or geodesic) triangles, and investigated expressions for the angles of a geodetic triangle whose sides are given, not certainly finite expressions, but approximations inclusive of small quantities of the fourth order, the side of the triangle or its ratio to the radius of the nearly spherical surface being a small quantity of the first order. The terms of the fourth order, as given by Gauss for any surface in general, are very complicated even when the surface is a spheroid. If we retain small quantities of the second order only, and put $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ for the angles of the geodetic triangle, while A, B, C are those of a plane triangle having sides equal respectively to those of the geodetic triangle, then, σ being the area of the plane triangle and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ the measures of curvature at the angular points,

$$\mathbf{A} = A + \sigma(2\mathbf{a} + \mathbf{b} + \mathbf{c}) / 12,$$
$$\mathbf{B} = B + \sigma(\mathbf{a} + 2\mathbf{b} + \mathbf{c}) / 12,$$
$$\mathbf{C} = C + \sigma(\mathbf{a} + \mathbf{b} + 2\mathbf{c}) / 12.$$

For the sphere $\mathbf{a} = \mathbf{b} = \mathbf{c}$, and making this simplification, we obtain the theorem previously given by A.M. Legendre. With the terms of the fourth order, we have (after Andrae):

$$\mathbf{A} - A = \frac{\varepsilon}{3} + \frac{\sigma}{3} k \left(\frac{m^2 - a^2}{20} k + \frac{\mathbf{a} - k}{4k} \right),$$
$$\mathbf{B} - B = \frac{\varepsilon}{3} + \frac{\sigma}{3} k \left(\frac{m^2 - b^2}{20} k + \frac{\mathbf{b} - k}{4k} \right),$$
$$\mathbf{C} - C = \frac{\varepsilon}{3} + \frac{\sigma}{3} k \left(\frac{m^2 - c^2}{20} k + \frac{\mathbf{c} - k}{4k} \right),$$

in which $\varepsilon = \sigma k \{1 + (m^2k / 8)\}$, $3m^2 = a^2 + b^2 + c^2$, $3k = \mathbf{a} + \mathbf{b} + \mathbf{c}$. For the ellipsoid of rotation the measure of curvature is equal to $1/\rho n$, ρ and n being the radii of curvature of the meridian and perpendicular.

It is rarely that the terms of the fourth order are required. As a rule spheroidal triangles are calculated as spherical (after Legendre), *i.e.* like plane triangles with a decrease of each angle of about $\varepsilon/3$; ε must, however, be calculated for each triangle separately with its mean measure of curvature k.

The geodetic line being the shortest that can be drawn on any surface between two given points, we may be conducted to its most important characteristics by the following considerations: let p, q be adjacent points on a curved surface; through s the middle point of the chord pq imagine a plane drawn perpendicular to pq, and let S be any point in the intersection of this plane with the surface; then pS + Sq is evidently least when sS is a minimum, which is when sS is a normal to the surface; hence it follows that of all plane curves on the surface joining p, q, when those points are indefinitely near to one another, that is the shortest which is made by the normal plane. That is to say, the osculating plane at any point of a geodetic line contains the normal to the surface at that point. Imagine now three points in space, A, B, C, such that AB = BC = c; let the direction cosines of AB be l, m, n, those of BC l′, m′, n′, then x, y, z being the co-ordinates of B, those of A and C will be respectively—

$$x - cl : y - cm : z - cn$$
$$x + cl' : y + cm' : z + cn'.$$

Hence the co-ordinates of the middle point M of AC are $x + \frac{1}{2}c(l' - l)$, $y + \frac{1}{2}c(m' - m)$, $z + \frac{1}{2}c(n' - n)$, and the direction cosines of BM are therefore proportional to $l' - l : m' - m : n' - n$. If the angle made by BC with AB be indefinitely small, the direction cosines of BM are as $\delta l : \delta m : \delta n$. Now if AB, BC be two contiguous elements of a geodetic, then BM must be a normal to the surface, and since δl, δm, δn are in this case represented by $\delta(dx/ds)$, $\delta(dy/ds)$, $\delta(dz/ds)$, and if the equation of the surface be $u = 0$, we have

$$\frac{d^2x}{ds^2} \Big/ \frac{du}{dx} = \frac{d^2y}{ds^2} \Big/ \frac{du}{dy} = \frac{d^2z}{ds^2} \Big/ \frac{du}{dz},$$

which, however, are equivalent to only one equation. In the case of the spheroid this equation becomes

$$y \frac{d^2x}{ds^2} - \frac{d^2y}{ds^2} = 0,$$

which integrated gives $ydx - xdy = Cds$. This again may be put in the form $r \sin a = C$, where a is the azimuth of the geodetic at any point—the angle between its direction and that of the meridian—and r the distance of the point from the axis of revolution.

From this it may be shown that the azimuth at A of the geodetic joining AB is not the same as the astronomical azimuth at A of B or that determined by the vertical plane AαB. Generally speaking, the geodetic lies between the two plane section curves joining A and B which are formed by the two vertical planes, supposing these points not far apart. If, however, A and B are nearly in the same latitude, the geodetic may cross (between A and B) that plane curve which lies nearest the adjacent pole of the spheroid. The condition of crossing is this. Suppose that for a moment we drop the consideration of the earth’s non-sphericity, and draw a perpendicular from the pole C on AB, meeting it in S between A and B. Then A being that point which is nearest the pole, the geodetic will cross the plane curve if AS be between ¼AB and ⅜AB. If AS lie between this last value and ½AB, the geodetic will lie wholly to the north of both plane curves, that is, supposing both points to be in the northern hemisphere.

The difference of the azimuths of the vertical section AB and of the geodetic AB, *i.e.* the astronomical and geodetic azimuths, is very small for all observable distances, being approximately:—

Geod. azimuth = Astr. azimuth $-1/12\ [e^2/(1-e^2)]\ [(s^2/\rho n\ (\cos^2\varphi \sin 2\alpha + (s/4a) \mid \sin 2\varphi \sin \alpha)]$, in which: e and a are the numerical eccentricity and semi-major axis respectively of the meridian ellipse, φ and α are the latitude and azimuth at A, s = AB, and ρ and n are the radii of curvature of the meridian and perpendicular at A. For s = 100 kilometres, only the first term is of moment; its value is $0''.028 \cos^2 \varphi \sin 2\alpha$, and it lies well within the errors of observation. If we imagine the geodetic AB, it will generally trisect the angles between the vertical sections at A and B, so that the geodetic at A is near the vertical section AB, and at B 613
near the section BA.[3] The greatest distance of the vertical sections one from another is $e^2 s^3 \cos^2 \varphi_0 \sin 2\alpha_0/16a^2$, in which φ_0 and α_0 are the mean latitude and azimuth respectively of the middle point of AB. For the value s = 64 kilometres, the maximum distance is 3 mm.

An idea of the course of a longer geodetic line may be gathered from the following example. Let the line be that joining Cadiz and St Petersburg, whose approximate positions are—

	Cadiz.	St Petersburg.
Lat.	36° 22′ N.	59° 56′ N.
Long.	6° 18′ W.	30° 17′ E.

If G be the point on the geodetic corresponding to F on that one of the plane curves which contains the normal at Cadiz (by “corresponding” we mean that F and G are on a meridian) then G is to the north of F; at a quarter of the whole distance from Cadiz GF is 458 ft., at half the distance it is 637 ft., and at three-quarters it is 473 ft. The azimuth of the geodetic at Cadiz differs 20″ from that of the vertical plane, which is the astronomical azimuth.

The azimuth of a geodetic line cannot be observed, so that the line does not enter of necessity into practical geodesy, although many formulae connected with its use are of great simplicity and elegance. The geodetic line has always held a more important place in the science of geodesy among the mathematicians of France, Germany and Russia than has been assigned to it in the operations of the English and Indian triangulations. Although the observed angles of a triangulation are not geodetic angles, yet in the calculation of the distance and reciprocal bearings of two points which are far apart, and are connected by a long chain of triangles, we may fall upon the geodetic line in this manner:—

If A, Z be the points, then to start the calculation from A, we obtain by some preliminary calculation the approximate azimuth of Z, or the angle made by the direction of Z with the side AB or AC of the first triangle. Let P_1 be the point where this line intersects BC; then, to find P_2, where the line cuts the next triangle side CD, we make the angle BP_1P_2 such that $BP_1P_2 + BP_1A = 180°$. This fixes P_2, and P_3 is fixed by a repetition of the same process; so for P_4, P_5 .... Now it is clear that the points P_1, P_2, P_3 so computed are those which would be actually fixed by an observer with a theodolite, proceeding in the following manner. Having set the instrument up at A, and turned the telescope in the direction of the computed bearing, an assistant places a mark P_1 on the line BC, adjusting it till bisected by the cross-hairs of the telescope at A. The theodolite is then placed over P_1, and the telescope turned to A; the horizontal circle is then moved through 180°. The assistant then places a mark P_2 on the line CD, so as to be bisected by the telescope, which is then moved to P_2, and in the same manner P_3 is fixed. Now it is clear that the series of points P_1, P_2, P_3 approaches to the geodetic line, for the plane of any two consecutive elements $P_{n-1} P_n$, $P_n P_{n+1}$ contains the normal at P_n.

If the objection be raised that not the geodetic azimuths but the astronomical azimuths are observed, it is necessary to consider that the observed vertical sections do not correspond to points on the sea-level but to elevated points. Since the normals of the ellipsoid of rotation do not in general intersect, there consequently arises an influence of the height on the azimuth. In the case of the measurement of the azimuth from A to B, the instrument is set to a point A′ over the surface of the ellipsoid (the sea-level), and it is then adjusted to a point B′, also over the surface, say at a height h′. The vertical plane containing A′ and B′ also contains A but not B: it must therefore be rotated through a small azimuth in order to contain B. The correction amounts approximately to $-e^2 h' \cos^2\varphi \sin 2\alpha/2a$; in the case of h′ = 1000 m., its value is $0''.108 \cos^2\varphi \sin 2\alpha$.

This correction is therefore of greater importance in the case of observed azimuths and horizontal angles than in the previously considered case of the astronomical and the geodetic azimuths. The observed azimuths and horizontal angles must therefore also be corrected in the case, where it is required to dispense with geodetic lines.

When the angles of a triangulation have been adjusted by the method of least squares, and the sides are calculated, the next process is to calculate the latitudes and longitudes of all the stations starting from one given point. The calculated latitudes, longitudes and azimuths, which are designated geodetic latitudes, longitudes and azimuths, are not to be confounded with the observed latitudes, longitudes and azimuths, for these last are subject to somewhat large errors. Supposing the latitudes of a number of stations in the triangulation to be observed, practically the mean of these determines the position in latitude of the network, taken as a whole. So the orientation or general azimuth of the whole is inferred from all the azimuth observations. The triangulation is then supposed to be projected on a spheroid of given elements, representing as nearly as one knows the real figure of the earth. Then, taking the latitude of one point and the direction of the meridian there as given—obtained, namely, from the astronomical observations there—one can compute the latitudes of all the other points with any degree of precision that may be considered desirable. It is necessary to employ for this purpose formulae which will give results true even for the longest distances to the second place of decimals of seconds, otherwise there will arise an accumulation of errors from imperfect calculation which should always be avoided. For very long distances, eight places of decimals should be employed in logarithmic calculations; if seven places only are available very great care will be required to keep the last place true. Now let φ, φ' be the latitudes of two stations A and B; α, α^* their mutual azimuths counted from north by east continuously from 0° to 360°; ω their difference of longitude measured from west to east; and s the distance AB.

First compute a latitude φ_1 by means of the formula $\varphi_1 = \varphi + (s \cos \alpha)/\rho$, where ρ is the radius of curvature of the meridian at the latitude φ; this will require but four places of logarithms. Then, in the first two of the following, five places are sufficient—

$$\varepsilon = \frac{s^2}{2\rho n} \sin \alpha \cos a, \qquad \eta = \frac{s^2}{2\rho n} \sin^2 \alpha \tan \varphi_1,$$

$$\varphi' - \varphi = \frac{s}{rho_0} \cos (\alpha - ⅔\varepsilon) - \eta,$$

$$\omega = \frac{s \sin (alpha - ⅓\varepsilon)}{n \cos (\varphi' + ⅓\eta)},$$

$$\alpha^* - \alpha = \omega \sin (\varphi' + ⅔\eta) - \varepsilon + 180°.$$

Here n is the normal or radius of curvature perpendicular to the meridian; both n and ρ correspond to latitude φ_1, and ρ_0 to latitude ½($\varphi + \varphi'$). For calculations of latitude and longitude, tables of the logarithmic values of $\rho \sin 1''$, $n \sin 1''$, and $2 n \rho \sin 1''$ are necessary. The following table contains these logarithms for every ten minutes of latitude from 52° to 53° computed with the elements a = 20926060 and a : b = 295 : 294 :—

Lat.	Log. 1/ρ sin 1″.	Log. 1/n sin 1″.	Log. 1/2ρn sin 1″.
° ′			
52 0	7.9939434	7.9928231	0.37131
10	9309	8190	29
20	9185	8148	28
30	9060	8107	26
40	8936	8065	24
50	8812	8024	23
53 0	8688	7982	22

The logarithm in the last column is that required also for the calculation of spherical excesses, the spherical excess of a triangle being expressed by $a b \sin C/(2\rho n) \sin 1''$.

It is frequently necessary to obtain the co-ordinates of one point with reference to another point; that is, let a perpendicular arc be drawn from B to the meridian of A meeting it in P, then, α being the azimuth of B at A, the co-ordinates of B with reference to A are

$$AP = s \cos (\alpha - ⅔\varepsilon), \quad BP = s \sin (\alpha - ⅓\varepsilon),$$

where ε is the spherical excess of APB, viz. $s^2 \sin \alpha \cos \alpha$ multiplied by the quantity whose logarithm is in the fourth column of the above table.

If it be necessary to determine the geographical latitude and longitude as well as the azimuths to a greater degree of accuracy than is given by the above formulae, we make use of the following formula: given the latitude φ of A, and the azimuth α and the distance s of B, to determine the latitude φ' and longitude ω of B, and the back azimuth α'. Here it is understood that α' is symmetrical to α, so that $\alpha^* + \alpha' = 360°$.

Let

$$\theta = s\Delta / a, \text{ where } \Delta = (1 - e^2 \sin^2 \varphi)^{1/2}$$

and

$$\xi = \frac{e^2\,\theta^2}{4\,(1 - e^2)} \cos^2\varphi \sin 2\alpha, \qquad \xi' = \frac{e^2\,\theta^3}{6\,(1 - e^2)} \cos^2\varphi \cos^2\alpha;$$

ξ, ξ' are always very minute quantities even for the longest distances; then, putting $\kappa = 90° - \varphi$,

$$\tan\frac{\alpha' + \xi - \omega}{2} = \frac{\sin \tfrac{1}{2}(\kappa - \theta - \xi')}{\sin \tfrac{1}{2}(\kappa + \theta + \xi')} \cot\frac{\alpha}{2}$$

$$\tan\frac{\alpha' + \xi - \omega}{2} = \frac{\cos \tfrac{1}{2}(\kappa - \theta - \xi')}{\cos \tfrac{1}{2}(\kappa + \theta + \xi')} \cot\frac{\alpha}{2}$$

$$\varphi' - \varphi = \frac{s \sin \tfrac{1}{2}(\alpha' + \xi - \alpha)}{\rho_0 \sin \tfrac{1}{2}(\alpha' + \xi + \alpha)} \left(1 + \frac{\theta^2}{12} \cos^2\frac{\alpha' - \alpha}{2}\right);$$

here ρ_0 is the radius of curvature of the meridian for the mean latitude $\tfrac{1}{2}(\varphi + \varphi')$. These formulae are approximate only, but they are sufficiently precise even for very long distances.

For lines of any length the formulae of F.W. Bessel (*Astr. Nach.*, 1823, iv. 241) are suitable.

If the two points A and B be defined by their geographical co-ordinates, we can accurately 614
calculate the corresponding astronomical azimuths, *i.e.* those of the vertical section, and then proceed, in the case of not too great distances, to determine the length and the azimuth of the shortest lines. For *any* distances recourse must again be made to Bessel's formula.[4]

Let α, α' be the mutual azimuths of two points A, B on a spheroid, k the chord line joining them, μ, μ' the angles made by the chord with the normals at A and B, φ, φ', ω their latitudes and difference of longitude, and $(x^2 + y^2)/a^2 + z^2\, b^2 = 1$ the equation of the surface; then if the plane xz passes through A the co-ordinates of A and B will be

$$\begin{aligned} x &= (a/\Delta) \cos\varphi, & x' &= (a/\Delta') \cos\varphi' \cos\omega, \\ y &= 0 & y' &= (a/\Delta') \cos\varphi' \sin\omega, \\ z &= (a/\Delta)\,(1 - e^2) \sin\varphi, & z' &= (a/\Delta')\,(1 - e^2) \sin\varphi', \end{aligned}$$

where $\Delta = (1 - e^2 \sin^2\varphi)^{1/2}$, $\Delta' = (1 - e^2 \sin^2\varphi')^{1/2}$, and e is the eccentricity. Let f, g, h be the direction cosines of the normal to that plane which contains the normal at A and the point B, and whose inclinations to the meridian plane of A is $= \alpha$; let also l, m, n and l′, m′, n′ be the direction cosines of the normal at A, and of the tangent to the surface at A which lies in the plane passing through B, then since the first line is perpendicular to each of the other two and to the chord k, whose direction cosines are proportional to $x' - x$, $y' - y$, $z' - z$, we have these three equations

$$\begin{aligned} f\,(x' - x) + gy' + h\,(z' - z) &= 0 \\ fl + gm + hn &= 0 \\ fl' + gm' + hn' &= 0. \end{aligned}$$

Eliminate f, g, h from these equations, and substitute

$$\begin{aligned} l &= \cos\varphi & l' &= -\sin\varphi\cos\alpha \\ m &= 0 & m' &= \sin\alpha \\ n &= \sin\varphi & n' &= \cos\varphi\cos\alpha, \end{aligned}$$

and we get

$$(x' - x)\sin\varphi + y'\cot\alpha - (z' - z)\cos\varphi = 0.$$

The substitution of the values of x, z, x′, y′, z′ in this equation will give immediately the value of $\cot\alpha$; and if we put ζ, ζ' for the corresponding azimuths on a sphere, or on the supposition $e = 0$, the following relations exist

$$\cot\alpha - \cot\zeta = e^2 \frac{\cos\varphi\, Q}{\cos\varphi'\, \Delta}$$

$$\cot\alpha' - \cot\zeta' = -e^2 \frac{\cos\varphi'\, Q}{\cos\varphi\, \Delta'}$$

$$\Delta' \sin\varphi - \Delta \sin\varphi' = Q \sin\omega.$$

If from B we let fall a perpendicular on the meridian plane of A, and from A let fall a perpendicular on the meridian plane of B, then the following equations become geometrically evident:

$$k \sin \mu \sin \alpha = (a/\Delta') \cos \varphi' \sin \omega$$
$$k \sin \mu' \sin \alpha' = (a/\Delta) \cos \varphi \sin \omega.$$

Now in any surface u = 0 we have

$$k^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$$

$$-\cos \mu = \left[(x' - x) \frac{du}{dx} + (y' - y) \frac{du}{dy} + (z' - z) \frac{du}{dz} \right] / k \left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2} + \frac{du^2}{dz^2} \right)^{1/2}$$

$$\cos \mu' = \left[(x' - x) \frac{du}{dx'} + (y' - y) \frac{du}{dy'} + (z' - z) \frac{du}{dz'} \right] / k \left(\frac{du^2}{dx'^2} + \frac{du^2}{dy'^2} + \frac{du^2}{dz'^2} \right)^{1/2} .$$

In the present case, if we put

$$1 - \frac{xx'}{a^2} - \frac{zz'}{b^2} = U,$$

then

$$\frac{k^2}{a^2} = 2U - e^2 \left(\frac{z' - z}{b} \right)^2$$

$$\cos \mu = (a/k)\, \Delta U;\ \cos \mu' = (a/k)\, \Delta' U.$$

Let u be such an angle that

$$(1 - e^2)^{1/2} \sin \varphi = \Delta \sin u$$
$$\cos \varphi = \Delta \cos u,$$

then on expressing x, x′, z, z′ in terms of u and u′,

$$U = 1 - \cos u \cos u' \cos \omega - \sin u \sin u';$$

also, if v be the third side of a spherical triangle, of which two sides are $\tfrac{1}{2}\pi - u$ and $\tfrac{1}{2}\pi - u'$ and the included angle ω, using a subsidiary angle ψ such that

$$\sin \psi \sin \tfrac{1}{2}v = e \sin \tfrac{1}{2} (u' - u) \cos \tfrac{1}{2} (u' + u),$$

we obtain finally the following equations:—

$$\begin{aligned} k &= 2a \cos \psi \sin \tfrac{1}{2}v \\ \cos \mu &= \Delta \sec \psi \sin \tfrac{1}{2}v \\ \cos \mu' &= \Delta' \sec \psi \sin \tfrac{1}{2}v \\ \sin \mu \sin \alpha &= (a/k) \cos u' \sin \omega \\ \sin \mu' \sin \alpha' &= (a/k) \cos u \sin \omega. \end{aligned}$$

These determine rigorously the distance, and the mutual zenith distances and azimuths, of any two points on a spheroid whose latitudes and difference of longitude are given.

By a series of reductions from the equations containing ζ, ζ′ it may be shown that

$$\alpha + \alpha' = \zeta + \zeta' + \tfrac{1}{4}e^4\omega\, (\varphi' - \varphi)^2 \cos^4 \varphi_0 \sin \varphi_0 + \ldots,$$

where φ_0 is the mean of φ and φ′, and the higher powers of e are neglected. A short computation will show that the small quantity on the right-hand side of this equation cannot amount even to the thousandth part of a second for $k < 0.1a$, which is, practically speaking, zero; consequently the sum of the azimuths α + α′ on the spheroid is equal to the sum of the spherical azimuths, whence follows this very important theorem (known as Dalby′s theorem). If φ, φ′ be the latitudes of two points on the surface of a spheroid, ω their difference of longitude, α, α′ their reciprocal azimuths,

$$\tan \tfrac{1}{2}\omega = \cot \tfrac{1}{2} (\alpha + \alpha') \{\cos \tfrac{1}{2} (\varphi' - \varphi) / \sin \tfrac{1}{2} (\varphi' + \varphi)\}.$$

The computation of the geodetic from the astronomical azimuths has been given above. From k we can now compute the length s of the vertical section, and from this the shortest length. The difference of length of the geodetic line and either of the plane curves is

$$e^4 s^5 \cos^4 \varphi_0 \sin^2 2\alpha_0 / 360\, a^4.$$

At least this is an approximate expression. Supposing $s = 0.1a$, this quantity would be less than one-hundredth of a millimetre. The line s is now to be calculated as a circular arc with a mean radius r along AB. If $\varphi_0 = \frac{1}{2}(\varphi + \varphi')$, $\alpha_0 = \frac{1}{2}(180° + \alpha - \alpha')$, $\Delta_0 = (1 - e^2 \sin^2 \varphi_0)^{1/2}$, then $1/r = \Delta_0/a\,[1 + (e^2/(1 - e^2)) \cos^2 \varphi_0 \cos^2 \alpha_0]$, and approximately $\sin (s/2r) = k/2r$. These formulae give, in the case of $k = 0.1a$, values certain to eight logarithmic decimal places. An excellent series of formulae for the solution of the problem, to determine the azimuths, chord and distance along the surface from the geographical co-ordinates, was given in 1882 by Ch. M. Schols (*Archives Néerlandaises*, vol. xvii.).

Irregularities of the Earth's Surface.

In considering the effect of unequal distribution of matter in the earth's crust on the form of the surface, we may simplify the matter by disregarding the considerations of rotation and eccentricity. In the first place, supposing the earth a sphere covered with a film of water, let the density ρ be a function of the distance from the centre so that surfaces of equal density are concentric spheres. Let now a disturbance of the arrangement of matter take place, so that the density is no longer to be expressed by ρ, a function of r only, but is expressed by $\rho + \rho'$, where ρ' is a function of three co-ordinates θ, φ, r. Then ρ' is the density of what may be designated disturbing matter; it is positive in some places and negative in others, and the whole quantity of matter whose density is ρ' is zero. The previously spherical surface of the sea of radius a now takes a new form. Let P be a point on the disturbed surface, P′ the corresponding point vertically below it on the undisturbed surface, $PP' = N$. The knowledge of N over the whole surface gives us the form of the disturbed or actual surface of the sea; it is an equipotential surface, and if V be the potential at P of the disturbing matter ρ', M the mass of the earth (the attraction-constant is assumed equal to unity)

$$\frac{M}{a + N} + V = C = \frac{M}{a} - \frac{M}{a^2} N + V.$$

As far as we know, N is always a very small quantity, and we have with sufficient approximation $N = 3V/4\pi\delta a$, where δ is the mean density of the earth. Thus we have the disturbance in elevation of the sea-level expressed in terms of the potential of the disturbing matter. If at any point P the value of N remain constant when we pass to any adjacent point, then the actual surface is there parallel to the ideal spherical surface; as a rule, however, the normal at P is inclined to that at P′, and astronomical observations have shown that this inclination, the deflection or deviation, amounting ordinarily to one or two seconds, may in some cases exceed 10″, or, as at the foot of the Himalayas, even 60″. By the expression "mathematical figure of the earth" we mean the surface of the sea produced in imagination so as to percolate the continents. We see then that the effect of the uneven distribution of matter in the crust of the earth is to produce small elevations and depressions on the mathematical surface which would be otherwise spheroidal. No geodesist can proceed far in his work without encountering the irregularities of the mathematical surface, and it is necessary that he should know how they affect his astronomical observations. The whole of this subject is dealt with in his usual elegant manner by Bessel in the *Astronomische Nachrichten*, Nos. 329, 330, 331, in a paper entitled "Ueber den Einfluss der Unregelmässigkeiten der Figur der Erde auf geodätische Arbeiten, &c." But without entering into further details it is not difficult to see how local attraction at any station affects the determinations of latitude, longitude and azimuth there.

Let there be at the station an attraction to the north-east throwing the zenith to the south-west, so that it takes in the celestial sphere a position Z′, its undisturbed position being Z. Let the rectangular components of the displacement ZZ′ be ξ measured southwards and η 615
measured westwards. Now the great circle joining Z′ with the pole of the heavens P makes there an angle with the meridian $PZ = \eta \operatorname{cosec} PZ' = \eta \sec \varphi$, where φ is the latitude of the station. Also this great circle meets the horizon in a point whose distance from the great circle PZ is $\eta \sec \varphi \sin \varphi = \eta \tan \varphi$. That is, a meridian mark, fixed by observations of the pole star, will be placed that amount to the east of north. Hence the observed latitude requires the correction ξ; the observed longitude a correction $\eta \sec \varphi$; and any observed azimuth a correction $\eta \tan \varphi$. Here it is supposed that azimuths are measured from north by east, and longitudes eastwards. The horizontal angles are also influenced by the deflections of the plumb-line, in fact, just as if the direction of the vertical axis of the theodolite varied by the same amount. This influence, however, is slight, so long as the sights point almost horizontally at the objects, which is always the case in the observation of distant points.

The expression given for N enables one to form an approximate estimate of the effect of a compact mountain in raising the sea-level. Take, for instance, Ben Nevis, which contains about a couple of cubic miles; a simple calculation shows that the elevation produced would only amount to about 3 in. In the case of a mountain mass like the Himalayas, stretching over some 1500 miles of country with a breadth of 300 and an average height of 3 miles, although it is difficult or impossible to find an expression for V, yet we may ascertain that an elevation amounting to several hundred feet may exist near their base. The geodetical operations, however, rather negative this idea, for it was shown by Colonel Clarke (*Phil. Mag.*, 1878) that the form of the sea-level along the Indian arc departs but slightly from that of the mean figure of the earth. If this be so, the action of the Himalayas must be counteracted by subterranean tenuity.

Suppose now that A, B, C, ... are the stations of a network of triangulation projected on or lying on a spheroid of semiaxis major and eccentricity a, e, this spheroid having its axis parallel to the axis of rotation of the earth, and its surface coinciding with the mathematical surface of the earth at A. Then basing the calculations on the observed elements at A, the calculated latitudes, longitudes and directions of the meridian at the other points will be the true latitudes, &c., of the points as projected on the spheroid. On comparing these geodetic elements with the corresponding astronomical determinations, there will appear a system of differences which represent the inclinations, at the various points, of the actual irregular surface to the surface of the spheroid of reference. These differences will suggest two things,—first, that we may improve the agreement of the two surfaces, by not restricting the spheroid of reference by the condition of making its surface coincide with the mathematical surface of the earth at A; and secondly, by altering the form and dimensions of the spheroid. With respect to the first circumstance, we may allow the spheroid two degrees of freedom, that is, the normals of the surfaces at A may be allowed to separate a small quantity, compounded of a meridional difference and a difference perpendicular to the same. Let the spheroid be so placed that its normal at A lies to the north of the normal to the earth's surface by the small quantity ξ and to the east by the quantity η. Then in starting the calculation of geodetic latitudes, longitudes and azimuths from A, we must take, not the observed elements φ, α, but for φ, $\varphi + \xi$, and for α, $\alpha + \eta \tan \varphi$, and zero longitude must be replaced by $\eta \sec \varphi$. At the same time suppose the elements of the spheroid to be altered from a, e to a + da, e + de. Confining our attention at first to the two points A, B, let (φ'), (α'), (ω) be the numerical elements at B as obtained in the first calculation, viz. before the shifting and alteration of the spheroid; they will now take the form

$$\begin{aligned}&(\varphi') + f\xi + g\eta + h da + k de,\\&(\alpha') + f'\xi + g'\eta + h' da + k' de,\\&\omega + f''\xi + g''\eta + h'' da + k'' de,\end{aligned}$$

where the coefficients f, g, ... &c. can be numerically calculated. Now these elements, corresponding to the projection of B on the spheroid of reference, must be equal severally to the astronomically determined elements at B, corrected for the inclination of the surfaces there. If ξ', η' be the components of the inclination at that point, then we have

$$\begin{aligned}\xi' &= (\varphi') - \varphi' + f\xi + g\eta + h da + k de,\\ \eta' \tan \varphi' &= (\alpha') - \alpha' + f'\xi + g'\eta + h' da + k' de,\\ \eta' \sec \varphi' &= (\omega) - \omega + f''\xi + g''\eta + h'' da + k'' de,\end{aligned}$$

where φ', α', ω are the observed elements at B. Here it appears that the observation of longitude gives no additional information, but is available as a check upon the azimuthal observations.

If now there be a number of astronomical stations in the triangulation, and we form equations such as the above for each point, then we can from them determine those values of ξ, η, da, de, which make the quantity $\xi^2 + \eta^2 + \xi'^2 + \eta'^2 + \dots$ a minimum. Thus we obtain that spheroid which best represents the surface covered by the triangulation.

In the *Account of the Principal Triangulation of Great Britain and Ireland* will be found the determination, from 75 equations, of the spheroid best representing the surface of the British Isles. Its elements are a = 20927005 ± 295 ft., b : a − b = 280 ± 8; and it is so placed that at Greenwich Observatory $\xi = 1''.864$, $\eta = -0''.546$.

Taking Durham Observatory as the origin, and the tangent plane to the surface (determined by $\xi = -0''.664$, $\eta = -4''.117$) as the plane of x and y, the former measured northwards, and z measured vertically downwards, the equation to the surface is

$$.99524953\, x^2 + .99288005\, y^2 + .99763052\, z^2 - 0.00671003xz - 41655070z = 0.$$

Altitudes.

The precise determination of the altitude of his station is a matter of secondary importance to the geodesist; nevertheless it is usual to observe the zenith distances of all trigonometrical points. Of great importance is a knowledge of the height of the base for its reduction to the sea-level. Again the height of a station does influence a little the observation of terrestrial angles, for a vertical line at B does not lie generally in the vertical plane of A (see above). The height above the sea-level also influences the geographical latitude, inasmuch as the centrifugal force is increased and the magnitude and direction of the attraction of the earth are altered, and the effect upon the latitude is a very small term expressed by the formula $h(g' - g) \sin 2\varphi/ag$, where g, g′ are the values of gravity at the equator and at the pole. This is $h \sin 2\varphi/5820$ seconds, h being in metres, a quantity which may be neglected, since for ordinary mountain heights it amounts to only a few hundredths of a second. We can assume this amount as joined with the northern component of the plumb-line perturbations.

The uncertainties of terrestrial refraction render it impossible to determine accurately by vertical angles the heights of distant points. Generally speaking, refraction is greatest at about daybreak; from that time it diminishes, being at a minimum for a couple of hours before and after mid-day; later in the afternoon it again increases. This at least is the general march of the phenomenon, but it is by no means regular. The vertical angles measured at the station on Hart Fell showed on one occasion in the month of September a refraction of double the average amount, lasting from 1 P.M. to 5 P.M. The mean value of the coefficient of refraction k determined from a very large number of observations of terrestrial zenith distances in Great Britain is .0792 ± .0047; and if we separate those rays which for a considerable portion of their length cross the sea from those which do not, the former give k = .0813 and the latter k = .0753. These values are determined from high stations and long distances; when the distance is short, and the rays graze the ground, the amount of refraction is extremely uncertain and variable. A case is noted in the Indian survey where the zenith distance of a station 10.5 miles off varied from a depression of 4′ 52″.6 at 4.30 P.M. to an elevation of 2′ 24″.0 at 10.50 P.M.

If h, h′ be the heights above the level of the sea of two stations, 90° + δ, 90° + δ′ their mutual zenith distances (δ being that observed at h), s their distance apart, the earth being regarded as a sphere of radius = a, then, with sufficient precision,

$$h' - h = s \tan \left(s \frac{1 - 2k}{2a} - \delta \right), \quad h - h' = s \tan \left(s \frac{1 - 2k}{2a} - \delta' \right).$$

If from a station whose height is h the horizon of the sea be observed to have a zenith distance 90° + δ, then the above formula gives for h the value

$$h = \frac{a}{2} \frac{\tan^2 \delta}{1 - 2k}$$

Suppose the depression δ to be n minutes, then $h = 1.054n^2$ if the ray be for the greater part of its length crossing the sea; if otherwise, $h = 1.040n^2$. To take an example: the mean of eight observations of the zenith distance of the sea horizon at the top of Ben Nevis is 91° 4′ 48″, or δ = 64.8; the ray is pretty equally disposed over land and water, and hence $h = 1.047n^2 = 4396$ ft. The actual height of the hill by spirit-levelling is 4406 ft., so that the error of the height thus obtained is only 10 ft.

The determination of altitudes by means of spirit-levelling is undoubtedly the most exact method, particularly in its present development as precise-levelling, by which there have been determined in all civilized countries close-meshed nets of elevated points covering the entire land.

(A. R. C; F. R. H.)

---

1 An arrangement acting similarly had been previously introduced by Borda.

2 *Geodetic Survey of South Africa*, vol. iii. (1905), p. viii; *Les Nouveaux Appareils pour la mesure rapide des bases géod.*, par J. René Benoît et Ch. Éd. Guillaume (1906).

3 See a paper "On the Course of Geodetic Lines on the Earth's Surface" in the *Phil. Mag.* 1870; Helmert, *Theorien der höheren Geodäsie*, 1. 321.

4 Helmert, Theorien der höheren Geodäsie, 1. 232, 247.

GEOFFREY, surnamed MARTEL (1006-1060), count of Anjou, son of the count Fulk Nerra (*q.v.*) and of the countess Hildegarde or Audegarde, was born on the 14th of October 1006. During his father's lifetime he was recognized as suzerain by Fulk l'Oison ("the Gosling"), count of Vendôme, the son of his half-sister Adela. Fulk having revolted, he confiscated the countship, which he did not restore till 1050. On the 1st of January 1032 he married Agnes, widow of William the Great, duke of Aquitaine, and taking arms against William the Fat, eldest son and successor of William the Great, defeated him and took him prisoner at Mont-Couër near Saint-Jouin-de-Marnes on the 20th of September 1033. He then tried to win recognition as dukes of Aquitaine for the sons of his wife Agnes by William the Great, who were still minors, but Fulk Nerra promptly took up arms to defend his suzerain William the Fat, from whom he held the Loudunois and Saintonge in fief against his son. In 1036 616 Geoffrey Martel had to liberate William the Fat, on payment of a heavy ransom, but the latter having died in 1038, and the second son of William the Great, Odo, duke of Gascony, having fallen in his turn at the siege of Mauzé (10th of March 1039) Geoffrey made peace with his father in the autumn of 1039, and had his wife's two sons recognized as dukes. About this time, also, he had interfered in the affairs of Maine, though without much result, for having sided against Gervais, bishop of Le Mans, who was trying to make himself guardian of the young count of Maine, Hugh, he had been beaten and forced to make terms with Gervais in 1038. In 1040 he succeeded his father in Anjou and was able to conquer Touraine (1044) and assert his authority over Maine (see Anjou). About 1050 he repudiated Agnes, his first wife, and married Grécie, the widow of Bellay, lord of Montreuil-Bellay (before August 1052), whom he subsequently left in order to marry Adela, daughter of a certain Count Odo. Later he returned to Grécie, but again left her to marry Adelaide the German. When, however, he died on the 14th of November 1060, at the monastery of St Nicholas at Angers, he left no children, and transmitted the countship to Geoffrey the Bearded, the eldest of his nephews (see ANJOU).

See Louis Halphen, *Le Comté d'Anjou au XI^e siècle* (Paris, 1906). A summary biography is given by Célestin Port, *Dictionnaire historique, géographique et biographique de Maine-et-Loire* (3 vols., Paris-Angers, 1874-1878), vol. ii. pp. 252-253, and a sketch of the wars by Kate Norgate, *England under the Angevin Kings* (2 vols., London, 1887), vol. i. chs. iii. iv.

(L. H.*)

GEOFFREY, surnamed Plantagenet [or Plantegenet] (1113-1151), count of Anjou, was the son of Count Fulk the Young and of Eremburge (or Arembourg of La Flèche); he was born on the 24th of August 1113. He is also called "le bel" or "the handsome," and received the surname of Plantagenet from the habit which he is said to have had of wearing in his cap a sprig of broom (*genêt*). In 1127 he was made a knight, and on the 2nd of June 1129 married Matilda, daughter of Henry I. of England, and widow of the emperor Henry V. Some months afterwards he succeeded to his father, who gave up the countship when he definitively went to the kingdom of Jerusalem. The years of his government were spent in subduing the Angevin barons and in conquering Normandy (see Anjou). In 1151, while returning from the siege of Montreuil-Bellay, he took cold, in consequence of bathing in the Loir at Château-du-Loir, and died on the 7th of September. He was buried in the cathedral of Le Mans. By his wife Matilda he had three sons: Henry Plantagenet, born at Le Mans on Sunday, the 5th of March 1133; Geoffrey, born at Argentan on the 1st of June 1134; and William Long-Sword, born on the 22nd of July 1136.

See Kate Norgate, *England under the Angevin Kings* (2 vols., London, 1887), vol. i. chs. v.-viii.; Célestin Port, *Dictionnaire historique, géographique et biographique de Maine-et-Loire* (3 vols., Paris-Angers, 1874-1878), vol. ii. pp. 254-256. A history of Geoffrey le Bel has yet to be written; there is a biography of him written in the 12th century by Jean, a monk of Marmoutier, *Historia Gaufredi, ducis Normannorum et comitis Andegavorum*, published by Marchegay et Salmon; "Chroniques des comtes d'Anjou" (*Société de l'histoire de France*, Paris, 1856), pp. 229-310.

(L. H.*)

GEOFFREY (1158-1186), duke of Brittany, fourth son of the English king Henry II. and his wife Eleanor of Aquitaine, was born on the 23rd of September 1158. In 1167 Henry suggested a marriage between Geoffrey and Constance (d. 1201), daughter and heiress of Conan IV., duke of Brittany (d. 1171); and Conan not only assented, perhaps under compulsion, to this proposal, but surrendered the greater part of his unruly duchy to the English king. Having received the homage of the Breton nobles, Geoffrey joined his brothers, Henry and Richard, who, in alliance with Louis VII. of France, were in revolt against their father; but he made his peace in 1174, afterwards helping to restore order in Brittany and Normandy, and aiding the new French king, Philip Augustus, to crush some rebellious vassals. In July 1181 his marriage with Constance was celebrated, and practically the whole of his subsequent life was spent in warfare with his brother Richard. In 1183 he made peace with his father, who had come to Richard's assistance; but a fresh struggle soon broke out for the possession of Anjou, and Geoffrey was in Paris treating for aid with Philip Augustus, when he died on the 19th of August 1186. He left a daughter, Eleanor, and his wife bore a posthumous son, the unfortunate Arthur.

GEOFFREY (*c.* 1152-1212), archbishop of York, was a bastard son of Henry II., king of England. He was distinguished from his legitimate half-brothers by his consistent attachment and fidelity to his father. He was made bishop of Lincoln at the age of twenty-one (1173); but though he enjoyed the temporalities he was never consecrated and resigned the see in 1183. He then became his father's chancellor, holding a large number of lucrative benefices in plurality. Richard nominated him archbishop of York in 1189, but he was not consecrated till 1191, or enthroned till 1194. Geoffrey, though of high character, was a man of uneven temper; his history in chiefly one of quarrels, with the see of Canterbury, with the chancellor William Longchamp, with his half-brothers Richard and John, and especially with his canons at York. This last dispute kept him in litigation before Richard and the pope for many years. He led the clergy in their refusal to be taxed by John and was forced to fly the kingdom in 1207. He died in Normandy on the 12th of December 1212.

See Giraldus Cambrensis, *Vita Galfridi*; Stubbs's prefaces to *Roger de Hoveden*, vols. iii. and iv. (Rolls Series).

(H. W. C. D.)

GEOFFREY DE MONTBRAY (d. 1093), bishop of Coutances (*Constantiensis*), a right-hand man of William the Conqueror, was a type of the great feudal prelate, warrior and administrator at need. He knew, says Orderic, more about marshalling mailed knights than edifying psalm-singing clerks. Obtaining, as a young man, in 1048, the see of Coutances, by his brother's influence (see Mowbray), he raised from his fellow nobles and from their Sicilian spoils funds for completing his cathedral, which was consecrated in 1056. With bishop Odo, a warrior like himself, he was on the battle-field of Hastings, exhorting the Normans to victory; and at William's coronation it was he who called on them to acclaim their duke as king. His reward in England was a mighty fief scattered over twelve counties. He accompanied William on his visit to Normandy (1067), but, returning, led a royal force to the relief of Montacute in September 1069. In 1075 he again took the field, leading with Bishop Odo a vast host against the rebel earl of Norfolk, whose stronghold at Norwich they besieged and captured.

Meanwhile the Conqueror had invested him with important judicial functions. In 1072 he had presided over the great Kentish suit between the primate and Bishop Odo, and about the same time over those between the abbot of Ely and his despoilers, and between the bishop of Worcester and the abbot of Ely, and there is some reason to think that he acted as a Domesday commissioner (1086), and was placed about the same time in charge of Northumberland. The bishop, who attended the Conqueror's funeral, joined in the great rising against William Rufus next year (1088), making Bristol, with which (as Domesday shows) he was closely connected and where he had built a strong castle, his base of operations. He burned Bath and ravaged Somerset, but had submitted to the king before the end of the year. He appears to have been at Dover with William in January 1090, but, withdrawing to Normandy, died at Coutances three years later. In his fidelity to Duke Robert he seems to have there held out for him against his brother Henry, when the latter obtained the Cotentin.

See E.A. Freeman, *Norman Conquest* and *William Rufus*; J.H. Round, *Feudal England*; and, for original authorities, the works of Orderic Vitalis and William of Poitiers, and of Florence of Worcester; the Anglo-Saxon Chronicle; William of Malmesbury's *Gesta pontificum*, and Lanfranc's works, ed. Giles; Domesday Book.

(J. H. R.)

GEOFFREY OF MONMOUTH (d. 1154), bishop of St Asaph and writer on early British history, was born about the year 1100. Of his early life little is known, except that he received a liberal education under the eye of his paternal uncle, Uchtryd, who was at that time archdeacon, and subsequently bishop, of Llandaff. In 1129 Geoffrey appears at Oxford among the witnesses of an Oseney charter. He subscribes himself Geoffrey Arturus; from this we may perhaps infer that he had already begun his experiments in the manufacture of Celtic mythology. A first edition of his *Historia Britonum* was in circulation by the year 1139, 617
although the text which we possess appears to date from 1147. This famous work, which the author has the audacity to place on the same level with the histories of William of Malmesbury and Henry of Huntingdon, professes to be a translation from a Celtic source; "a very old book in the British tongue" which Walter, archdeacon of Oxford, had brought from Brittany. Walter the archdeacon is a historical personage; whether his book has any real existence may be fairly questioned. There is nothing in the matter or the style of the *Historia* to preclude us from supposing that Geoffrey drew partly upon confused traditions, partly on his own powers of invention, and to a very slight degree upon the accepted authorities for early British history. His chronology is fantastic and incredible; William of Newburgh justly remarks that, if we accepted the events which Geoffrey relates, we should have to suppose that they had happened in another world. William of Newburgh wrote, however, in the reign of Richard I. when the reputation of Geoffrey's work was too well established to be shaken by such criticisms. The fearless romancer had achieved an immediate success. He was patronized by Robert, earl of Gloucester, and by two bishops of Lincoln; he obtained, about 1140, the archdeaconry of Llandaff "on account of his learning"; and in 1151 was promoted to the see of St Asaph.

Before his death the *Historia Britonum* had already become a model and a quarry for poets and chroniclers. The list of imitators begins with Geoffrey Gaimar, the author of the *Estorie des Engles* (*c.* 1147), and Wace, whose *Roman de Brut* (1155) is partly a translation and partly a free paraphrase of the *Historia*. In the next century the influence of Geoffrey is unmistakably attested by the *Brut* of Layamon, and the rhyming English chronicle of Robert of Gloucester. Among later historians who were deceived by the *Historia Britonum* it is only needful to mention Higdon, Hardyng, Fabyan (1512), Holinshed (1580) and John Milton. Still greater was the influence of Geoffrey upon those writers who, like Warner in *Albion's England* (1586), and Drayton in *Polyolbion* (1613), deliberately made their accounts of English history as poetical as possible. The stories which Geoffrey preserved or invented were not infrequently a source of inspiration to literary artists. The earliest English tragedy, *Gorboduc* (1565), the *Mirror for Magistrates* (1587), and Shakespeare's Lear, are instances in point. It was, however, the Arthurian legend which of all his fabrications attained the greatest vogue. In the work of expanding and elaborating this theme the successors of Geoffrey went as far beyond him as he had gone beyond Nennius; but he retains the credit due to the founder of a great school. Marie de France, who wrote at the court of Henry II., and Chrétien de Troyes, her French contemporary, were the earliest of the avowed romancers to take up the theme. The succeeding age saw the Arthurian story popularized, through translations of the French romances, as far afield as Germany and Scandinavia. It produced in England the *Roman du Saint Graal* and the *Roman de Merlin*, both from the pen of Robert de Borron; the *Roman de Lancelot*; the *Roman de Tristan*, which is attributed to a fictitious Lucas de Gast. In the reign of Edward IV. Sir Thomas Malory paraphrased and arranged the best episodes of these romances in English prose. His *Morte d'Arthur*, printed by Caxton in 1485, epitomizes the rich mythology which Geoffrey's work had first called into life, and gave the Arthurian story a lasting place in the English imagination. The influence of the *Historia Britonum* may be illustrated in another way, by enumerating the more familiar of the legends to which it first gave popularity. Of the twelve books into which it is divided only three (Bks. IX., X., XI.) are concerned with Arthur. Earlier in the work, however, we have the adventures of Brutus; of his follower Corineus, the vanquisher of the Cornish giant Goemagol (Gogmagog); of Locrinus and his daughter Sabre (immortalized in Milton's *Comus*); of Bladud the builder of Bath; of Lear and his daughters; of the three pairs of brothers, Ferrex and Porrex, Brennius and Belinus, Elidure and Peridure. The story of Vortigern and Rowena takes its final form in the *Historia Britonum*; and Merlin makes his first appearance in the prelude to the Arthur legend. Besides the *Historia Britonum* Geoffrey is also credited with a *Life of Merlin* composed in Latin verse. The authorship of this work has, however, been disputed, on the ground that the style is distinctly superior to that of the *Historia*. A minor composition, the *Prophecies of Merlin*, was written before 1136, and afterwards incorporated with the *Historia*, of which it forms the seventh book.

For a discussion of the manuscripts of Geoffrey's work, see Sir T.D. Hardy's *Descriptive Catalogue* (Rolls Series), i. pp. 341 ff. The *Historia Britonum* has been critically edited by San Marte (Halle, 1854). There is an English translation by J.A. Giles (London, 1842). The *Vita Merlini* has been edited by F. Michel and T. Wright (Paris, 1837). See also the *Dublin Univ. Magazine* for April 1876, for an article by T. Gilray on the literary influence of Geoffrey; G. Heeger's *Trojanersage der Britten* (1889); and La Borderie's *Études historiques bretonnes* (1883).

(H. W. C. D.)

GEOFFREY OF PARIS (d. *c.* 1320), French chronicler, was probably the author of the *Chronique métrique de Philippe le Bel, or Chronique rimée de Geoffroi de Paris*. This work, which deals with the history of France from 1300 to 1316, contains 7918 verses, and is valuable as that of a writer who had a personal knowledge of many of the events which he relates. Various short historical poems have also been attributed to Geoffrey, but there is no certain information about either his life or his writings.

The *Chronique* was published by J.A. Buchon in his *Collection des chroniques*, tome ix. (Paris, 1827), and it has also been printed in tome xxii. of the *Recueil des historiens des Gaules et de la France* (Paris, 1865). See G. Paris, *Histoire de la littérature française au moyen âge* (Paris, 1890); and A. Molinier, *Les Sources de l'histoire de France*, tome iii. (Paris, 1903).

GEOFFREY THE BAKER (d. *c.* 1360), English chronicler, is also called Walter of Swinbroke, and was probably a secular clerk at Swinbrook in Oxfordshire. He wrote a *Chronicon Angliae temporibus Edwardi II. et Edwardi III.*, which deals with the history of England from 1303 to 1356. From the beginning until about 1324 this work is based upon Adam Murimuth's *Continuatio chronicarum*, but after this date it is valuable and interesting, containing information not found elsewhere, and closing with a good account of the battle of Poitiers. The author obtained his knowledge about the last days of Edward II. from William Bisschop, a companion of the king's murderers, Thomas Gurney and John Maltravers. Geoffrey also wrote a *Chroniculum* from the creation of the world until 1336, the value of which is very slight. His writings have been edited with notes by Sir E.M. Thompson as the *Chronicon Galfridi le Baker de Swynebroke* (Oxford, 1889). Some doubt exists concerning Geoffrey's share in the compilation of the *Vita et mors Edwardi II.*, usually attributed to Sir Thomas de la More, or Moor, and printed by Camden in his *Anglica scripta*. It has been maintained by Camden and others that More wrote an account of Edward's reign in French, and that this was translated into Latin by Geoffrey and used by him in compiling his *Chronicon*. Recent scholarship, however, asserts that More was no writer, and that the *Vita et mors* is an extract from Geoffrey's *Chronicon*, and was attributed to More, who was the author's patron. In the main this conclusion substantiates the verdict of Stubbs, who has published the *Vita et mors* in his *Chronicles of the reigns of Edward I. and Edward II.* (London, 1883). The manuscripts of Geoffrey's works are in the Bodleian library at Oxford.

GEOFFRIN, MARIE THÉRÈSE RODET (1699-1777), a Frenchwoman who played an interesting part in French literary and artistic life, was born in Paris in 1699. She married, on the 19th of July 1713, Pierre François Geoffrin, a rich manufacturer and lieutenant-colonel of the National Guard, who died in 1750. It was not till Mme Geoffrin was nearly fifty years of age that we begin to hear of her as a power in Parisian society. She had learned much from Mme de Tencin, and about 1748 began to gather round her a literary and artistic circle. She had every week two dinners, on Monday for artists, and on Wednesday for her friends the Encyclopaedists and other men of letters. She received many foreigners of distinction, Hume and Horace Walpole among others. Walpole spent much time in her society before he was finally attached to Mme du Deffand, and speaks of her in his letters as a model of common sense. She was indeed somewhat of a small tyrant in her circle. She had adopted the pose of an old woman earlier than necessary, and her coquetry, if such it can be called, took the form of being mother and mentor to her guests, many of whom were indebted to her generosity for substantial help. Although her aim appears to have been to have the *Encyclopédie* in conversation and action around her, she was extremely displeased with any of her friends who were so rash as to incur open disgrace. Marmontel lost her favour after the official censure of *Bélisaire*, and her advanced views did not prevent her from observing the forms of religion. A devoted Parisian, Mme Geoffrin rarely left the city, so that her journey to Poland in 1766 to visit the king, Stanislas Poniatowski, whom she had known in his early days in Paris, was a great event in her life. Her experiences induced a sensible gratitude that she had been born “*Française*” and “*particulière*.” In her last illness her daughter, Thérèse, marquise de la Ferté Imbault, excluded her mother’s old friends so that she might die as a good Christian, a proceeding wittily described by the old lady: “My daughter is like Godfrey de Bouillon, she wished to defend my tomb from the infidels.” Mme Geoffrin died in Paris on the 6th of October 1777. 618

See *Correspondance inédite du roi Stanislas Auguste Poniatowski et de Madame Geoffrin*, edited by the comte de Mouÿ (1875); P. de Ségur, *Le Royaume de la rue Saint-Honoré, Madame Geoffrin et sa fille* (1897); A. Tornezy, *Un Bureau d’esprit au XVIII^e siècle: le salon de Madame Geoffrin* (1895); and Janet Aldis, *Madame Geoffrin, her Salon and her Times, 1750-1777* (1905).

GEOFFROY, ÉTIENNE FRANÇOIS (1672-1731), French chemist, born in Paris on the 13th of February 1672, was first an apothecary and then practised medicine. After studying at Montpellier he accompanied Marshal Tallard on his embassy to London in 1698 and thence travelled to Holland and Italy. Returning to Paris he became professor of chemistry at the Jardin du Roi and of pharmacy and medicine at the Collège de France, and dean of the faculty of medicine. He died in Paris on the 6th of January 1731. His name is best known in connexion with his tables of affinities (*tables des rapports*), which he presented to the French Academy in 1718 and 1720. These were lists, prepared by collating observations on the actions of substances one upon another, showing the varying degrees of affinity exhibited by analogous bodies for different reagents, and they retained their vogue for the rest of the century, until displaced by the profounder conceptions introduced by C.L. Berthollet. Another of his papers dealt with the delusions of the philosopher’s stone, but nevertheless he believed that iron could be artificially formed in the combustion of vegetable matter. His *Tractatus de materia medica*, published posthumously in 1741, was long celebrated.

His brother Claude Joseph, known as Geoffroy the younger (1685-1752), was also an apothecary and chemist who, having a considerable knowledge of botany, devoted himself especially to the study of the essential oils in plants.

GEOFFROY, JULIEN LOUIS (1743-1814), French critic, was born at Rennes in 1743. He studied in the school of his native town and at the Collège Louis le Grand in Paris. He took orders and fulfilled for some time the humble functions of an usher, eventually becoming professor of rhetoric at the *Collège Mazarin*. A bad tragedy, Caton, was accepted at the *Théâtre Français*, but was never acted. On the death of Élie Fréron in 1776 the other collaborators in the *Année littéraire* asked Geoffroy to succeed him, and he conducted the journal until in 1792 it ceased to appear. Geoffroy was a bitter critic of Voltaire and his followers, and made for himself many enemies. An enthusiastic royalist, he published with Fréron's brother-in-law, the abbé Thomas Royou (1741-1792), a journal, *L'Ami du roi* (1790-1792), which possibly did more harm than good to the king's cause by its ill-advised partisanship. During the Terror Geoffroy hid in the neighbourhood of Paris, only returning in 1799. An attempt to revive the *Année littéraire* failed, and Geoffroy undertook the dramatic feuilleton of the *Journal des débats*. His scathing criticisms had a success of notoriety, but their popularity was ephemeral, and the publication of them (5 vols., 1819-1820) as *Cours de littérature dramatique* proved a failure. He was also the author of a perfunctory *Commentaire* on the works of Racine prefixed to Lenormant's edition (1808). He died in Paris on the 27th of February 1814.

GEOFFROY SAINT-HILAIRE, ÉTIENNE (1772-1844), French naturalist, was the son of Jean Gèrard Geoffroy, procurator and magistrate of Étampes, Seine-et-Oise, where he was born on the 15th of April 1772. Destined for the church he entered the college of Navarre, in Paris, where he studied natural philosophy under M.J. Brisson; and in 1788 he obtained one of the canonicates of the chapter of Sainte Croix at Étampes, and also a benefice. Science, however, offered him a more congenial career, and he gained from his father permission to remain in Paris, and to attend the lectures at the Collège de France and the Jardin des Plantes, on the condition that he should also read law. He accordingly took up his residence at Cardinal Lemoine's college, and there became the pupil and soon the esteemed associate of Brisson's friend, the abbé Haüy, the mineralogist. Having, before the close of the year 1790, taken the degree of bachelor in law, he became a student of medicine, and attended the lectures of A.F. de Fourcroy at the Jardin des Plantes, and of L.J.M. Daubenton at the Collège de France. His studies at Paris were at length suddenly interrupted, for, in August 1792, Haüy and the other professors of Lemoine's college, as also those of the college of Navarre, were arrested by the revolutionists as priests, and confined in the prison of St Firmin. Through the influence of Daubenton and others Geoffroy on the 14th of August obtained an order for the release of Haüy in the name of the Academy; still the other professors of the two colleges, save C.F. Lhomond, who had been rescued by his pupil J.L. Tallien, remained in confinement. Geoffroy, foreseeing their certain destruction if they remained in the hands of the revolutionists, determined if possible to secure their liberty by stratagem. By bribing one of the officials at St Firmin, and disguising himself as a commissioner of prisons, he gained admission to his friends, and entreated them to effect their escape by following him. All, however, dreading lest their deliverance should render the doom of their fellow-captives the more certain, refused the offer, and one priest only, who was unknown to Geoffroy, left the prison. Already on the night of the 2nd of September the massacre of the proscribed had begun, when Geoffroy, yet intent on saving the life of his friends and teachers, repaired to St Firmin. At 4 o'clock on the morning of the 3rd of September, after eight hours' waiting, he by means of a ladder assisted the escape of twelve ecclesiastics, not of the number of his acquaintance, and then the approach of dawn and the discharge of a gun directed at him warned him, his chief purpose unaccomplished, to return to his lodgings. Leaving Paris he retired to Étampes, where, in consequence of the anxieties of which he had lately been the prey, and the horrors which he had witnessed, he was for some time seriously ill. At the beginning of the winter of 1792 he returned to his studies in Paris, and in March of the following year Daubenton, through the interest of Bernardin de Saint Pierre, procured him the office of sub-keeper and assistant demonstrator of the cabinet of natural history, vacant by the resignation of B.G.E. Lacépède. By a law passed in June 1793, Geoffroy was appointed one of the twelve professors of the newly constituted museum of natural history, being assigned the chair of zoology. In the same year he busied himself with the formation of a menagerie at that institution.

In 1794 through the introduction of A.H. Tessier he entered into correspondence with Georges Cuvier, to whom, after the perusal of some of his manuscripts, he wrote: "Venez jouer parmi nous le rôle de Linné, d'un autre législateur de l'histoire naturelle." Shortly after the appointment of Cuvier as assistant at the Muséum d'Histoire Naturelle, Geoffroy received him into his house. The two friends wrote together five memoirs on natural history, one of which, on the classification of mammals, puts forward the idea of the subordination of characters upon which Cuvier based his zoological system. It was in a paper entitled "Histoire des Makis, ou singes de Madagascar," written in 1795, that Geoffroy first gave expression to his views on "the unity of organic composition," the influence of which is perceptible in all his subsequent writings; nature, he observes, presents us with only one plan of construction, the same in principle, but varied in its accessory parts.

In 1798 Geoffroy was chosen a member of the great scientific expedition to Egypt, and on the capitulation of Alexandria in August 1801, he took part in resisting the claim made by the British general to the collections of the expedition, declaring that, were that demand persisted in, history would have to record that he also had burnt a library in Alexandria. Early in January 1802 Geoffroy returned to his accustomed labours in Paris. He was elected a member of the academy of sciences of that city in September 1807. In March of the following year the emperor, who had already recognized his national services by the award of the cross of the legion of honour, selected him to visit the museums of Portugal, for the purpose of procuring collections from them, and in the face of considerable opposition from the British he eventually was successful in retaining them as a permanent possession for his country. In 1809, the year after his return to France, he was made professor of zoology at the faculty of sciences at Paris, and from that period he devoted himself more exclusively than before to anatomical study. In 1818 he gave to the world the first part of his celebrated *Philosophie anatomique*, the second volume of which, published in 1822, and subsequent memoirs account for the formation of monstrosities on the principle of arrest of development, and of the attraction of similar parts. When, in 1830, Geoffroy proceeded to apply to the invertebrata his views as to the unity of animal composition, he found a vigorous opponent in Georges Cuvier, and the discussion between them, continued up to the time of the death of the latter, soon attracted the attention of the scientific throughout Europe. Geoffroy, a synthesist, contended, in accordance with his theory of unity of plan in organic composition, that all animals are formed of the same elements, in the same number, and with the same connexions: homologous parts, however they differ in form and size, must remain associated in the same invariable order. With Goethe he held that there is in nature a law of compensation or balancing of growth, so that if one organ take on an excess of development, it is at the expense of some other part; and he maintained that, since nature takes no sudden leaps, even organs which are superfluous in any given species, if they have played an important part in other species of the same family, are retained as rudiments, which testify to the permanence of the general plan of creation. It was his conviction that, owing to the conditions of life, the same forms had not been perpetuated since the origin of all things, although it was not his belief that existing species are becoming modified. Cuvier, who was an analytical observer of facts, admitted only the prevalence of "laws of co-existence" or "harmony" in animal organs, and maintained the absolute invariability of species, which he declared had been created with a regard to the circumstances in which they were placed, each organ contrived with a view to the function it had to fulfil, thus putting, in Geoffroy's considerations, the effect for the cause. 619

In July 1840 Geoffroy became blind, and some months later he had a paralytic attack. From that time his strength gradually failed him. He resigned his chair at the museum in 1841, and died at Paris on the 19th of June 1844.

Geoffroy wrote: *Catalogue des mammifères du Muséum National d'Histoire Naturelle* (1813), not quite completed; *Philosophie anatomique*—t. i., *Des organes respiratoires* (1818), and t. ii., *Des monstruosités humaines* (1822); *Système dentaire des mammifères et des oiseaux* (1st pt., 1824); *Sur le principe de l'unité de composition organique* (1828); *Cours de l'histoire naturelle des mammifères* (1829); *Principes de philosophie zoologique* (1830); *Études progressives d'un naturaliste* (1835); *Fragments biographiques* (1832); *Notions synthétiques, historiques et physiologiques de philosophie naturelle* (1838), and other works; also part of the *Description de l'Égypte par la commission des sciences* (1821-1830); and, with Frédéric Cuvier (1773-1838), a younger brother of G. Cuvier, *Histoire naturelle des mammifères* (4 vols., 1820-1842); besides numerous papers on such subjects as the anatomy of marsupials, ruminants and electrical fishes, the vertebrate theory of the skull, the opercula of fishes, teratology, palaeontology and the influence of surrounding conditions in modifying animal forms.

See *Vie, travaux, et doctrine scientifique d'Étienne Geoffroy Saint-Hilaire, par son fils M. Isidore Geoffroy Saint-Hilaire* (Paris and Strasburg, 1847), to which is appended a list of Geoffroy's works; and Joly, in *Biog. universelle*, t. xvi. (1856).

GEOFFROY SAINT-HILAIRE, ISIDORE (1805-1861), French zoologist, son of the preceding, was born at Paris on the 16th of December 1805. In his earlier years he showed an aptitude for mathematics, but eventually he devoted himself to the study of natural history and of medicine, and in 1824 he was appointed assistant naturalist to his father. On the occasion of his taking the degree of doctor of medicine in September 1829, he read a thesis entitled *Propositions sur la monstruosité, considérée chez l'homme et les animaux*; and in 1832-1837 was published his great teratological work, *Histoire générale et particulière des anomalies de l'organisation chez l'homme et les animaux*, 3 vols. 8vo. with 20 plates. In 1829 he delivered for his father the second part of a course of lectures on ornithology, and during the three following years he taught zoology at the Athénée, and teratology at the École pratique. He was elected a member of the academy of sciences at Paris in 1833, was in 1837 appointed to act as deputy for his father at the faculty of sciences in Paris, and in the following year was sent to Bordeaux to organize a similar faculty there. He became successively inspector of the academy of Paris (1840), professor of the museum on the retirement of his father (1841), inspector-general of the university (1844), a member of the royal council for public instruction (1845), and on the death of H.M.D. de Blainville, professor of zoology at the faculty of sciences (1850). In 1854 he founded the Acclimatization Society of Paris, of which he was president. He died at Paris on the 10th of November 1861.

Besides the above-mentioned works, he wrote: *Essais de zoologie générale* (1841); *Vie ... d'Étienne Geoffroy Saint-Hilaire* (1847); *Acclimatation et domestication des animaux utiles* (1849; 4th ed., 1861); *Lettres sur les substances alimentaires et particulièrement sur la viande de cheval* (1856); and *Histoire naturelle générale des règnes organiques* (3 vols., 1854-1862), which was not quite completed. He was the author also of various papers on zoology, comparative anatomy and palaeontology.

GEOGRAPHY (Gr. γῆ, earth, and γράφειν, to write), the exact and organized knowledge of the distribution of phenomena on the surface of the earth. The fundamental basis of geography is the vertical relief of the earth's crust, which controls all mobile distributions. The grander features of the relief of the lithosphere or stony crust of the earth control the distribution of the hydrosphere or collected waters which gather into the hollows, filling them up to a height corresponding to the volume, and thus producing the important practical division of the surface into land and water. The distribution of the mass of the atmosphere over the surface of the earth is also controlled by the relief of the crust, its greater or lesser density at the surface corresponding to the lesser or greater elevation of the surface. The simplicity of the zonal distribution of solar energy on the earth's surface, which would characterize a uniform globe, is entirely destroyed by the dissimilar action of land and water with regard to radiant heat, and by the influence of crust-forms on the direction of the resulting circulation. The influence of physical environment becomes clearer and stronger when the distribution of plant and animal life is considered, and if it is less distinct in the case of man, the reason is found in the modifications of environment consciously produced by human effort. Geography is a synthetic science, dependent for the data with which it deals on the results of specialized sciences such as astronomy, geology, oceanography, meteorology, biology and anthropology, as well as on topographical description. The physical and natural sciences are concerned in geography only so far as they deal with the forms of the earth's surface, or as regards the distribution of phenomena. The distinctive task of geography as a science is to investigate the control exercised by the crust-forms directly or indirectly upon the various mobile distributions. This gives to it unity and definiteness, and renders superfluous the attempts that have been made from time to time to define the limits which divide geography from geology on the one hand and from history on the other. It is essential to classify the subject-matter of geography in such a manner as to give prominence not only to facts, but to their mutual relations and their natural and inevitable order.

The fundamental conception of geography is form, including the figure of the earth and the varieties of crustal relief. Hence mathematical geography (see Map), including cartography as a practical application, comes first. It merges into physical geography, which takes account of the forms of the lithosphere (geomorphology), and also of the distribution of the hydrosphere and the rearrangements resulting from the workings of solar energy 620 throughout the hydrosphere and atmosphere (oceanography and climatology). Next follows the distribution of plants and animals (biogeography), and finally the distribution of mankind and the various artificial boundaries and redistributions (anthropogeography). The applications of anthropogeography to human uses give rise to political and commercial geography, in the elucidation of which all the earlier departments or stages have to be considered, together with historical and other purely human conditions. The evolutionary idea has revolutionized and unified geography as it did biology, breaking down the old hard-and-fast partitions between the various departments, and substituting the study of the nature and influence of actual terrestrial environments for the earlier motive, the discovery and exploration of new lands.

History of Geographical Theory

The earliest conceptions of the earth, like those held by the primitive peoples of the present day, are difficult to discover and almost impossible fully to grasp. Early generalizations, as far as they were made from known facts, were usually expressed in symbolic language, and for our present purpose it is not profitable to speculate on the underlying truths which may sometimes be suspected in the old mythological cosmogonies.

Early Greek ideas. Flat earth of Homer.

The first definite geographical theories to affect the western world were those evolved, or at least first expressed, by the Greeks.[1] The earliest theoretical problem of geography was the form of the earth. The natural supposition that the earth is a flat disk, circular or elliptical in outline, had in the time of Homer acquired a special definiteness by the introduction of the idea of the ocean river bounding the whole, an application of imperfectly understood observations. Thales of Miletus is claimed as the first exponent of the idea of a spherical earth; but, although this does not appear to be warranted, his disciple Anaximander (*c.* 580 B.C.) put forward the theory that the earth had the figure of a solid body hanging freely in the centre of the hollow sphere of the starry heavens. The Pythagorean school of philosophers adopted the theory of a spherical earth, but from metaphysical rather than scientific reasons; their convincing argument was that a sphere being the most perfect solid figure was the only one worthy to circumscribe the dwelling-place of man. The division of the sphere into parallel zones and some of the consequences of this generalization seem to have presented themselves to Parmenides (*c.* 450 B.C.); but these ideas did not influence the Ionian school of philosophers, who in their treatment of geography preferred to deal with facts demonstrable by travel rather than with speculations. Thus Hecataeus, claimed by H.F. Tozer[2] as the father of geography on account of his *Periodos*, or general treatise on the earth, did not advance beyond the primitive conception of a circular disk. He systematized the form of the land within the ring of ocean—the οἰκουμένη, or habitable world—by recognizing two continents: Europe to the north, and Asia to the south of the midland sea. Herodotus, equally oblivious of the sphere, criticized and ridiculed the circular outline of the *oekumene*, which he knew to be longer from east to west than it was broad from north to south. He also pointed out reasons for accepting a division of the land into three continents—Europe, Asia and Africa. Beyond the limits of his personal travels Herodotus applied the characteristically Greek theory of symmetry to complete, in the unknown, outlines of lands and rivers analogous to those which had been explored. Symmetry was in fact the first geographical theory, and the effect of Herodotus's hypothesis that the Nile must flow from west to east before turning north in order to balance the Danube running from west to east before turning south lingered in the maps of Africa down to the time of Mungo Park.[3]

Hecataeus.

Herodotus.

The idea of symmetry.

Aristotle and the sphere.

To Aristotle (384-322 B.C.) must be given the distinction of founding scientific geography. He demonstrated the sphericity of the earth by three arguments, two of which could be tested by observation. These were: (1) that the earth must be spherical, because of the tendency of matter to fall together towards a common centre; (2) that only a sphere could always throw a circular shadow on the moon during an eclipse; and (3) that the shifting of the horizon and the appearance of new constellations, or the disappearance of familiar stars, as one travelled from north to south, could only be explained on the hypothesis that the earth was a sphere. Aristotle, too, gave greater definiteness to the idea of zones conceived by Parmenides, who had pictured a torrid zone uninhabitable by reason of heat, two frigid zones uninhabitable by reason of cold, and two intermediate temperate zones fit for human occupation. Aristotle defined the temperate zone as extending from the tropic to the arctic circle, but there is some uncertainty as to the precise meaning he gave to the term "arctic circle." Soon after his time, however, this conception was clearly established, and with so large a generalization the mental horizon was widened to conceive of a geography which was a science. Aristotle had himself shown that in the southern temperate zone winds similar to those of the northern temperate zone should blow, but from the opposite direction.

Fitting the oekumene to the sphere.

While the theory of the sphere was being elaborated the efforts of practical geographers were steadily directed towards ascertaining the outline and configuration of the *oekumene*, or habitable world, the only portion of the terrestrial surface known to the ancients and to the medieval peoples, and still retaining a shadow of its old monopoly of geographical attention in its modern name of the “Old World.” The fitting of the *oekumene* to the sphere was the second theoretical problem. The circular outline had given way in geographical opinion to the elliptical with the long axis lying east and west, and Aristotle was inclined to view it as a very long and relatively narrow band almost encircling the globe in the temperate zone. His argument as to the narrowness of the sea between West Africa and East Asia, from the occurrence of elephants at both extremities, is difficult to understand, although it shows that he looked on the distribution of animals as a problem of geography.

Problem of the Antipodes.

Pythagoras had speculated as to the existence of antipodes, but it was not until the first approximately accurate measurements of the globe and estimates of the length and breadth of the *oekumene* were made by Eratosthenes (*c.* 250 B.C.) that the fact that, as then known, it occupied less than a quarter of the surface of the sphere was clearly recognized. It was natural, if not strictly logical, that the ocean river should be extended from a narrow stream to a world-embracing sea, and here again Greek theory, or rather fancy, gave its modern name to the greatest feature of the globe. The old instinctive idea of symmetry must often have suggested other *oekumene* balancing the known world in the other quarters of the globe. The Stoic philosophers, especially Crates of Mallus, arguing from the love of nature for life, placed an *oekumene* in each quarter of the sphere, the three unknown world-islands being those of the Antoeci, Perioeci and Antipodes. This was a theory not only attractive to the philosophical mind, but eminently adapted to promote exploration. It had its opponents, however, for Herodotus showed that sea-basins existed cut off from the ocean, and it is still a matter of controversy how far the pre-Ptolemaic geographers believed in a water-connexion between the Atlantic and Indian oceans. It is quite clear that Pomponius Mela (*c.* A.D. 40), following Strabo, held that the southern temperate zone contained a habitable land, which he designated by the name *Antichthones*.

Aristotle’s geographical views.

Aristotle left no work on geography, so that it is impossible to know what facts he associated with the science of the earth’s surface. The word geography did not appear before Aristotle, the first use of it being in the Περὶ κόσμων, which is one of the writings doubtfully ascribed to him, and H. Berger considers that the expression was introduced by Eratosthenes.[4] Aristotle was certainly conversant with many facts, such as the formation of deltas, coast-erosion, and to a certain extent the dependence of plants and animals on their physical surroundings. He formed a comprehensive theory of the variations of climate with latitude and season, and was convinced of the necessity of a circulation of water between the sea and rivers, though, like Plato, he held that this took place by water rising from the sea through crevices in the rocks, losing its dissolved salts in the process. He speculated on the differences in the character of races of mankind living in different climates, and correlated the political forms of communities with their situation on a seashore, or in the neighbourhood of natural strongholds.

Strabo.

Strabo (*c.* 50 B.C.-A.D. 24) followed Eratosthenes rather than Aristotle, but with sympathies which went out more to the human interests than the mathematical basis of geography. He compiled a very remarkable work dealing, in large measure from personal travel, with the countries surrounding the Mediterranean. He may be said to have set the pattern which was followed in succeeding ages by the compilers of “political geographies” dealing less with theories than with facts, and illustrating rather than formulating the principles of the science. 621

Ptolemy.

Claudius Ptolemaeus (*c.* A.D. 150) concentrated in his writings the final outcome of all Greek geographical learning, and passed it across the gulf of the middle ages by the hands of the Arabs, to form the starting-point of the science in modern times. His geography was based more immediately on the work of his predecessor, Marinus of Tyre, and on that of Hipparchus, the follower and critic of Eratosthenes. It was the ambition of Ptolemy to describe and represent accurately the surface of the *oekumene*, for which purpose he took immense trouble to collect all existing determinations of the latitude of places, all estimates of longitude, and to make every possible rectification in the estimates of distances by land or sea. His work was mainly cartographical in its aim, and theory was as far as possible excluded. The symmetrically placed hypothetical islands in the great continuous ocean disappeared, and the *oekumene* acquired a new form by the representation of the Indian Ocean as a larger Mediterranean completely cut off by land from the Atlantic. The *terra incognita* uniting Africa and Farther Asia was an unfortunate hypothesis which helped to retard exploration. Ptolemy used the word *geography* to signify the description of the whole *oekumene* on mathematical principles, while *chorography* signified the fuller description of a particular region, and *topography* the very detailed description of a smaller locality. He introduced the simile that geography represented an artist's sketch of a whole portrait, while chorography corresponded to the careful and detailed drawing of an eye or an ear.[5]

The Caliph al-Mamūn (*c.* A.D. 815), the son and successor of Hārūn al-Rashīd, caused an Arabic version of Ptolemy's great astronomical work (Σύνταξις μεγίστη) to be made, which is known as the *Almagest*, the word being nothing more than the Gr. μεγίστη with the Arabic article *al* prefixed. The geography of Ptolemy was also known and is constantly referred to by Arab writers. The Arab astronomers measured a degree on the plains of Mesopotamia, thereby deducing a fair approximation to the size of the earth. The caliph's librarian, Abu Jafar Muhammad Ben Musa, wrote a geographical work, now unfortunately lost, entitled *Rasm el Arsi* ("A Description of the World"), which is often referred to by subsequent writers as having been composed on the model of that of Ptolemy.

Geography in the middle ages.

The middle ages saw geographical knowledge die out in Christendom, although it retained, through the Arabic translations of Ptolemy, a certain vitality in Islam. The verbal interpretation of Scripture led Lactantius (*c.* A.D. 320) and other ecclesiastics to denounce the spherical theory of the earth as heretical. The wretched subterfuge of Cosmas (*c.* A.D. 550) to explain the phenomena of the apparent movements of the sun by means of an earth modelled on the plan of the Jewish Tabernacle gave place ultimately to the wheel-maps—the T in an O—which reverted to the primitive ignorance of the times of Homer and Hecataeus.[6]

The journey of Marco Polo, the increasing trade to the East and the voyages of the Arabs in the Indian Ocean prepared the way for the reacceptance of Ptolemy's ideas when the sealed books of the Greek original were translated into Latin by Angelus in 1410.

Revival of geography.

The old arguments of Aristotle and the old measurements of Ptolemy were used by Toscanelli and Columbus in urging a westward voyage to India; and mainly on this account did the crossing of the Atlantic rank higher in the history of scientific geography than the laborious feeling out of the coast-line of Africa. But not until the voyage of Magellan shook the scales from the eyes of Europe did modern geography begin to advance. Discovery had outrun theory; the rush of new facts made Ptolemy practically obsolete in a generation, after having been the fount and origin of all geography for a millennium.

Apianus.

The earliest evidence of the reincarnation of a sound theoretical geography is to be found in the text-books by Peter Apian and Sebastian Münster. Apian in his *Cosmographicus liber*, published in 1524, and subsequently edited and added to by Gemma Frisius under the title of *Cosmographia*, based the whole science on mathematics and measurement. He followed Ptolemy closely, enlarging on his distinction between geography and chorography, and expressing the artistic analogy in a rough diagram. This slender distinction was made much of by most subsequent writers until Nathanael Carpenter in 1625 pointed out that the difference between geography and chorography was simply one of degree, not of kind.

Münster.

Sebastian Münster, on the other hand, in his *Cosmographia universalis* of 1544, paid no regard to the mathematical basis of geography, but, following the model of Strabo, described the world according to its different political divisions, and entered with great zest into the question of the productions of countries, and into the manners and costumes of the various peoples. Thus early commenced the separation between what were long called mathematical and political geography, the one subject appealing mainly to mathematicians, the other to historians.

Throughout the 16th and 17th centuries the rapidly accumulating store of facts as to the extent, outline and mountain and river systems of the lands of the earth were put in order by the generation of cartographers of which Mercator was the chief; but the writings of Apian and Münster held the field for a hundred years without a serious rival, unless the many annotated editions of Ptolemy might be so considered. Meanwhile the new facts were the subject of original study by philosophers and by practical men without reference to classical traditions. Bacon argued keenly on geographical matters and was a lover of maps, in which he observed and reasoned upon such resemblances as that between the outlines of South America and Africa.

Philip Cluver's *Introductio in geographiam universam tam veterem quam novam* was published in 1624. Geography he defined as "the description of the whole earth, so far as it

Cluverius.

is known to us.” It is distinguished from cosmography by dealing with the earth alone, not with the universe, and from chorography and topography by dealing with the whole earth, not with a country or a place. The first book, of fourteen short chapters, is concerned with the general properties of the globe; the remaining six books treat in considerable detail of the countries of Europe and of the other continents. Each country is described with particular regard to its people as well as to its surface, and the prominence given to the human element is of special interest.

Carpenter.

A little-known book which appears to have escaped the attention of most writers on the history of modern geography was published at Oxford in 1625 by Nathanael Carpenter, fellow of Exeter College, with the title *Geographie delineated forth in Two Bookes, containing the Sphericall and Topicall parts thereof*. It is discursive in its style and verbose; but, considering the period at which it appeared, it is remarkable for the strong common sense displayed by the author, his comparative freedom from prejudice, and his firm application of the methods of scientific reasoning to the interpretation of phenomena. Basing his work on the principles of Ptolemy, he brings together illustrations from the most recent travellers, and does not hesitate to take as illustrative examples the familiar city of Oxford and his native county of Devon. He divides geography into *The Spherical Part*, or that for the study of which mathematics alone is required, and *The Topical Part*, or the description of the physical relations of parts of the earth’s surface, preferring this division to that favoured by the ancient geographers—into general and special. It is distinguished from other English geographical books of the period by confining attention to the principles of geography, and not describing the countries of the world.

Varenius.

A much more important work in the history of geographical method is the *Geographia generalis* of Bernhard Varenius, a German medical doctor of Leiden, who died at the age of twenty-eight in 1650, the year of the publication of his book. Although for a time it was lost sight of on the continent, Sir Isaac Newton thought so highly of this book that he prepared an annotated edition which was published in Cambridge in 1672, with the addition of the plates which had been planned by Varenius, but not produced by the original publishers. “The reason why this great man took so much care in correcting and publishing our author was, because he thought him necessary to be read by his audience, the young gentlemen of Cambridge, while he was delivering lectures on the same subject from the Lucasian Chair.”[7] The treatise of Varenius is a model of logical arrangement and terse expression; it is a work of science and of genius; one of the few of that age which can still be studied with profit. The English translation renders the definition thus: “Geography is that part of *mixed mathematics* which explains the state of the earth and of its parts, depending on quantity, viz. its figure, place, magnitude and motion, with the celestial appearances, &c. By some it is taken in too limited a sense, for a bare description of the several countries; and by others too extensively, who along with such a description would have their political constitution.”

Varenius was reluctant to include the human side of geography in his system, and only allowed it as a concession to custom, and in order to attract readers by imparting interest to the sterner details of the science. His division of geography was into two parts—(i.) General or universal, dealing with the earth in general, and explaining its properties without regard to particular countries; and (ii.) Special or particular, dealing with each country in turn from the chorographical or topographical point of view. General geography was divided into—(1) the *Absolute* part, dealing with the form, dimensions, position and substance of the earth, the distribution of land and water, mountains, woods and deserts, hydrography (including all the waters of the earth) and the atmosphere; (2) the *Relative* part, including the celestial properties, *i.e.* latitude, climate zones, longitude, &c.; and (3) the *Comparative* part, which “considers the particulars arising from comparing one part with another”; but under this head the questions discussed were longitude, the situation and distances of places, and navigation. Varenius does not treat of special geography, but gives a scheme for it under three heads—(1) *Terrestrial*, including position, outline, boundaries, mountains, mines, woods and deserts, waters, fertility and fruits, and living creatures; (2) *Celestial*, including appearance of the heavens and the climate; (3) *Human*, but this was added out of deference to popular usage. 622

This system of geography founded a new epoch, and the book—translated into English, Dutch and French—was the unchallenged standard for more than a century. The framework was capable of accommodating itself to new facts, and was indeed far in advance of the knowledge of the period. The method included a recognition of the causes and effects of phenomena as well as the mere fact of their occurrence, and for the first time the importance of the vertical relief of the land was fairly recognized.

The physical side of geography continued to be elaborated after Varenius’s methods, while the historical side was developed separately. Both branches, although enriched by new facts, remained stationary so far as method is concerned until nearly the end of the 18th century. The compilation of “geography books” by uninstructed writers led to the pernicious habit, which is not yet wholly overcome, of reducing the general or “physical” part to a few pages of concentrated information, and expanding the particular or “political” part by including unrevised travellers’ stories and uncritical descriptions of the various countries of the world. Such books were in fact not geography, but merely compressed travel.

Bergman.

The next marked advance in the theory of geography may be taken as the nearly simultaneous studies of the physical earth carried out by the Swedish chemist, Torbern Bergman, acting under the impulse of Linnaeus, and by the German philosopher, Immanuel Kant. Bergman’s *Physical Description of the Earth* was published in Swedish in 1766, and translated into English in 1772 and into German in 1774. It is a plain, straightforward description of the globe, and of the various phenomena of the surface, dealing only with definitely ascertained facts in the natural order of their relationships, but avoiding any systematic classification or even definitions of terms.

Kant.

The problems of geography had been lightened by the destructive criticism of the French cartographer D’Anville (who had purged the map of the world of the last remnants of traditional fact unverified by modern observations) and rendered richer by the dawn of the new era of scientific travel, when Kant brought his logical powers to bear upon them. Kant’s lectures on physical geography were delivered in the university of Königsberg from 1765 onwards.[8] Geography appealed to him as a valuable educational discipline, the joint foundation with anthropology of that “knowledge of the world” which was the result of reason and experience. In this connexion he divided the communication of experience from one person to another into two categories—the narrative or historical and the descriptive or geographical; both history and geography being viewed as descriptions, the former a description in order of time, the latter a description in order of space.

Physical geography he viewed as a summary of nature, the basis not only of history but also of “all the other possible geographies,” of which he enumerates five, viz. (1) *Mathematical geography*, which deals with the form, size and movements of the earth and its place in the solar system; (2) *Moral geography*, or an account of the different customs and characters of mankind according to the region they inhabit; (3) *Political geography*, the divisions according to their organized governments; (4) *Mercantile geography*, dealing with the trade in the surplus products of countries; (5) *Theological geography*, or the distribution of religions. Here there is a clear and formal statement of the interaction and causal relation of all the phenomena of distribution on the earth’s surface, including the influence of physical geography upon the various activities of mankind from the lowest to the highest. Notwithstanding the form of this classification, Kant himself treats mathematical geography as preliminary to, and therefore not dependent on, physical geography. Physical geography itself is divided into two parts: a general, which has to do with the earth and all that belongs to it—water, air and land; and a particular, which deals with special products of the earth—mankind, animals, plants and minerals. Particular importance is given to the vertical relief of the land, on which the various branches of human geography are shown to depend.

Humboldt.

Alexander von Humboldt (1769-1859) was the first modern geographer to become a great traveller, and thus to acquire an extensive stock of first-hand information on which an improved system of geography might be founded. The impulse given to the study of natural history by the example of Linnaeus; the results brought back by Sir Joseph Banks, Dr Solander and the two Forsters, who accompanied Cook in his voyages of discovery; the studies of De Saussure in the Alps, and the lists of desiderata in physical geography drawn up by that investigator, combined to prepare the way for Humboldt. The theory of geography was advanced by Humboldt mainly by his insistence on the great principle of the unity of nature. He brought all the “observable things,” which the eager collectors of the previous century had been heaping together regardless of order or system, into relation with the vertical relief and the horizontal forms of the earth’s surface. Thus he demonstrated that the forms of the land exercise a directive and determining influence on climate, plant life, animal life and on man himself. This was no new idea; it had been familiar for centuries in a less definite form, deduced from a priori considerations, and so far as regards the influence of surrounding circumstances upon man, Kant had already given it full expression. Humboldt’s concrete illustrations and the remarkable power of his personality enabled him to enforce these principles in a way that produced an immediate and lasting effect. The treatises on physical geography by Mrs Mary Somerville and Sir John Herschel (the latter written for the eighth edition of the *Encyclopaedia Britannica*) showed the effect produced in Great Britain by the stimulus of Humboldt’s work.

Humboldt’s contemporary, Carl Ritter (1779-1859), extended and disseminated the same views, and in his interpretation of “Comparative Geography” he laid stress on the importance of forming conclusions, not from the study of one region by itself, but from the comparison of the phenomena of many places. Impressed by the influence of terrestrial relief and climate on human movements, Ritter was led deeper and deeper into the study of history and archaeology. His monumental *Vergleichende Geographie*, which was to have made the whole world its theme, died out in a wilderness of detail in twenty-one volumes before it had covered more of the earth’s surface than Asia and a portion of Africa. Some of his followers showed a tendency to look on geography rather as an auxiliary to history than as a study of intrinsic worth.

Ritter.

During the rapid development of physical geography many branches of the study of nature, which had been included in the cosmography of the early writers, the physiography of Linnaeus and even the *Erdkunde* of Ritter, had been so much advanced by the labours of specialists that their connexion was apt to be forgotten. Thus geology, meteorology, oceanography and anthropology developed into distinct sciences. The absurd attempt was, and sometimes is still, made by geographers to include all natural science in geography; but it is more common for specialists in the various detailed sciences to think, and sometimes to assert, that the ground of physical geography is now fully occupied by these sciences. Political geography has been too often looked on from both sides as a mere summary of guide-book knowledge, useful in the schoolroom, a poor relation of physical geography that it was rarely necessary to recognize.

Geography as a natural science.

The science of geography, passed on from antiquity by Ptolemy, re-established by Varenius and Newton, and systematized by Kant, included within itself definite aspects of all those terrestrial phenomena which are now treated exhaustively under the heads of geology, meteorology, oceanography and anthropology; and the inclusion of the requisite portions of the perfected results of these sciences in geography is simply the gathering in of fruit matured from the seed scattered by geography itself.

The study of geography was advanced by improvements in cartography (see Map), not only in the methods of survey and projection, but in the representation of the third dimension by means of contour lines introduced by Philippe Buache in 1737, and the more remarkable because less obvious invention of isotherms introduced by Humboldt in 1817.

The “argument from design” had been a favourite form of reasoning amongst Christian theologians, and, as worked out by Paley in his *Natural Theology*, it served the useful purpose of emphasizing the fitness which exists between all the inhabitants of the earth and their physical environment. It was held that the earth had been created so as to fit the wants of man in every particular. This argument was tacitly accepted or explicitly avowed by almost every writer on the theory of geography, and Carl Ritter distinctly recognized and adopted it as the unifying principle of his system. As a student of nature, however, he did not fail to see, and as professor of geography he always taught, that man was in very large measure conditioned by his physical environment. The apparent opposition of the observed fact to the assigned theory he overcame by looking upon the forms of the land and the arrangement of land and sea as instruments of Divine Providence for guiding the destiny as well as for supplying the requirements of man. This was the central theme of Ritter’s philosophy; his religion and his geography were one, and the consequent fervour with which he pursued his mission goes far to account for the immense influence he acquired in Germany.

The teleological argument in geography.

The evolutionary theory, more than hinted at in Kant’s “Physical Geography,” has, since the writings of Charles Darwin, become the unifying principle in geography. The conception of the development of the plan of the earth from the first cooling of the surface of the planet throughout the long geological periods, the guiding power of environment on the circulation of water and of air, on the distribution of plants and animals, and finally on the movements of man, give to geography a philosophical dignity and a scientific completeness 623 which it never previously possessed. The influence of environment on the organism may not be quite so potent as it was once believed to be, in the writings of Buckle, for instance,[9] and certainly man, the ultimate term in the series, reacts upon and greatly modifies his environment; yet the fact that environment does influence all distributions is established beyond the possibility of doubt. In this way also the position of geography, at the point where physical science meets and mingles with mental science, is explained and justified. The change which took place during the 19th century in the substance and style of geography may be well seen by comparing the eight volumes of Malte-Brun’s *Géographie universelle* (Paris, 1812-1829) with the twenty-one volumes of Reclus’s *Géographie universelle* (Paris, 1876-1895).

The theory of evolution in geography.

In estimating the influence of recent writers on geography it is usual to assign to Oscar Peschel (1826-1875) the credit of having corrected the preponderance which Ritter gave to the historical element, and of restoring physical geography to its old pre-eminence.[10] As a matter of fact, each of the leading modern exponents of theoretical geography—such as Ferdinand von Richthofen, Hermann Wagner, Friedrich Ratzel, William M. Davis, A. Penck, A. de Lapparent and Elisée Reclus—has his individual point of view, one devoting more attention to the results of geological processes, another to anthropological conditions, and the rest viewing the subject in various blendings of the extreme lights.

The two conceptions which may now be said to animate the theory of geography are the genetic, which depends upon processes of origin, and the morphological, which depends on facts of form and distribution.

Progress of Geographical Discovery

Exploration and geographical discovery must have started from more than one centre, and to deal justly with the matter one ought to treat of these separately in the early ages before the whole civilized world was bound together by the bonds of modern intercommunication. At the least there should be some consideration of four separate systems of discovery—the Eastern, in which Chinese and Japanese explorers acquired knowledge of the geography of Asia, and felt their way towards Europe and America; the Western, in which the dominant races of the Mexican and South American plateaus extended their knowledge of the American continent before Columbus; the Polynesian, in which the conquering races of the Pacific Islands found their way from group to group; and the Mediterranean. For some of these we have no certain information, and regarding others the tales narrated in the early records are so hard to reconcile with present knowledge that they are better fitted to be the battle-ground of scholars championing rival theories than the basis of definite history. So it has come about that the only practicable history of geographical exploration starts from the Mediterranean centre, the first home of that civilization which has come to be known as European, though its field of activity has long since overspread the habitable land of both temperate zones, eastern Asia alone in part excepted.

From all centres the leading motives of exploration were probably the same—commercial intercourse, warlike operations, whether resulting in conquest or in flight, religious zeal expressed in pilgrimages or missionary journeys, or, from the other side, the avoidance of persecution, and, more particularly in later years, the advancement of knowledge for its own sake. At different times one or the other motive predominated.

Before the 14th century B.C. the warrior kings of Egypt had carried the power of their arms southward from the delta of the Nile well-nigh to its source, and eastward to the confines of Assyria. The hieroglyphic inscriptions of Egypt and the cuneiform inscriptions of Assyria are rich in records of the movements and achievements of armies, the conquest of towns and the subjugation of peoples; but though many of the recorded sites have been identified, their discovery by wandering armies was isolated from their subsequent history and need not concern us here.

The Phoenicians.

The Phoenicians are the earliest Mediterranean people in the consecutive chain of geographical discovery which joins pre-historic time with the present. From Sidon, and later from its more famous rival Tyre, the merchant adventurers of Phoenicia explored and colonized the coasts of the Mediterranean and fared forth into the ocean beyond. They traded also on the Red sea, and opened up regular traffic with India as well as with the ports of the south and west, so that it was natural for Solomon to employ the merchant navies of Tyre in his oversea trade. The western emporium known in the scriptures as Tarshish was probably situated in the south of Spain, possibly at Cadiz, although some writers contend that it was Carthage in North Africa. Still more diversity of opinion prevails as to the southern gold-exporting port of Ophir, which some scholars place in Arabia, others at one or another point on the east coast of Africa. Whether associated with the exploitation of Ophir (*q.v.*) or not the first great voyage of African discovery appears to have been accomplished by the Phoenicians sailing the Red Sea. Herodotus (himself a notable traveller in the 5th century B.C.) relates that the Egyptian king Necho of the XXVIth Dynasty (*c.* 600 B.C.) built a fleet on the Red Sea, and confided it to Phoenician sailors with the orders to sail southward and return to Egypt by the Pillars of Hercules and the Mediterranean sea. According to the tradition, which Herodotus quotes sceptically, this was accomplished; but the story is too vague to be accepted as more than a possibility.

The great Phoenician colony of Carthage, founded before 800 B.C., perpetuated the commercial enterprise of the parent state, and extended the sphere of practical trade to the ocean shores of Africa and Europe. The most celebrated voyage of antiquity undertaken for the express purpose of discovery was that fitted out by the senate of Carthage under the command of Hanno, with the intention of founding new colonies along the west coast of Africa. According to Pliny, the only authority on this point, the period of the voyage was that of the greatest prosperity of Carthage, which may be taken as somewhere between 570 and 480 B.C. The extent of this voyage is doubtful, but it seems probable that the farthest point reached was on the east-running coast which bounds the Gulf of Guinea on the north. Himilco, a contemporary of Hanno, was charged with an expedition along the west coast of Iberia northward, and as far as the uncertain references to this voyage can be understood, he seems to have passed the Bay of Biscay and possibly sighted the coast of England.

The Greeks.

The sea power of the Greek communities on the coast of Asia Minor and in the Archipelago began to be a formidable rival to the Phoenician soon after the time of Hanno and Himilco, and peculiar interest attaches to the first recorded Greek voyage beyond the Pillars of Hercules. Pytheas, a navigator of the Phocean colony of Massilia (Marseilles), determined the latitude of that port with considerable precision by the somewhat clumsy method of ascertaining the length of the longest day, and when, about 330 B.C., he set out on exploration to the northward in search of the lands whence came gold, tin and amber, he followed this system of ascertaining his position from time to time. If on each occasion he himself made the observations his voyage must have extended over six years; but it is not impossible that he ascertained the approximate length of the longest day in some cases by questioning the natives. Pytheas, whose own narrative is not preserved, coasted the Bay of Biscay, sailed up the English Channel and followed the coast of Britain to its most northerly point. Beyond this he spoke of a land called *Thule*, which, if his estimate of the length of the longest day is correct, may have been Shetland, but was possibly Iceland; and from some confused statements as to a sea which could not be sailed through, it has been assumed that Pytheas was the first of the Greeks to obtain direct knowledge of the Arctic regions. During this or a second voyage Pytheas entered the Baltic, discovered the coasts where amber is obtained and returned to the Mediterranean. It does not seem that any maritime trade followed these discoveries, and indeed it is doubtful whether his contemporaries accepted the truth of Pytheas’s narrative; Strabo four hundred years later certainly did not, but the critical studies of modern scholars have rehabilitated the Massilian explorer.

Alexander the Great.

The Greco-Persian wars had made the remoter parts of Asia Minor more than a name to the Greek geographers before the time of Alexander the Great, but the campaigns of that conqueror from 329 to 325 B.C. opened up the greater Asia to the knowledge of Europe. His armies crossed the plains beyond the Caspian, penetrated the wild mountain passes north-west of India, and did not turn back until they had entered on the Indo-Gangetic plain. This was one of the few great epochs of geographical discovery.

The world was henceforth viewed as a very large place stretching far on every side beyond the Midland or Mediterranean Sea, and the land journey of Alexander resulted in a voyage of discovery in the outer ocean from the mouth of the Indus to that of the Tigris, thus opening direct intercourse between Grecian and Hindu civilization. The Greeks who accompanied Alexander described with care the towns and villages, the products and the aspect of the country. The conqueror also intended to open up trade by sea between Europe and India, and the narrative of his general Nearchus records this famous voyage of discovery, the detailed accounts of the chief pilot Onesicritus being lost. At the beginning of October 326 B.C. Nearchus left the Indus with his fleet, and the anchorages sought for each night are carefully recorded. He entered the Persian Gulf, and rejoined Alexander at Susa, when he was ordered to prepare another expedition for the circumnavigation of Arabia. Alexander died at Babylon in 323 B.C., and the fleet was dispersed without making the voyage.

The dynasties founded by Alexander’s generals, Seleucus, Antiochus and Ptolemy, encouraged the same spirit of enterprise which their master had fostered, and extended geographical knowledge in several directions. Seleucus Nicator established the Greco-Bactrian empire and continued the intercourse with India. Authentic information respecting the great valley of the Ganges was supplied by Megasthenes, an ambassador sent by Seleucus, who reached the remote city of Patali-putra, the modern Patna.

The Ptolemies.

The Ptolemies in Egypt showed equal anxiety to extend the bounds of geographical knowledge. Ptolemy Euergetes (247-222 B.C.) rendered the greatest service to geography by the protection and encouragement of Eratosthenes, whose labours gave the first approximate knowledge of the true size of the spherical earth. The second Euergetes and his successor Ptolemy Lathyrus (118-115 B.C.) furnished Eudoxus with a fleet to explore the Arabian sea. After two successful voyages, Eudoxus, impressed with the idea that Africa was surrounded by ocean on the south, left the Egyptian service, and proceeded to Cadiz and other Mediterranean centres of trade seeking a patron who would finance an expedition for the purpose of African discovery; and we learn from Strabo that the veteran explorer made at least two voyages southward along the coast of Africa. The Ptolemies continued to send fleets annually from their Red Sea ports of Berenice and Myos Hormus to Arabia, as well as to ports on the coasts of Africa and India.

The Romans.

The Romans did not encourage navigation and commerce with the same ardour as their predecessors; still the luxury of Rome, which gave rise to demands for the varied products of all the countries of the known world, led to an active trade both by ships and caravans. But it was the military genius of Rome, and the ambition for universal empire, which led, not only to the discovery, but also to the survey of nearly all Europe, and of large tracts in Asia and Africa. Every new war produced a new survey and itinerary of the countries which were conquered, and added one more to the imperishable roads that led from every quarter of the known world to Rome. In the height of their power the Romans had surveyed and explored all the coasts of the Mediterranean, Italy, Greece, the Balkan Peninsula, Spain, Gaul, western Germany and southern Britain. In Africa their empire included Egypt, Carthage, Numidia and Mauritania. In Asia they held Asia Minor and Syria, had sent expeditions into Arabia, and were acquainted with the more distant countries formerly invaded by Alexander, including Persia, Scythia, Bactria and India. Roman intercourse with India especially led to the extension of geographical knowledge.

Before the Roman legions were sent into a new region to extend the limits of the empire, it was usual to send out exploring expeditions to report as to the nature of the country. It is narrated by Pliny and Seneca that the emperor Nero sent out two centurions on such a mission towards the source of the Nile (probably about A.D. 60), and that the travellers pushed southwards until they reached vast marshes through which they could not make their way either on foot or in boats. This seems to indicate that they had penetrated to about 9° N. Shortly before A.D. 79 Hippalus took advantage of the regular alternation of the monsoons to make the voyage from the Red Sea to India across the open ocean out of sight of land. Even though this sea-route was known, the author of the *Periplus of the Erythraean Sea*, published after the time of Pliny, recites the old itinerary around the coast of the Arabian Gulf. It was, however, in the reigns of Severus and his immediate successors that Roman intercourse with India was at its height, and from the writings of Pausanias (*c.* 174) it appears that direct communication between Rome and China had already taken place.

After the division of the Roman empire, Constantinople became the last refuge of learning, arts and taste; while Alexandria continued to be the emporium whence were imported the commodities of the East. The emperor Justinian (483-565), in whose reign the greatness of the Eastern empire culminated, sent two Nestorian monks to China, who returned with eggs of the silkworm concealed in a hollow cane, and thus silk manufactures were established in the Peloponnesus and the Greek islands. It was also in the reign of Justinian that Cosmas Indicopleustes, an Egyptian merchant, made several voyages, and afterwards composed his Χριστιανικὴ τοπογραφία (Christian Topography), containing, in addition to his absurd cosmogony, a tolerable description of India.

The Arabs.

The great outburst of Mahommedan conquest in the 7th century was followed by the Arab civilization, having its centres at Bagdad and Cordova, in connexion with which geography again received a share of attention. The works of the ancient Greek geographers were translated into Arabic, and starting with a sound basis of theoretical knowledge, exploration once more made progress. From the 9th to the 13th century intelligent Arab travellers wrote accounts of what they had seen and heard in distant lands. The earliest Arabian traveller whose observations have come down to us is the merchant Sulaiman, who embarked in the Persian Gulf and made several voyages to India and China, in the middle of the 9th century. Abu Zaid also wrote on India, and his work is the most important that we possess before the epoch-making discoveries of Marco Polo. Masudi, a great traveller who knew from personal experience all the countries between Spain and China, described the plains, mountains and seas, the dynasties and peoples, in his *Meadows of Gold*, an abstract made by himself of his larger work *News of the Time*. He died in 956, and was known, from the comprehensiveness of his survey, as the Pliny of the East. Amongst his contemporaries were Istakhri, who travelled through all the Mahommedan countries and wrote his *Book of Climates* in 950, and Ibn Haukal, whose *Book of Roads and Kingdoms*, based on the work of Istakhri, was written in 976. Idrisi, the best known of the Arabian geographical authors, after travelling far and wide in the first half of the 12th century, settled in Sicily, where he wrote a treatise descriptive of an armillary sphere which he had constructed for Roger II., the Norman king, and in this work he incorporated all accessible results of contemporary travel.

The Northmen of Denmark and Norway, whose piratical adventures were the terror of all the coasts of Europe, and who established themselves in Great Britain and Ireland, in France and Sicily, were also geographical explorers in their rough but practical way during the

The Northmen.

darkest period of the middle ages. All Northmen were not bent on rapine and plunder; many were peaceful merchants. Alfred the Great, king of the Saxons in England, not only educated his people in the learning of the past ages; he inserted in the geographical works he translated many narratives of the travel of his own time. Thus he placed on record the voyages of the merchant Ulfsten in the Baltic, including particulars of the geography of Germany. And in particular he told of the remarkable voyage of Other, a Norwegian of Helgeland, who was the first authentic Arctic explorer, the first to tell of the rounding of the North Cape and the sight of the midnight sun. This voyage of the middle of the 9th century deserves to be held in happy memory, for it unites the first Norwegian polar explorer with the first English collector of travels. Scandinavian merchants brought the products of India to England and Ireland. From the 8th to the 11th century a commercial route from India passed through Novgorod to the Baltic, and Arabian coins found in Sweden, and particularly in the island of Gotland, prove how closely the enterprise of the Northmen and of the Arabs intertwined. Five-sixths of these coins preserved at Stockholm were from the mints of the Samanian dynasty, which reigned in Khorasan and Transoxiana from about A.D. 900 to 1000. It was the trade with the East that originally gave importance to the city of Visby in Gotland.

In the end of the 9th century Iceland was colonized from Norway; and about 985 the intrepid viking, Eric the Red, discovered Greenland, and induced some of his Icelandic countrymen to settle on its inhospitable shores. His son, Leif Ericsson, and others of his followers were concerned in the discovery of the North American coast (see Vinland), which, but for the isolation of Iceland from the centres of European awakening, would have had momentous consequences. As things were, the importance of this discovery passed unrecognized. The story of two Venetians, Nicolo and Antonio Zeno, who gave a vague account of voyages in the northern seas in the end of the 13th century, is no longer to be accepted as history.

Close of the dark ages.

At length the long period of barbarism which accompanied and followed the fall of the Roman empire drew to a close in Europe. The Crusades had a favourable influence on the intellectual state of the Western nations. Interesting regions, known only by the scant reports of pilgrims, were made the objects of attention and study; while religious zeal, and the hope of gain, combined with motives of mere curiosity, induced several persons to travel by land into remote regions of the East, far beyond the countries to which the operations of the crusaders extended. Among these was Benjamin of Tudela, who set out from Spain in 1160, travelled by land to Constantinople, and having visited India and some of the eastern islands, returned to Europe by way of Egypt after an absence of thirteen years.

Asiatic journeys.

Joannes de Plano Carpini, a Franciscan monk, was the head of one of the missions despatched by Pope Innocent to call the chief and people of the Tatars to a better mind. He reached the headquarters of Batu, on the Volga, in February 1246; and, after some stay, went on to the camp of the great khan near Karakorum in central Asia, and returned safely in the autumn of 1247. A few years afterwards, a Fleming named Rubruquis was sent on a similar mission, and had the merit of being the first traveller of this era who gave a correct account of the Caspian Sea. He ascertained that it had no outlet. At nearly the same time Hayton, king of Armenia, made a journey to Karakorum in 1254, by a route far to the north of that followed by Carpini and Rubruquis. He was treated with honour and hospitality, and returned by way of Samarkand and Tabriz, to his own territory. The curious narrative of King Hayton was translated by Klaproth.

While the republics of Italy, and above all the state of Venice, were engaged in distributing the rich products of India and the Far East over the Western world, it was impossible that motives of curiosity, as well as a desire of commercial advantage, should not be awakened to such a degree as to impel some of the merchants to visit those remote lands. Among these were the brothers Polo, who traded with the East and themselves visited Tatary. The recital of their travels fired the youthful imagination of young Marco Polo, son of Nicolo, and he set out for the court of Kublai Khan, with his father and uncle, in 1265. Marco remained for seventeen years in the service of the Great Khan, and was employed on many important missions. Besides what he learnt from his own observation, he collected much information from others concerning countries which he did not visit. He returned to Europe possessed of a vast store of knowledge respecting the eastern parts of the world, and, being afterwards made a prisoner by the Genoese, he dictated the narrative of his travels during his captivity. The work of Marco Polo is the most valuable narrative of travels that appeared during the middle ages, and despite a cold reception and many denials of the accuracy of the record, its substantial truthfulness has been abundantly proved.

Missionaries continued to do useful geographical work. Among them were John of Monte 625
Corvino, a Franciscan monk, Andrew of Perugia, John Marignioli and Friar Jordanus, who

visited the west coast of India, and above all Friar Odoric of Pordenone. Odoric set out on his travels about 1318, and his journeys embraced parts of India, the Malay Archipelago, China and even Tibet, where he was the first European to enter Lhasa, not yet a forbidden city.

Ibn Batuta, the great Arab traveller, is separated by a wide space of time from his countrymen already mentioned, and he finds his proper place in a chronological notice after the days of Marco Polo, for he did not begin his wanderings until 1325, his career thus coinciding in time with the fabled journeyings of Sir John Mandeville. While Arab learning flourished during the darkest ages of European ignorance, the last of the Arab geographers lived to see the dawn of the great period of the European awakening. Ibn Batuta went by land from Tangier to Cairo, then visited Syria, and performed the pilgrimages to Medina and Mecca. After exploring Persia, and again residing for some time at Mecca, he made a voyage down the Red sea to Yemen, and travelled through that country to Aden. Thence he visited the African coast, touching at Mombasa and Quiloa, and then sailed across to Ormuz and the Persian Gulf. He crossed Arabia from Bahrein to Jidda, traversed the Red sea and the desert to Syene, and descended the Nile to Cairo. After this he revisited Syria and Asia Minor, and crossed the Black sea, the desert from Astrakhan to Bokhara, and the Hindu Kush. He was in the service of Muhammad Tughluk, ruler of Delhi, about eight years, and was sent on an embassy to China, in the course of which the ambassadors sailed down the west coast of India to Calicut, and then visited the Maldive Islands and Ceylon. Ibn Batuta made the voyage through the Malay Archipelago to China, and on his return he proceeded from Malabar to Bagdad and Damascus, ultimately reaching Fez, the capital of his native country, in November 1349. After a journey into Spain he set out once more for Central Africa in 1352, and reached Timbuktu and the Niger, returning to Fez in 1353. His narrative was committed to writing from his dictation.

Spanish exploration.

The European country which had come the most completely under the influence of Arab culture now began to send forth explorers to distant lands, though the impulse came not from the Moors but from Italian merchant navigators in Spanish service. The peaceful reign of Henry III. of Castile is famous for the attempts of that prince to extend the diplomatic relations of Spain to the remotest parts of the earth. He sent embassies to all the princes of Christendom and to the Moors. In 1403 the Spanish king sent a knight of Madrid, Ruy Gonzalez de Clavijo, to the distant court of Timur, at Samarkand. He returned in 1406, and wrote a valuable narrative of his travels.

Italians continued to make important journeys in the East during the 15th century. Among them was Nicolo Conti, who passed through Persia, sailed along the coast of Malabar, visited Sumatra, Java and the south of China, returned by the Red sea, and got home to Venice in 1444 after an absence of twenty-five years. He related his adventures to Poggio Bracciolini, secretary to Pope Eugenius IV.; and the narrative contains much interesting information. One of the most remarkable of the Italian travellers was Ludovico di Varthema, who left his native land in 1502. He went to Egypt and Syria, and for the sake of visiting the holy cities became a Mahommedan. He was the first European who gave an account of the interior of Yemen. He afterwards visited and described many places in Persia, India and the Malay Archipelago, returning to Europe in a Portuguese ship after an absence of five years.

Portuguese exploration—Prince Henry the Navigator.

In the 15th century the time was approaching when the discovery of the Cape of Good Hope was to widen the scope of geographical enterprise. This great event was preceded by the general utilization in Europe of the polarity of the magnetic needle in the construction of the mariner’s compass. Portugal took the lead along this new path, and foremost among her pioneers stands Prince Henry the Navigator (1394-1460), who was a patron both of exploration and of the study of geographical theory. The great westward projection of the coast of Africa, and the islands to the north-west of that continent, were the principal scene of the work of the mariners sent out at his expense; but his object was to push onward and reach India from the Atlantic. The progress of discovery received a check on his death, but only for a time. In 1462 Pedro de Cintra extended Portuguese exploration along the African coast and discovered Sierra Leone. Fernan Gomez followed in 1469, and opened trade with the Gold Coast; and in 1484 Diogo Cão discovered the mouth of the Congo. The king of Portugal next despatched Bartolomeu Diaz in 1486 to continue discoveries southwards; while, in the following year, he sent Pedro de Covilhão and Affonso de Payva to discover the country of Prester John. Diaz succeeded in rounding the southern point of Africa, which he named Cabo Tormentoso—the Cape of Storms—but King João II., foreseeing the realization of the long-sought passage to India, gave it the stimulating and enduring name of the Cape of Good Hope. Payva died at Cairo; but Covilhão, having heard that a Christian ruler reigned in the mountains of Ethiopia, penetrated into Abyssinia in 1490. He delivered the letter which João II. had addressed to Prester John to the Negus Alexander of Abyssinia, but he was detained by that prince and never allowed to leave the country.

The Portuguese, following the lead of Prince Henry, continued to look for the road to India by the Cape of Good Hope. The same end was sought by Christopher Columbus, following the suggestion of Toscanelli, and under-estimating the diameter of the globe, by sailing due west. The voyages of Columbus (1492-1498) resulted in the discovery of the West Indies and North America which barred the way to the Far East. In 1493 the pope, Alexander VI., issued a bull instituting the famous "line of demarcation" running from N. to S. 100 leagues W. of the Azores, to the west of which the Spaniards were authorized to explore and to the east of which the Portuguese received the monopoly of discovery. The direct line of Portuguese exploration resulted in the discovery of the Cape route to India by Vasco da Gama (1498), and in 1500 to the independent discovery of South America by Pedro Alvarez Cabral. The voyages of Columbus and of Vasco da Gama were so important that it is unnecessary to detail their results in this place. See Columbus, Christopher; Gama, Vasco da.

Columbus.

The three voyages of Vasco da Gama (who died on the scene of his labours, at Cochin, in 1524) revolutionized the commerce of the East. Until then the Venetians held the carrying trade of India, which was brought by the Persian Gulf and Red sea into Syria and Egypt, the Venetians receiving the products of the East at Alexandria and Beirut and distributing them over Europe. This commerce was a great source of wealth to Venice; but after the discovery of the new passage round the Cape, and the conquests of the Portuguese, the trade of the East passed into other hands.

Vasco da Gama.

The discoveries of Columbus awakened a spirit of enterprise in Spain which continued in full force for a century; adventurers flocked eagerly across the Atlantic, and discovery followed discovery in rapid succession. Many of the companions of Columbus continued his work. Vicente Yañez Pinzon in 1500 reached the mouth of the Amazon. In the same year Alonso de Ojeda, accompanied by Juan de la Cosa, from whose maps we learn much of the discoveries of the 16th century navigators, and by a Florentine named Amerigo Vespucci, touched the coast of South America somewhere near Surinam, following the shore as far as the Gulf of Maracaibo. Vespucci afterwards made three voyages to the Brazilian coast; and in 1504 he wrote an account of his four voyages, which was widely circulated, and became the means of procuring for its author at the hands of the cartographer Waldseemüller in 1507 the disproportionate distinction of giving his name to the whole continent. In 1508 Alonso de Ojeda obtained the government of the coast of South America from Cabo de la Vela to the Gulf of Darien; Ojeda landed at Cartagena in 1510, and sustained a defeat from the natives, in which his lieutenant, Juan de la Cosa, was killed. After another reverse on the east side of the Gulf of Darien Ojeda returned to Hispaniola and died there. The Spaniards in the Gulf of Darien were left by Ojeda under the command of Francisco Pizarro, the future conqueror of Peru. After suffering much from famine and disease, Pizarro resolved to leave, and embarked the survivors in small vessels, but outside the harbour they met a ship which proved to be that of Martin Fernandez Enciso, Ojeda's partner, coming with provisions and reinforcements. One of the crew of Enciso's ship, Vasco Nuñez de Balboa, the future discoverer of the Pacific Ocean, induced his commander to form a settlement on the other side of the Gulf of Darien. The soldiers became discontented and deposed Enciso, who was a man of learning and an accomplished cosmographer. His work *Suma de Geografia*, which was printed in 1519, is the first Spanish book which gives an account of America. Vasco Nuñez, the new commander, entered upon a career of conquest in the neighbourhood of Darien, which ended in the discovery of the Pacific Ocean on the 25th of September 1513. Vasco Nuñez was beheaded in 1517 by Pedrarias de Avila, who was sent out to supersede him. This was one of the greatest calamities that could have happened to South America; for the discoverer of the South sea was on the point of sailing with a little fleet into his unknown ocean, and a humane and judicious man would probably have been the conqueror of Peru, instead of the cruel and ignorant Pizarro. In the year 1519 Panama was founded by Pedrarias; and the conquest of Peru by Pizarro followed a few years afterwards. Hernan Cortes overran and conquered Mexico from 1518 to 1521, and the discovery and conquest of Guatemala by Alvarado, the invasion of Florida by De Soto, and of Nueva Granada by Quesada, followed in rapid succession. The first detailed account of the west coast of South America was written by a keenly observant old soldier, Pedro de Cieza de Leon, who was travelling in South America from 1533 to 1550, and published his story at Seville in 1553.

Spaniards in America.

The great desire of the Spanish government at that time was to find a westward route to the Moluccas. For this purpose Juan Diaz de Solis was despatched in October 1515, and in January 1516 he discovered the mouth of the Rio de la Plata. He was, however, killed by the natives, and his ships returned. In the following year the Portuguese Ferdinando Magalhães, familiarly known as Magellan,

Pacific Ocean.

laid before Charles V., at Valladolid, a scheme for reaching the Spice Islands by sailing westward. He started on the 21st of September 1519, entered the strait which now bears his name in October 1520, worked his way through between Patagonia and Tierra del Fuego, and entered on the vast Pacific which he crossed without sighting any of its innumerable 626
island groups. This was unquestionably the greatest of the voyages which followed from the impulse of Prince Henry, and it was rendered possible only by the magnificent courage of the commander in spite of rebellion, mutiny and starvation. It was the 6th of March 1521 when he reached the Ladrone Islands. Thence Magellan proceeded to the Philippines, and there his career ended in an unimportant encounter with hostile natives. Eventually a Biscayan named Sebastian del Cano, sailing home by way of the Cape of Good Hope, reached San Lucar in command of the "Victoria" on the 6th of September 1522, with eighteen survivors; this one ship of the squadron which sailed on the quest succeeded in accomplishing the first circumnavigation of the globe. Del Cano was received with great distinction by the emperor, who granted him a globe for his crest, and the motto *Primus circumdedisti me*.

While the Spaniards were circumnavigating the world and completing their knowledge of the coasts of Central and South America, the Portuguese were actively engaged on similar work as regards Africa and the East Indies.

Portuguese in Africa and the East.

With Abyssinia the mission of Covilhão led to further intercourse. In April 1520 Vasco da Gama, as viceroy of the Indies, took a fleet into the Red sea, and landed an embassy consisting of Dom Rodriguez de Lima and Father Francisco Alvarez, a priest whose detailed narrative is the earliest and not the least interesting account we possess of Abyssinia. It was not until 1526 that the embassy was dismissed; and not many years afterwards the negus entreated the help of the Portuguese against Mahommedan invaders, and the viceroy sent an expeditionary force, commanded by his brother Cristoforo da Gama, with 450 musketeers. Da Gama was taken prisoner and killed, but his followers enabled the Christians of Abyssinia to regain their power, and a Jesuit mission remained in the country. The Portuguese also established a close connexion with the kingdom of Congo on the west side of Africa, and obtained much information respecting the interior of the continent. Duarte Lopez, a Portuguese settled in the country, was sent on a mission to Rome by the king of Congo, and Pope Sixtus V. caused him to recount to his chamberlain, Felipe Pigafetta, all he had learned during the nine years he had been in Africa, from 1578 to 1587. This narrative, under the title of *Description of the Kingdom of Congo*, was published at Rome by Pigafetta in 1591. A map was attached on which several great equatorial lakes are shown, and the empire of Monomwezi or Unyamwezi is laid down. The most valuable work on Africa about this time is, however, that written by the Moor Leo Africanus in the early part of the 16th century. Leo travelled extensively in the north and west of Africa, and was eventually taken by pirates and sold to a master who presented him to Pope Leo X. At the pope's desire he translated his work on Africa into Italian.

In Further India and the Malay Archipelago the Portuguese acquired predominating influence at sea, establishing factories on the Malabar coast, in the Persian Gulf, at Malacca, and in the Spice Islands, and extending their commercial enterprises from the Red sea to China. Their missionaries were received at the court of Akbar, and Benedict Goes, a native of the Azores, was despatched on a journey overland from Agra to China. He started in 1603, and, after traversing the least-known parts of Central Asia, he reached the confines of China. He appears to have ascended from Kabul to the plateau of the Pamir, and thence onwards by Yarkand, Khotan and Aksu. He died on the journey in March 1607; and thus, as one of the brethren pronounced his epitaph, "seeking Cathay he found heaven."

English, Dutch and French.

The activity and love of adventure, which became a passion for two or three generations in Spain and Portugal, spread to other countries. It was the spirit of the age; and England, Holland and France were fired by it. English enterprise was first aroused by John and Sebastian Cabot, father and son, who came from Venice and settled at Bristol in the time of Henry VII. The Cabots received a patent in 1496, empowering them to seek unknown lands; and John Cabot discovered Newfoundland and part of the coast of America. Sebastian afterwards made a voyage to Rio de la Plata in the service of Spain, but he returned to England in 1548 and received a pension from Edward VI. At his suggestion a voyage was undertaken for the discovery of a north-east passage to Cathay, with Sir Hugh Willoughby as captain-general of the fleet and Richard Chancellor as pilot-major. They sailed in May 1553, but Willoughby and all his crew perished on the Lapland coast. Chancellor, however, was more fortunate. He reached the White Sea, performed the journey overland to Moscow, where he was well received, and may be said to have been the founder of the trade between Russia and England. He returned to Archangel and brought his ship back in safety to England. On a second voyage, in 1556, Chancellor was drowned; and three subsequent voyages, led by Stephen Burrough, Arthur Pet and Charles Jackman, in small craft of 50 tons and under, carried on an examination of the straits which lead into the Kara sea.

The French followed closely on the track of John Cabot, and Norman and Breton fishermen frequented the banks of Newfoundland at the beginning of the 16th century. In 1524 Francis I. sent Giovanni da Verazzano of Florence on an expedition of discovery to the coast of North America; and the details of his voyage were embodied in a letter addressed by him to the king of France from Dieppe, in July 1524. In 1534 Jacques Cartier set out to continue the discoveries of Verazzano, and visited Newfoundland and the Gulf of St Lawrence. In the following year he made another voyage, discovered the island of Anticosti, and ascended the St Lawrence to Hochelaga, now Montreal. He returned, after passing two winters in Canada; and on another occasion he also failed to establish a colony. Admiral de Coligny made several unsuccessful endeavours to form a colony in Florida under Jean Ribault of Dieppe, René de Laudonnière and others, but the settlers were furiously assailed by the Spaniards and the attempt was abandoned.

The Elizabethan era.

The reign of Elizabeth is famous for the gallant enterprises that were undertaken by sea and land to discover and bring to light the unknown parts of the earth. The great promoter of geographical discovery in the Elizabethan period was Richard Hakluyt (1553-1616), who was active in the formation of the two companies for colonizing Virginia in 1606; and devoted his life to encouraging and recording similar undertakings. He published much, and left many valuable papers at his death, most of which, together with many other narratives, were published in 1622 in the great work of the Rev. Samuel Purchas, entitled *Hakluytus Posthumus, or Purchas his Pilgrimes*.

It is from these works that our knowledge of the gallant deeds of the English and other explorers of the Elizabethan age is mainly derived. The great and splendidly illustrated collections of voyages and travels of Theodorus de Bry and Hulsius served a similar useful purpose on the continent of Europe. One important object of English maritime adventurers of those days was to discover a route to Cathay by the north-west, a second was to settle Virginia, and a third was to raid the Spanish settlements in the West Indies. Nor was the trade to Muscovy and Turkey neglected; while latterly a resolute and successful attempt was made to establish direct commercial relations with India.

The conception of the north-western route to Cathay now leads the story of exploration, for the first time as far as important and sustained efforts are concerned, towards the Arctic seas. This part of the story is fully told under the heading of Polar Regions, and only the names of Martin Frobisher (1576), John Davis (1585), Henry Hudson (1607) and William Baffin (1616) need be mentioned here in order to preserve the complete conspectus of the history of discovery. The Dutch emulated the British in the Arctic seas during this period, directing their efforts mainly towards the discovery of a north-east passage round the northern end of Novaya Zemlya; and William Barents or Barendsz (1594-1597) is the most famous name in this connexion, his boat voyage along the coast of Novaya Zemlya after losing his ship and wintering in a high latitude, being one of the most remarkable achievements in polar annals.

Many English voyages were also made to Guinea and the West Indies, and twice English vessels followed in the track of Magellan, and circumnavigated the globe. In 1577 Francis Drake, who had previously served with Hawkins in the West Indies, undertook his celebrated voyage round the world. Reaching the Pacific through the Strait of Magellan, Drake proceeded northward along the west coast of America, resolved to attempt the discovery of a northern passage from the Pacific to the Atlantic. The coast from the southern extremity of the Californian peninsula to Cape Mendocino had been discovered by Juan Rodriguez Cabrillo and Francisco de Ulloa in 1539. Drake’s discoveries extended from Cape Mendocino to 48° N., in which latitude he gave up his quest, sailed across the Pacific and reached the Philippine Islands, returning home round the Cape of Good Hope in 1580.

Thomas Cavendish, emulous of Drake’s example, fitted out three vessels for an expedition to the South sea in 1586. He took the same route as Drake along the west coast of America. From Cape San Lucas Cavendish steered across the Pacific, seeing no land until he reached the Ladrone Islands. He returned to England in 1588. The third English voyage into the Pacific was not so fortunate. Sir Richard Hawkins (1593) on reaching the bay of Atacames, in 1°N. in 1594, was attacked by a Spanish fleet, and, after a desperate naval engagement, was forced to surrender. Hawkins declared his object to be discovery and the survey of unknown lands, and his voyage, though terminating in disaster, bore good fruit. *The Observations of Sir Richard Hawkins in his Voyage into the South Sea*, published in 1622, are very valuable. It was long before another British ship entered the Pacific Ocean. Sir John Narborough took two ships through the Strait of Magellan in 1670 and touched on the coast of Chile, but it was not until 1685 that Dampier sailed over the part of the Pacific where Hawkins met his defeat.

The exploring enterprise of the Spanish nation did not wane after the conquest of Peru and Mexico, and the acquisition of the vast empire of the Indies. It was spurred into renewed activity by the audacity of Sir John Hawkins in the West Indies, and by the appearance of Drake, Cavendish and Richard Hawkins in the Pacific.

In the interior of South America the Spanish conquerors had explored the region of the Andes from the isthmus of Panama to Chile. Pedro de Valdivia in 1540 made an expedition into the country of the Araucanian Indians of Chile, and was the first to explore the eastern 627
base of the Andes in what is now Argentine Patagonia. In 1541 Francisco de Orellana discovered the whole course of the Amazon from its source in the Andes to the Atlantic. A second voyage on the Amazon was made in 1561 by the mad pirate Lope de Aguirre; but it was not until 1639 that a full account was written of the great river by Father Cristoval de Acuña, who ascended it from its mouth and reached the city of Quito.

Spaniards in the Pacific.

The voyage of Drake across the Pacific was preceded by that of Alvaro de Mendaña, who was despatched from Peru in 1567 to discover the great Antarctic continent which was believed to extend far northward into the South sea, the search for which now became one of the leading motives of exploration. After a voyage of eighty days across the Pacific, Mendaña discovered the Solomon Islands; and the expedition returned in safety to Callao. The appearance of Drake on the Peruvian coast led to an expedition being fitted out at Callao, to go in chase of him, under the command of Pedro Sarmiento. He sailed from Callao in October 1579, and made a careful survey of the Strait of Magellan, with the object of fortifying that entrance to the South sea. The colony which he afterwards took out from Spain was a complete failure, and is only remembered now from the name of "Port Famine," which Cavendish gave to the site at which he found the starving remnant of Sarmiento's settlers. In June 1595 Mendaña sailed from the coast of Peru in command of a second expedition to colonize the Solomon Islands. After discovering the Marquesas, he reached the island of Santa Cruz of evil memory, where he and many of the settlers died. His young widow took command of the survivors and brought them safely to Manila. The viceroys of Peru still persevered in their attempts to plant a colony in the hypothetical southern continent. Pedro Fernandez de Quiros, who was pilot under Mendaña and Luis Vaez de Torres, were sent in command of two ships to continue the work of exploration. They sailed from Callao in December 1605, and discovered several islands of the New Hebrides group. They anchored in a bay of a large island which Quiros named "Australia del Espiritu Santo." From this place Quiros returned to America, but Torres continued the voyage, passed through the strait between Australia and New Guinea which bears his name, and explored and mapped the southern and eastern coasts of New Guinea.

The Portuguese, in the early part of the 17th century (1578-1640), were under the dominion of Spain, and their enterprise was to some extent damped; but their missionaries extended geographical knowledge in Africa. Father Francisco Paez acquired great influence in Abyssinia, and explored its highlands from 1600 to 1622. Fathers Mendez and Lobo traversed the deserts between the coast of the Red sea and the mountains, became acquainted with Lake Tsana, and discovered the sources of the Blue Nile in 1624-1633.

Rivalry in the East.

But the attention of the Portuguese was mainly devoted to vain attempts to maintain their monopoly of the trade of India against the powerful rivalry of the English and Dutch. The English enterprises were persevering, continuous and successful. James Lancaster made a voyage to the Indian Ocean from 1591 to 1594; and in 1599 the merchants and adventurers of London resolved to form a company, with the object of establishing a trade with the East Indies. On the 31st of December 1599 Queen Elizabeth granted the charter of incorporation to the East India Company, and Sir James Lancaster, one of the directors, was appointed general of their first fleet. He was accompanied by John Davis, the great Arctic navigator, as pilot-major. This voyage was eminently successful. The ships touched at Achin in Sumatra and at Java, returning with full ladings of pepper in 1603. The second voyage was commanded by Sir Henry Middleton; but it was in the third voyage, under Keelinge and Hawkins, that the mainland of India was first reached in 1607. Captain Hawkins landed at Surat and travelled overland to Agra, passing some time at the court of the Great Mogul. In the voyage of Sir Edward Michelborne in 1605, John Davis lost his life in a fight with a Japanese junk. The eighth voyage, led by Captain Saris, extended the operations of the company to Japan; and in 1613 the Japanese government granted privileges to the company; but the British retired in 1623, giving up their factory. The chief result of this early intercourse between Great Britain and Japan was the interesting series of letters written by William Adams from 1611 to 1617. From the tenth voyage of the East India Company, commanded by Captain Best, who left England in 1612, dates the establishment of permanent British factories on the coast of India. It was Captain Best who secured a regular *firman* for trade from the Great Mogul.

From that time a fleet was despatched every year, and the company’s operations greatly increased geographical knowledge of India and the Eastern Archipelago. British visits to Eastern countries, at this time, were not confined to the voyages of the company. Journeys were also made by land, and, among others, the entertaining author of the *Crudities*, Thomas Coryate, of Odcombe in Somersetshire, wandered on foot from France to India, and died (1617) in the company’s factory at Surat. In 1561 Anthony Jenkinson arrived in Persia with a letter from Queen Elizabeth to the shah. He travelled through Russia to Bokhara, and returned by the Caspian and Volga. In 1579 Christopher Burroughs built a ship at Nizhniy Novgorod and traded across the Caspian to Baku; and in 1598 Sir Anthony and Robert Shirley arrived in Persia, and Robert was afterwards sent by the shah to Europe as his ambassador. He was followed by a Spanish mission under Garcia de Silva, who wrote an interesting account of his travels; and to Sir Dormer Cotton’s mission, in 1628, we are indebted for Sir Thomas Herbert’s charming narrative. In like manner Sir Thomas Roe’s mission to India resulted not only in a large collection of valuable reports and letters of his own, but also in the detailed account of his chaplain Terry. But the most learned and intelligent traveller in the East, during the 17th century, was the German, Engelbrecht Kaempfer, who accompanied an embassy to Persia, in 1684, and was afterwards a surgeon in the service of the Dutch East India Company. He was in the Persian Gulf, India and Java, and resided for more than two years in Japan, of which he wrote a history.

Dutch exploration, 16th-17th centuries.

The Dutch nation, as soon as it was emancipated from Spanish tyranny, displayed an amount of enterprise, which, for a long time, was fully equal to that of the British. The Arctic voyages of Barents were quickly followed by the establishment of a Dutch East India Company; and the Dutch, ousting the Portuguese, not only established factories on the mainland of India and in Japan, but acquired a preponderating influence throughout the Malay Archipelago. In 1583 Jan Hugen van Linschoten made a voyage to India with a Portuguese fleet, and his full and graphic descriptions of India, Africa, China and the Malay Archipelago must have been of no small use to his countrymen in their distant voyages. The first of the Dutch Indian voyages was performed by ships which sailed in April 1595, and rounded the Cape of Good Hope. A second large Dutch fleet sailed in 1598; and, so eager was the republic to extend her commerce over the world that another fleet, consisting of five ships of Rotterdam, was sent in the same year by way of Magellan’s Strait, under Jacob Mahu as admiral, with William Adams as pilot. Mahu died on the passage out, and was succeeded by Simon de Cordes, who was killed on the coast of Chile. In September 1599 the fleet had entered the Pacific. The ships were then steered direct for Japan, and anchored off Bungo in April 1600. In the same year, 1598, a third expedition was despatched under Oliver van Noort, a native of Utrecht, but the voyage contributed nothing to geography. The Dutch Company in 1614 again resolved to send a fleet to the Moluccas by the westward route, and Joris Spilbergen was appointed to the command as admiral, with a commission from the States-General. He was furnished with four ships of Amsterdam, two of Rotterdam and one from Zeeland. On the 6th of May 1615 Spilbergen entered the Pacific Ocean, and touched at several places on the coast of Chile and Peru, defeating the Spanish fleet in a naval engagement off Chilca. After plundering Payta and making requisitions at Acapulco, the Dutch fleet crossed the Pacific and reached the Moluccas in March 1616.

The Dutch now resolved to discover a passage into the Pacific to the south of Tierra del Fuego, the insular nature of which had been ascertained by Sir Francis Drake. The vessels fitted out for this purpose were the “Eendracht,” of 360 tons, commanded by Jacob Lemaire, and the “Hoorn,” of 110 tons, under Willem Schouten. They sailed from the Texel on the 14th of June 1615, and by the 20th of January 1616 they were south of the entrance of Magellan’s Strait. Passing through the strait of Lemaire they came to the southern extremity of Tierra del Fuego, which was named Cape Horn, in honour of the town of Hoorn in West Friesland, of which Schouten was a native. They passed the cape on the 31st of January, encountering the usual westerly winds. The great merit of this discovery of a second passage into the South sea lies in the fact that it was not accidental or unforeseen, but was due to the sagacity of those who designed the voyage. On the 1st of March the Dutch fleet sighted the island of Juan Fernandez; and, having crossed the Pacific, the explorers sailed along the north coast of New Guinea and arrived at the Moluccas on the 17th of September 1616.

There were several early indications of the existence of the great Australian continent, and the Dutch endeavoured to obtain further knowledge concerning the country and its extent; but only its northern and western coasts had been visited before the time of Governor van Diemen. Dirk Hartog had been on the west coast in latitude 26° 30′ S. in 1616. Pelsert struck on a reef called “Houtman’s Abrolhos” on the 4th of June 1629. In 1697 the Dutch captain Vlamingh landed on the west coast of Australia, then called New Holland, in 31° 43′ S., and named the Swan river from the black swans he discovered there. In 1642 the governor and council of Batavia fitted out two ships to prosecute the discovery of the south land, then believed to be part of a vast Antarctic continent, and entrusted the command to Captain Abel

Jansen Tasman. This voyage proved to be the most important to geography that had been undertaken since the first circumnavigation of the globe. Tasman sailed from Batavia in 1642, and on the 24th of November sighted high land in 42° 30′ S., which was named van Diemen’s Land, and after landing there proceeded to the discovery of the western coast of New Zealand; at first called Staten Land, and supposed to be connected with the Antarctic continent from which this voyage proved New Holland to be separated. He then reached Tongatabu, one of the Friendly Islands of Cook; and returned by the north coast of New Guinea to Batavia. In 1644 Tasman made a second voyage to effect a fuller discovery of New Guinea.

The French directed their enterprise more in the direction of North America than of the 628
Indies. One of their most distinguished explorers was Samuel Champlain, a captain in the navy, who, after a remarkable journey through Mexico and the West Indies from 1599 to 1602, established his historic connexion with Canada, to the geographical knowledge of which he made a very large addition.

French in North America.

The principles and methods of surveying and position finding had by this time become well advanced, and the most remarkable example of the early application of these improvements is to be found in the survey of China by Jesuit missionaries. They first prepared a map of the country round Peking, which was submitted to the emperor Kang-hi, and, being satisfied with the accuracy of the European method of surveying, he resolved to have a survey made of the whole empire on the same principles. This great work was begun in July 1708, and the completed maps were presented to the emperor in 1718. The records preserved in each city were examined, topographical information was diligently collected, and the Jesuit fathers checked their triangulation by meridian altitudes of the sun and pole star and by a system of remeasurements. The result was a more accurate map of China than existed, at that time, of any country in Europe. Kang-hi next ordered a similar map to be made of Tibet, the survey being executed by two lamas who were carefully trained as surveyors by the Jesuits at Peking. From these surveys were constructed the well-known maps which were forwarded to Duhalde, and which D’Anville utilized for his atlas.

Missionaries in the East.

Several European missionaries had previously found their way from India to Tibet. Antonio Andrada, in 1624, was the first European to enter Tibet since the visit of Friar Odoric in 1325. The next journey was that of Fathers Grueber and Dorville about 1660, who succeeded in passing from China, through Tibet, into India. In 1715 Fathers Desideri and Freyre made their way from Agra, across the Himalayas, to Lhasa, and the Capuchin Friar Orazio della Penna resided in that city from 1735 until 1747. But the most remarkable journey in this direction was performed by a Dutch traveller named Samuel van de Putte. He left Holland in 1718, went by land through Persia to India, and eventually made his way to Lhasa, where he resided for a long time. He went thence to China, returned to Lhasa, and was in India in time to be an eye-witness of the sack of Delhi by Nadir Shah in 1737. In 1743 he left India and died at Batavia on the 27th of September 1745. The premature death of this illustrious traveller is the more to be lamented because his vast knowledge died with him. Two English missions sent by Warren Hastings to Tibet, one led by George Bogle in 1774, and the other by Captain Turner in 1783, complete Tibetan exploration in the 18th century.

The 18th century.

Asia.

From Persia much new information was supplied by Jean Chardin, Jean Tavernier, Charles Hamilton, Jean de Thévenot and Father Jude Krusinski, and by English traders on the Caspian. In 1738 John Elton traded between Astrakhan and the Persian port of Enzelî on the Caspian, and undertook to build a fleet for Nadir Shah. Another English merchant, named Jonas Hanway, arrived at Astrabad from Russia, and travelled to the camp of Nadir at Kazvin. One lasting and valuable result of Hanway’s wanderings was a charming book of travels. In 1700 Guillaume Delisle published his map of the continents of the Old World; and his successor D’Anville produced his map of India in 1752. D’Anville’s map contained all that was then known, but ten years afterwards Major Rennell began his surveying labours, which extended over the period from 1763 to 1782. His survey covered an area 900 m. long by 300 wide, from the eastern confines of Bengal to Agra, and from the Himalayas to Calpi. Rennell was indefatigable in collecting geographical information; his Bengal atlas appeared in 1781, his famous map of India in 1788 and the memoir in 1792. Surveys were also made along the Indian coasts.

Arabia received very careful attention, in the 18th century, from the Danish scientific mission, which included Carsten Niebuhr among its members. Niebuhr landed at Loheia, on the coast of Yemen, in December 1762, and went by land to Sana. All the other members of the mission died, but he proceeded from Mokha to Bombay. He then made a journey through Persia and Syria to Constantinople, returning to Copenhagen in 1767. His valuable work, the *Description of Arabia*, was published in 1772, and was followed in 1774-1778 by two volumes of travels in Asia. The great traveller survived until 1815, when he died at the age of eighty-two.

James Bruce of Kinnaird, the contemporary of Niebuhr, was equally devoted to Eastern travel; and his principal geographical work was the tracing of the Blue Nile from its source to its junction with the White Nile. Before the death of Bruce an African

Africa.

Association was formed, in 1788, for collecting information respecting the interior of that continent, with Major Rennell and Sir Joseph Banks as leading members. The association first employed John Ledyard (who had previously made an extraordinary journey into Siberia) to cross Africa from east to west on the parallel of the Niger, and William Lucas to cross the Sahara to Fezzan. Lucas went from Tripoli to Mesurata, obtained some information respecting Fezzan and returned in 1789. One of the chief problems the association wished to solve was that of the existence and course of the river Niger, which was believed by some authorities to be identical with the Congo. Mungo Park, then an assistant surgeon of an Indiaman, volunteered his services, which were accepted by the association, and in 1795 he succeeded in reaching the town of Segu on the Niger, but was prevented from continuing his journey to Timbuktu. Five years later he accepted an offer from the government to command an expedition into the interior of Africa, the plan being to cross from the Gambia to the Niger and descend the latter river to the sea. After losing most of his companions he himself and the rest perished in a rapid on the Niger at Busa, having been attacked from the shore by order of a chief who thought he had not received suitable presents. His work, however, had established the fact that the Niger was not identical with the Congo.

While the British were at work in the direction of the Niger, the Portuguese were not unmindful of their old exploring fame. In 1798 Dr F.J.M. de Lacerda, an accomplished astronomer, was appointed to command a scientific expedition of discovery to the north of the Zambesi. He started in July, crossed the Muchenja Mountains, and reached the capital of the Cazembe, where he died of fever. Lacerda left a valuable record of his adventurous journey; but with Mungo Park and Lacerda the history of African exploration in the 18th century closes.

In South America scientific exploration was active during this period. The great geographical event of the century, as regards that continent, was the measurement of an arc of the meridian. The undertaking was proposed by the French Academy as

South America.

part of an investigation with the object of ascertaining the length of the degree near the equator and near the pole respectively so as to determine the figure of the earth. A commission left Paris in 1735, consisting of Charles Marie de la Condamine, Pierre Bouguer, Louis Godin and Joseph de Jussieu the naturalist. Spain appointed two accomplished naval officers, the brothers Ulloa, as coadjutors. The operations were carried on during eight years on a plain to the south of Quito; and, in addition to his memoir on this memorable measurement, La Condamine collected much valuable geographical information during a voyage down the Amazon. The arc measured was 3° 7′ 3″ in length; and the work consisted of two measured bases connected by a series of triangles, one north and the other south of the equator, on the meridian of Quito. Contemporaneously, in 1738, Pierre Louis Moreau de Maupertuis, Alexis Claude Clairaut, Charles Etienne Louis Camus, Pierre Charles Lemonnier and the Swedish physicist Celsius measured an arc of the meridian in Lapland.

The British and French governments despatched several expeditions of discovery into the Pacific and round the world during the 18th century. They were preceded by the wonderful and romantic voyages of the buccaneers. The narratives of such men as

The Pacific Ocean.

Woodes Rogers, Edward Davis, George Shelvocke, Clipperton and William Dampier, can never fail to interest, while they are not without geographical value. The works of Dampier are especially valuable, and the narratives of William Funnell and Lionel Wafer furnished the best accounts then extant of the Isthmus of Darien. Dampier’s literary ability eventually secured for him a commission in the king’s service; and he was sent on a voyage of discovery, during which he explored part of the coasts of Australia and New Guinea, and discovered the strait which bears his name between New Guinea and New Britain, returning in 1701. In 1721 Jacob Roggewein was despatched on a voyage of some importance across the Pacific by the Dutch West India Company, during which he discovered Easter Island on the 6th of April 1722.

The voyage of Lord Anson to the Pacific in 1740-1744 was of a predatory character, and he lost more than half his men from scurvy; while it is not pleasant to reflect that at the very time when the French and Spaniards were measuring an arc of the meridian at Quito, the British under Anson were pillaging along the coast of the Pacific and burning the town of Payta. But a romantic interest attaches to the wreck of the “Wager,” one of Anson’s fleet, on a desert island near Chiloe, for it bore fruit in the charming narrative of Captain John Byron, which will endure for all time. In 1764 Byron himself was sent on a voyage of discovery round the world, which led immediately after his return to the despatch of another to complete his work, under the command of Captain Samuel Wallis.

The expedition, consisting of the “Dolphin” commanded by Wallis, and the “Swallow” under Captain Philip Carteret, sailed in September 1766, but the ships were separated on entering the Pacific from the Strait of Magellan. Wallis discovered Tahiti on the 19th of June 1767, and he gave a detailed account of that island. He returned to England in May 1768. Carteret discovered the Charlotte and Gloucester Islands, and Pitcairn Island on the 2nd of July 1767; revisited the Santa Cruz group, which was discovered by Mendaña and Quiros; and discovered the strait separating New Britain from New Ireland. He reached Spithead again in February 1769. Wallis and Carteret were followed very closely by the French expedition of Bougainville, which sailed from Nantes in November 1766. Bougainville had first to perform the unpleasant task of delivering up the Falkland Islands, where he had encouraged the formation of a French settlement, to the Spaniards. He then entered the Pacific, and reached Tahiti in April 1768. Passing through the New Hebrides group he 629
touched at Batavia, and arrived at St Malo after an absence of two years and four months.

Captain Cook.

The three voyages of Captain James Cook form an era in the history of geographical discovery. In 1767 he sailed for Tahiti, with the object of observing the transit of Venus, accompanied by two naturalists, Sir Joseph Banks and Dr Solander, a pupil of Linnaeus, as well as by two astronomers. The transit was observed on the 3rd of June 1769. After exploring Tahiti and the Society group, Cook spent six months surveying New Zealand, which he discovered to be an island, and the coast of New South Wales from latitude 38° S. to the northern extremity. The belief in a vast Antarctic continent stretching far into the temperate zone had never been abandoned, and was vehemently asserted by Charles Dalrymple, a disappointed candidate nominated by the Royal Society for the command of the Transit expedition of 1769. In 1772 the French explorer Yves Kerguelen de Tremarec had discovered the land that bears his name in the South Indian Ocean without recognizing it to be an island, and naturally believed it to be part of the southern continent.

Cook’s second voyage was mainly intended to settle the question of the existence of such a continent once for all, and to define the limits of any land that might exist in navigable seas towards the Antarctic circle. James Cook at his first attempt reached a south latitude of 57° 15′. On a second cruise from the Society Islands, in 1773, he, first of all men, crossed the Antarctic circle, and was stopped by ice in 71° 10′ S. During the second voyage Cook visited Easter Island, discovered several islands of the New Hebrides and New Caledonia; and on his way home by Cape Horn, in March 1774, he discovered the Sandwich Island group and described South Georgia. He proved conclusively that any southern continent that might exist lay under the polar ice. The third voyage was intended to attempt the passage from the Pacific to the Atlantic by the north-east. The “Resolution” and “Discovery” sailed in 1776, and Cook again took the route by the Cape of Good Hope. On reaching the North American coast, he proceeded northward, fixed the position of the western extremity of America and surveyed Bering Strait. He was stopped by the ice in 70° 41′ N., and named the farthest visible point on the American shore Icy Cape. He then visited the Asiatic shore and discovered Cape North. Returning to Hawaii, Cook was murdered by the natives. On the 14th of February 1779, his second, Captain Edward Clerke, took command, and proceeding to Petropavlovsk in the following summer, he again examined the edge of the ice, but only got as far as 70° 33′ N. The ships returned to England in October 1780.

In 1785 the French government carefully fitted out an expedition of discovery at Brest, which was placed under the command of François La Pérouse, an accomplished and experienced officer. After touching at Concepcion in Chile and at Easter Island, La Pérouse proceeded to Hawaii and thence to the coast of California, of which he has given a very interesting account. He then crossed the Pacific to Macao, and in July 1787 he proceeded to explore the Gulf of Tartary and the shores of Sakhalin, remaining some time at Castries Bay, so named after the French minister of marine. Thence he went to the Kurile Islands and Kamchatka, and sailed from the far north down the meridian to the Navigator and Friendly Islands. He was in Botany Bay in January 1788; and sailing thence, the explorer, his ship and crew were never seen again. Their fate was long uncertain. In September 1791 Captain Antoine d’Entrecasteaux sailed from Brest with two vessels to seek for tidings. He visited the New Hebrides, Santa Cruz, New Caledonia and Solomon Islands, and made careful though rough surveys of the Louisiade Archipelago, islands north of New Britain and part of New Guinea. D’Entrecasteaux died on board his ship on the 20th of July 1793, without ascertaining the fate of La Pérouse. Captain Peter Dillon at length ascertained, in 1828, that the ships of La Pérouse had been wrecked on the island of Vanikoro during a hurricane.

The work of Captain Cook bore fruit in many ways. His master, Captain William Bligh, was sent in the “Bounty” to convey breadfruit plants from Tahiti to the West Indies. He reached Tahiti in October 1788, and in April 1789 a mutiny broke out, and he, with several officers and men, was thrust into an open boat in mid-ocean. During the remarkable voyage he then made to Timor, Bligh passed amongst the northern islands of the New Hebrides, which he named the Banks Group, and made several running surveys. He reached England in March 1790. The “Pandora,” under Captain Edwards, was sent out in search of the “Bounty,” and discovered the islands of Cherry and Mitre, east of the Santa Cruz group, but she was eventually lost on a reef in Torres Strait. In 1796-1797 Captain Wilson, in the missionary ship “Duff,” discovered the Gambier and other islands, and rediscovered the islands known to and seen by Quiros, but since called the Duff Group. Another result of Captain Cook’s work was the colonization of Australia. On the 18th of January 1788 Admiral Phillip and Captain Hunter arrived in Botany Bay in the “Supply” and “Sirius,” followed by six transports, and established a colony at Port Jackson. Surveys were then undertaken in several directions. In 1795 and 1796 Matthew Flinders and George Bass were engaged on exploring work in a small boat called the “Tom Thumb.” In 1797 Bass, who had been a surgeon, made an expedition southwards, continued the work of Cook from Ram Head, and explored the strait which bears his name, and in 1798 he and Flinders were surveying on the east coast of Van Diemen’s land.

Yet another outcome of Captain Cook’s work was the voyage of George Vancouver, who had served as a midshipman in Cook’s second and third voyages. The Spaniards under Quadra had begun a survey of north-western America and occupied Nootka Sound, which their government eventually agreed to surrender. Captain Vancouver was sent out to receive the cession, and to survey the coast from Cape Mendocino northwards. He commanded the old “Discovery,” and was at work during the seasons of 1792, 1793 and 1794, wintering at Hawaii. Returning home in 1795, he completed his narrative and a valuable series of charts.

Arctic regions.

The 18th century saw the Arctic coast of North America reached at two points, as well as the first scientific attempt to reach the North Pole. The Hudson Bay Company had been incorporated in 1670, and its servants soon extended their operations over a wide area to the north and west of Canada. In 1741 Captain Christopher Middleton was ordered to solve the question of a passage from Hudson Bay to the westward. Leaving Fort Churchill in July 1742, he discovered the Wager river and Repulse Bay. He was followed by Captain W. Moor in 1746, and Captain Coats in 1751, who examined the Wager Inlet up to the end. In November 1769 Samuel Hearne was sent by the Hudson Bay Company to discover the sea on the north side of America, but was obliged to return. In February 1770 he set out again from Fort Prince of Wales; but, after great hardships, he was again forced to return to the fort. He started once more in December 1771, and at length reached the Coppermine river, which he surveyed to its mouth, but his observations are unreliable. With the same object Alexander Mackenzie, with a party of Canadians, set out from Fort Chippewyan on the 3rd of June 1789, and descending the great river which now bears the explorer’s name reached the Arctic sea.

In February 1773 the Royal Society submitted a proposal to the king for an expedition towards the North Pole. The expedition was fitted out under Captains Constantine Phipps and Skeffington Lutwidge, and the highest latitude reached was 80° 48′ N., but no opening was discovered in the heavy Polar pack. The most important Arctic work in the 18th century was performed by the Russians, for they succeeded in delineating the whole of the northern coast of Siberia. Some of this work was possibly done at a still earlier date. The Cossack Simon Dezhneff is thought to have made a voyage, in the summer of 1648, from the river Kolyma, through Bering Strait (which was rediscovered by Vitus Bering in 1728) to Anadyr. Between 1738 and 1750 Manin and Sterlegoff made their way in small sloops from the mouth of the Yenesei as far north as 75° 15′ N. The land from Taimyr to Cape Chelyuskin, the most northern extremity of Siberia, was mapped in many years of patient exploration by Chelyuskin, who reached the extreme point (77° 34′ N.) in May 1742. To the east of Cape Chelyuskin the Russians encountered greater difficulties. They built small vessels at Yakutsk on the Lena, 900 m. from its mouth, whence the first expedition was despatched under Lieut. Prontschichev in 1735. He sailed from the mouth of the Lena to the mouth of the Olonek, where he wintered, and on the 1st of September 1736 he got as far as 77° 29′ N., within 5 m. of Cape Chelyuskin. Both he and his young wife died of scurvy, and the vessel returned. A second expedition, under Lieut. Laptyev, started from the Lena in 1739, but encountered masses of drift ice in Chatanga bay, and with this ended the voyages to the westward of the Lena. Several attempts were also made to navigate the sea from the Lena to the Kolyma. In 1736 Lieut. Laptyev sailed, but was stopped by the drift ice in August, and in 1739, during another trial, he reached the mouth of the Indigirka, where he wintered. In the season of 1740 he continued his voyage to beyond the Kolyma, wintering at Nizhni Kolymsk. In September 1740 Vitus Bering sailed from Okhotsk on a second Arctic voyage with George William Steller on board as naturalist. In June 1741 he named the magnificent peak on the coast of North America Mount St Elias and explored the Aleutian Islands. In November the ship was wrecked on Bering Island; and the gallant Dane, worn out with scurvy, died there on the 8th of December 1741. In March 1770 a merchant named Liakhov saw a large herd of reindeer coming from the north to the Siberian coast, which induced him to start in a sledge in the direction whence they came. Thus he reached the New Siberian or Liakhov Islands, and for years afterwards the seekers for fossil ivory resorted to them. The Russian Captain Vassili Chitschakov in 1765 and 1766 made two persevering attempts to penetrate the ice north of Spitsbergen, and reached 80° 30′ N., while Russian parties twice wintered at Bell Sound.

Geographical societies.

In reviewing the progress of geographical discovery thus far, it has been possible to keep fairly closely to a chronological order. But in the 19th century and after exploring work was so generally and steadily maintained in all directions, and was in so many cases narrowed down from long journeys to detailed surveys within relatively small areas, that it becomes desirable to cover the whole period at one view for certain great divisions of the world. (See Africa; Asia; Australia; Polar Regions; &c.) Here, however, may be noticed the development of geographical societies devoted to the encouragement of exploration and research. The first of the existing geographical societies was that of Paris, founded in 1825 under the title of La Société de Géographie. The Berlin Geographical Society (Gesellschaft für Erdkunde) is second in order of seniority, having been founded in 1827. The Royal Geographical Society, which was founded in London in 1830, comes third on the list; but it may be viewed as a direct result of the earlier African Association founded in 1788. Sir John Barrow, Sir John Cam Hobhouse (Lord Broughton), Sir Roderick Murchison, Mr Robert Brown and Mr Bartle Frere formed the foundation committee of the Royal Geographical Society, and the first president was Lord Goderich. The action of the society in supplying practical instruction to intending travellers, in astronomy, surveying and the various branches of science useful to collectors, has had much to do with advancement of discovery. Since the war of 1870 many geographical societies have been established on the continent of Europe. At the close of the 19th century there were upwards of 100 such societies in the world, with more than 50,000 members, and over 150 journals were devoted entirely to geographical subjects.[11] The great development of photography has been a notable aid to explorers, not only by placing at their disposal a faithful and ready means of recording the features of a country and the types of inhabitants, but by supplying a method of quick and accurate topographical surveying. 630

The Principles of Geography

As regards the scope of geography, the order of the various departments and their interrelation, there is little difference of opinion, and the principles of geography[12] are now generally accepted by modern geographers. The order in which the various subjects are treated in the following sketch is the natural succession from fundamental to dependent facts, which corresponds also to the evolution of the diversities of the earth’s crust and of its inhabitants.

Mathematical geography.

The fundamental geographical conceptions are mathematical, the relations of space and form. The figure and dimensions of the earth are the first of these. They are ascertained by a combination of actual measurement of the highest precision on the surface and angular observations of the positions of the heavenly bodies. The science of geodesy is part of mathematical geography, of which the arts of surveying and cartography are applications. The motions of the earth as a planet must be taken into account, as they render possible the determination of position and direction by observations of the heavenly bodies. The diurnal rotation of the earth furnishes two fixed points or poles, the axis joining which is fixed or nearly so in its direction in space. The rotation of the earth thus fixes the directions of north and south and defines those of east and west. The angle which the earth’s axis makes with the plane in which the planet revolves round the sun determines the varying seasonal distribution of solar radiation over the surface and the mathematical zones of climate. Another important consequence of rotation is the deviation produced in moving bodies relatively to the surface. In the form known as Ferrell’s Law this runs: “If a body moves in any direction on the earth’s surface, there is a deflecting force which arises from the earth’s rotation which tends to deflect it to the right in the northern hemisphere but to the left in the southern hemisphere.” The deviation is of importance in the movement of air, of ocean currents, and to some extent of rivers.[13]

Physical geography.

In popular usage the words “physical geography” have come to mean geography viewed from a particular standpoint rather than any special department of the subject. The popular meaning is better conveyed by the word physiography, a term which appears to have been introduced by Linnaeus, and was reinvented as a substitute for the cosmography of the middle ages by Professor Huxley. Although the term has since been limited by some writers to one particular part of the subject, it seems best to maintain the original and literal meaning. In the stricter sense, physical geography is that part of geography which involves the processes of contemporary change in the crust and the circulation of the fluid envelopes. It thus draws upon physics for the explanation of the phenomena with the space-relations of which it is specially concerned. Physical geography naturally falls into three divisions, dealing respectively with the surface of the lithosphere—geomorphology; the hydrosphere—oceanography; and the atmosphere—climatology. All these rest upon the facts of mathematical geography, and the three are so closely inter-related that they cannot be rigidly separated in any discussion.

Geomorphology.

Geomorphology is the part of geography which deals with terrestrial relief, including the submarine as well as the subaërial portions of the crust. The history of the origin of the various forms belongs to geology, and can be completely studied only by geological methods. But the relief of the crust is not a finished piece of sculpture; the forms are for the most part transitional, owing their characteristic outlines to the process by which they are produced; therefore the geographer must, for strictly geographical purposes, take some account of the processes which are now in action modifying the forms of the crust. Opinion still differs as to the extent to which the geographer's work should overlap that of the geologist.

The primary distinction of the forms of the crust is that between elevations and depressions. Granting that the geoid or mean surface of the ocean is a uniform spheroid, the distribution of land and water approximately indicates a division of the surface of the globe into two areas, one of elevation and one of depression. The increasing number of measurements of the height of land in all continents and islands, and the very detailed levellings in those countries which have been thoroughly surveyed, enable the average elevation of the land above sea-level to be fairly estimated, although many vast gaps in accurate knowledge remain, and the estimate is not an exact one. The only part of the sea-bed the configuration of which is at all well known is the zone bordering the coasts where the depth is less than about 100 fathoms or 200 metres, *i.e.* those parts which sailors speak of as "in soundings." Actual or projected routes for telegraph cables across the deep sea have also been sounded with extreme accuracy in many cases; but beyond these lines of sounding the vast spaces of the ocean remain unplumbed save for the rare researches of scientific expeditions, such as those of the "Challenger," the "Valdivia," the "Albatross" and the "Scotia." Thus the best approximation to the average depth of the ocean is little more than an expert guess; yet a fair approximation is probable for the features of sub-oceanic relief are so much more uniform than those of the land that a smaller number of fixed points is required to determine them.

Crustal relief.

The chief element of uncertainty as to the largest features of the relief of the earth's crust is due to the unexplored area in the Arctic region and the larger regions of the Antarctic, of which we know nothing. We know that the earth's surface if unveiled of water would exhibit a great region of elevation arranged with a certain rough radiate symmetry round the north pole, and extending southwards in three unequal arms which taper to points in the south. A depression surrounds the little-known south polar region in a continuous ring and extends northwards in three vast hollows lying between the arms of the elevated area. So far only is it possible to speak with certainty, but it is permissible to take a few steps into the twilight of dawning knowledge and indicate the chief subdivisions which are likely to be established in the great crust-hollow and the great crust-heap. The boundary between these should obviously be the mean surface of the sphere.

Sir John Murray deduced the mean height of the land of the globe as about 2250 ft. above sea-level, and the mean depth of the oceans as 2080 fathoms or 12,480 ft. below sea-level.[14] Calculating the area of the land at 55,000,000 sq. m. (or 28.6% of the surface), and that of the oceans as 137,200,000 sq. m. (or 71.4% of the surface), he found that the volume of the land above sea-level was 23,450,000 cub. m., the volume of water below sea-level 323,800,000, and the total volume of the water equal to about $\frac{1}{666}$th of the volume of the whole globe. From these data, as revised by A. Supan,[15] H.R. Mill calculated the position of mean sphere-level at about 10,000 ft. or 1700 fathoms below sea-level. He showed that an imaginary spheroidal shell, concentric with the earth and cutting the slope between the elevated and depressed areas at the contour-line of 1700 fathoms, would not only leave above it a volume of the crust equal to the volume of the hollow left below it, but would also divide the surface of the earth so that the area of the elevated region was equal to that of the depressed region.[16]

Areas of the

A similar observation was made almost simultaneously by Romieux,[17] who further speculated on the equilibrium between the weight of the elevated land mass and that of the total waters of the ocean, and deduced some interesting relations between them. Murray, as the result of his study, divided the earth's surface into

crust according to Murray.

three zones—the *continental area* containing all dry land, the *transitional area* including the submarine slopes down to 1000 fathoms, and the *abysmal area* consisting of the floor of the ocean beyond that depth; and Mill proposed to take the line of mean-sphere level, instead of the empirical depth of 1000 fathoms, as the boundary between the transitional and abysmal areas.

An elaborate criticism of all the existing data regarding the volume relations of the vertical relief of the globe was made in 1894 by Professor Hermann Wagner, whose recalculations of volumes and mean heights—the best results which have yet been obtained—led to the following conclusions.[18] 631

Areas of the crust according to Wagner.

The area of the dry land was taken as 28.3% of the surface of the globe, and that of the oceans as 71.7%. The mean height deduced for the land was 2300 ft. above sea-level, the mean depth of the sea 11,500 ft. below, while the position of mean-sphere level comes out as 7500 ft. (1250 fathoms) below sea-level. From this it would appear that 43% of the earth's surface was above and 57% below the mean level. It must be noted, however, that since 1895 the soundings of Nansen in the north polar area, of the "Valdivia," "Belgica," "Gauss" and "Scotia" in the Southern Ocean, and of various surveying ships in the North and South Pacific, have proved that the mean depth of the ocean is considerably greater than had been supposed, and mean-sphere level must therefore lie deeper than the calculations of 1895 show; possibly not far from the position deduced from the freer estimate of 1888. The whole of the available data were utilized by the prince of Monaco in 1905 in the preparation of a complete bathymetrical map of the oceans on a uniform scale, which must long remain the standard work for reference on ocean depths.

By the device of a hypsographic curve co-ordinating the vertical relief and the areas of the earth's surface occupied by each zone of elevation, according to the system introduced by Supan,[19] Wagner showed his results graphically.

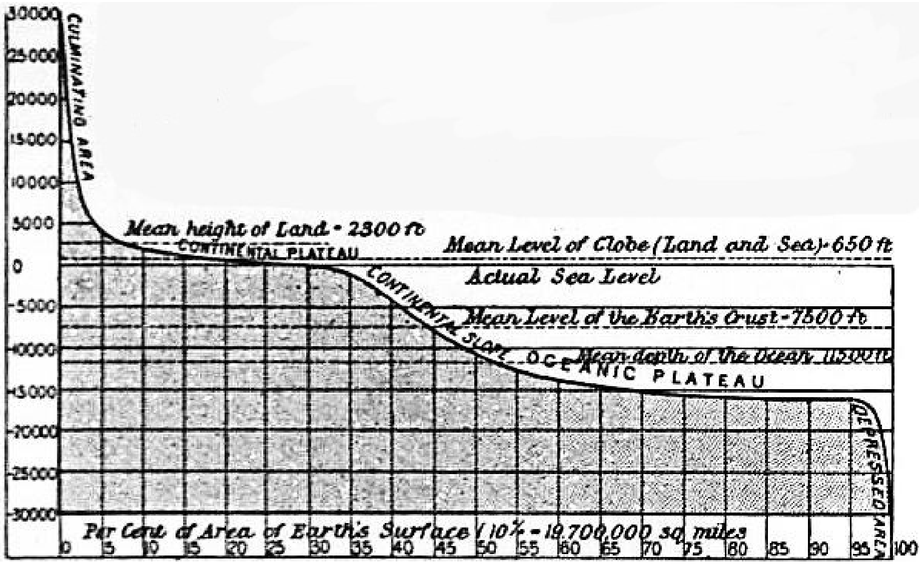
This curve with the values reduced from metres to feet is reproduced below.

Wagner subdivides the earth's surface, according to elevation, into the following five regions:

Wagner's Divisions of the Earth's Crust:

Name.	Per cent of Surface.	From	To
Depressed area	3	Deepest.	−16,400 feet.
Oceanic plateau	54	−16,400 feet.	− 7,400 feet.
Continental slope	9	− 7,400 feet.	− 660 feet.
Continental plateau	28	− 660 feet.	+ 3,000 feet.
Culminating area	6	+ 3,300 feet.	Highest.

The continental plateau might for purposes of detailed study be divided into the *continental shelf* from -660 ft. to sea-level, and *lowlands* from sea-level to +660 ft. (corresponding to the mean level of the whole globe).[20] *Uplands* reaching from 660 ft. to 2300 (the approximate mean level of the land), and *highlands*, from 2300 upwards, might also be distinguished.



A striking fact in the configuration of the crust is that each continent, or elevated mass of the crust, is diametrically opposite to an ocean basin or great depression; the only partial exception being in the case of southern South America, which is antipodal to eastern Asia. Professor C. Lapworth has generalized the grand features of crustal relief in a scheme of attractive simplicity. He sees throughout all the chaos of irregular crust-forms the recurrence of a certain harmony, a succession of folds or waves which build up all the minor features.[21] One great series of crust waves from east to west is crossed by a second great series of crust waves from north to south, giving rise by their interference to six great elevated masses (the continents), arranged in three groups, each consisting of a northern and a southern member separated by a minor depression. These elevated masses are divided from one another by similar great depressions.

Arrangement of world-ridges and hollows.

He says: "The surface of each of our great continental masses of land resembles that of a long and broad arch-like form, of which we see the simplest type in the New World. The surface of the North American arch is sagged downwards in the middle into a central depression which lies between two long marginal plateaus, and these plateaus are finally crowned by the wrinkled crests which form its two modern mountain systems. The surface of each of our ocean floors exactly resembles that of a continent turned upside down. Taking the Atlantic as our simplest type, we may say that the surface of an ocean basin resembles that of a mighty trough or syncline, buckled up more or less centrally in a medial ridge, which is bounded by two long and deep marginal hollows, in the cores of which still deeper grooves sink to the profoundest depths. This complementary relationship descends even to the minor features of the two. Where the great continental sag sinks below the ocean level, we have our gulfs and our Mediterraneans, seen in our type continent, as the Mexican Gulf and Hudson Bay. Where the central oceanic buckle attains the water-line we have our oceanic islands, seen in our type ocean, as St Helena and the Azores. Although the apparent crust-waves are neither equal in size nor symmetrical in form, this complementary relationship between them is always discernible. The broad Pacific depression seems to answer to the broad elevation of the Old World—the narrow trough of the Atlantic to the narrow continent of America."

Lapworth's fold-theory.

The most thorough discussion of the great features of terrestrial relief in the light of their origin is that by Professor E. Suess,[22] who points out that the plan of the earth is the result of two movements of the crust—one, subsidence over wide areas, giving rise to oceanic depressions and leaving the continents protuberant; the other, folding along comparatively narrow belts, giving rise to mountain ranges. This theory of crust blocks dropped by subsidence is opposed to Lapworth's theory of vast crust-folds, but geology is the science which has to decide between them.

Suess's theory.

Geomorphology is concerned, however, in the suggestions which have been made as to the cause of the distribution of heap and hollow in the larger features of the crust. Élie de Beaumont, in his speculations on the relation between the direction of mountain ranges and their geological age and character, was feeling towards a comprehensive theory of the forms of crustal relief; but his ideas were too geometrical, and his theory that the earth is a spheroid built up on a rhombic dodecahedron, the pentagonal faces of which determined the direction of mountain ranges, could not be proved.[23] The "tetrahedral theory" brought forward by Lowthian Green,[24] that the form of the earth is a spheroid based on a regular tetrahedron, is more serviceable, because it accounts for three very interesting facts of the terrestrial plan—(1) the antipodal position of continents and ocean basins; (2) the triangular outline of the continents; and (3) the excess of sea in the southern hemisphere. Recent investigations have recalled attention to the work of Lowthian Green, but the question is still in the controversial stage.[25] The study of tidal strain in the earth's crust by Sir George Darwin has led that physicist to indicate the possibility of the triangular form and southerly direction of the continents being a result of the differential or tidal attraction of the sun and moon. More recently Professor A.E.H. Love has shown that the great features of the relief of the lithosphere may be expressed by spherical harmonics of the first, second and third degrees, and their formation related to gravitational action in a sphere of unequal density.[26]

In any case it is fully recognized that the plan of the earth is so clear as to leave no doubt as to its being due to some general cause which should be capable of detection.

If the level of the sea were to become coincident with the mean level of the lithosphere, there would result one tri-radiate land-mass of nearly uniform outline and one continuous sheet of water broken by few islands. The actual position of sea-level lies so near the summit 632
of the crust-heap that the varied relief of the upper portion leads to the formation of a complicated coast-line and a great number of detached portions of land. The hydrosphere is, in fact, continuous, and the land is all in insular masses: the largest is the Old World of Europe, Asia and Africa; the next in size, America; the third, possibly, Antarctica; the fourth, Australia; the fifth, Greenland. After this there is a considerable gap before New Guinea, Borneo, Madagascar, Sumatra and the vast multitude of smaller islands descending in size by regular gradations to mere rocks. The contrast between island and mainland was natural enough in the days before the discovery of Australia, and the mainland of the Old World was traditionally divided into three continents. These "continents," "parts of the earth," or "quarters of the globe," proved to be convenient divisions; America was added as a fourth, and subsequently divided into two, while Australia on its discovery was classed sometimes as a new continent, sometimes merely as an island, sometimes compromisingly as an island-continent, according to individual opinion. The discovery of the insularity of Greenland might again give rise to the argument as to the distinction between island and continent. Although the name of continent was not applied to large portions of land for any physical reasons, it so happens that there is a certain physical similarity or homology between them which is not shared by the smaller islands or peninsulas.

The continents.

The typical continental form is triangular as regards its sea-level outline. The relief of the surface typically includes a central plain, sometimes dipping below sea-level, bounded by lateral highlands or mountain ranges, loftier on one side than on the other, the higher enclosing a plateau shut in by mountains. South America and North America follow this type most closely; Eurasia (the land mass of Europe and Asia) comes next, while Africa and Australia are farther removed from the type, and the structure of Antarctica and Greenland is unknown.

Homology of continents.

If the continuous, unbroken, horizontal extent of land in a continent is termed its *trunk*,[27] and the portions cut up by inlets or channels of the sea into islands and peninsulas the *limbs*, it is possible to compare the continents in an instructive manner.

The following table is from the statistics of Professor H. Wagner,[28] his metric measurements being transposed into British units:

Comparison of the Continents.

	Area total mil. sq. m.	Mean height, feet.	Area trunk, mil. sq. m.	Area penin-sulas, mil. sq. m.	Area islands, mil. sq. m.	Area limbs, mil. sq. m.	Area limbs, per cent.
Old World	35.8	2360					
New World	16.2	2230					
Eurasia	20.85	2620	15.42	4.09	1.34	5.43	26
Africa	11.46	2130	11.22	..	0.24	0.24	2.1
North America	9.26	2300	6.92	0.78	1.56	2.34	25
South America	6.84	1970	6.76	0.02	0.06	0.08	1.1
Australia	3.43	1310	2.77	0.16	0.50	0.66	19
Asia	17.02	3120	12.93	3.05	1.04	4.09	24
Europe	3.83	980	2.49	1.04	0.30	1.34	35

The usual classification of islands is into continental and oceanic. The former class includes all those which rise from the continental shelf, or show evidence in the character of their rocks of having at one time been continuous with a neighbouring continent. The latter rise abruptly from the oceanic abysses. Oceanic islands are divided according to their geological character into volcanic islands and those of organic origin, including coral islands. More elaborate subdivisions according to structure, origin and position have been proposed.[29] In some cases a piece of land is only an island at high water, and by imperceptible gradation the form passes into a peninsula. The typical peninsula is connected with the mainland by a relatively narrow isthmus; the name is, however, extended to any limb projecting from the trunk of the mainland, even when, as in the Indian peninsula, it is connected by its widest part.

Islands.

Small peninsulas are known as promontories or headlands, and the extremity as a cape. The opposite form, an inlet of the sea, is known when wide as a gulf, bay or bight, according to size and degree of inflection, or as a fjord or ria when long and narrow. It is convenient to employ a specific name for a projection of a coast-line less pronounced than a peninsula, and for an inlet less pronounced than a bay or bight; outcurve and incurve may serve the turn. The varieties of coast-lines were reduced to an exact classification by Richthofen, who grouped them according to the height and slope of the land into cliff-coasts (*Steilküsten*)—narrow beach coasts with cliffs, wide beach coasts with cliffs, and low coasts, subdividing each group according as the coast-line runs parallel to or crosses the line of strike of the mountains, or is not related to mountain structure. A further subdivision depends on the character of the inter-relation of land and sea along the shore producing such types as a fjord-coast, ria-coast or lagoon-coast. This extremely elaborate subdivision may be reduced, as Wagner points out, to three types—the continental coast where the sea comes up to the solid rock-material of the land; the marine coast, which is formed entirely of soft material sorted out by the sea; and the composite coast, in which both forms are combined.

Coasts.

On large-scale maps it is necessary to show two coast-lines, one for the highest, the other for the lowest tide; but in small-scale maps a single line is usually wider than is required to represent the whole breadth of the inter-tidal zone. The measurement of a coast-line is difficult, because the length will necessarily be greater when measured on a large-scale map where minute irregularities can be taken into account. It is usual to distinguish between the general coast-line measured from point to point of the headlands disregarding the smaller bays, and the detailed coast-line which takes account of every inflection shown by the map employed, and follows up river entrances to the point where tidal action ceases. The ratio between these two coast-lines represents the “coastal development” of any region.

Coast-lines.

While the forms of the sea-bed are not yet sufficiently well known to admit of exact classification, they are recognized to be as a rule distinct from the forms of the land, and the importance of using a distinctive terminology is felt. Efforts have been made to arrive at a definite international agreement on this subject, and certain terms suggested by a committee were adopted by the Eighth International Geographical Congress at New York in 1904.[30] The forms of the ocean floor include the “shelf,” or shallow sea margin, the “depression,” a general term applied to all submarine hollows, and the “elevation.” A depression when of great extent is termed a “basin,” when it is of a more or less round form with approximately equal diameters, a “trough” when it is wide and elongated with gently sloping borders, and a “trench” when narrow and elongated with steeply sloping borders, one of which rises higher than the other. The extension of a trough or basin penetrating the land or an elevation is termed an “embayment” when wide, and a “gully” when long and narrow; and the deepest part of a depression is termed a “deep.” A depression of small extent when steep-sided is termed a “caldron,” and a long narrow depression crossing a part of the continental border is termed a “furrow.” An elevation of great extent which rises at a very gentle angle from a surrounding depression is termed a “rise,” one which is relatively narrow and steep-sided a “ridge,” and one which is approximately equal in length and breadth but steep-sided a “plateau,” whether it springs direct from a depression or from a rise. An elevation of small extent is distinguished as a “dome” when it is more than 100 fathoms from the surface, a “bank” when it is nearer the surface than 100 fathoms but deeper than 6 fathoms, and a “shoal” when it comes within 6 fathoms of the surface and so becomes a serious danger to shipping. The highest point of an elevation is termed a “height,” if it does not form an island or one of the minor forms.

Submarine forms.

The forms of the dry land are of infinite variety, and have been studied in great detail.[31] From the descriptive or topographical point of view, geometrical form alone should be considered; but the origin and geological structure of land forms must in many cases be taken into account when dealing with the function they

Land forms.

exercise in the control of mobile distributions. The geographers who have hitherto given most attention to the forms of the land have been trained as geologists, and consequently there is a general tendency to make origin or structure the basis of classification rather than form alone.

The fundamental form-elements may be reduced to the six proposed by Professor Penck as the basis of his double system of classification by form and origin.[32] These may be looked upon as being all derived by various modifications or arrangements of the single form-unit, the *slope* or inclined plane surface. No one form occurs alone, but always grouped together with others in various ways to make up districts, regions and lands of distinctive characters. The form-elements are:

The six elementary land forms.

1. The *plain* or gently inclined uniform surface. 633

2. The *scarp* or steeply inclined slope; this is necessarily of small extent except in the direction of its length.

3. The *valley*, composed of two lateral parallel slopes inclined towards a narrow strip of plain at a lower level which itself slopes downwards in the direction of its length. Many varieties of this fundamental form may be distinguished.

4. The *mount*, composed of a surface falling away on every side from a particular place. This place may either be a point, as in a volcanic cone, or a line, as in a mountain range or ridge of hills.

5. The *hollow* or form produced by a land surface sloping inwards from all sides to a particular lowest place, the converse of a mount.

6. The *cavern* or space entirely surrounded by a land surface.

These forms never occur scattered haphazard over a region, but always in an orderly subordination depending on their mode of origin. The dominant forms result from crustal movements, the subsidiary from secondary reactions during the action of the primitive forms on mobile distributions. The geological structure and the mineral composition of the rocks are often the chief causes determining the character of the land forms of a region. Thus the scenery of a limestone country depends on the solubility and permeability of the rocks, leading to the typical Karst-formations of caverns, swallow-holes and underground stream courses, with the contingent phenomena of dry valleys and natural bridges. A sandy beach or desert owes its character to the mobility of its constituent sand-grains, which are readily drifted and piled up in the form of dunes. A region where volcanic activity has led to the embedding of dykes or bosses of hard rock amongst softer strata produces a plain broken by abrupt and isolated eminences.[33]

Geology and land forms.

It would be impracticable to go fully into the varieties of each specific form; but, partly as an example of modern geographical classification, partly because of the exceptional importance of mountains amongst the features of the land, one exception may be made. The classification of mountains into types has usually had regard rather to geological structure than to external form, so that some geologists would even apply the name of a mountain range to a region not distinguished by relief from the rest of the country if it bear geological evidence of having once been a true range. A mountain may be described (it cannot be defined) as an elevated region of irregular surface rising comparatively abruptly from lower ground. The actual elevation of a summit above sea-level does not necessarily affect its mountainous character; a gentle eminence, for instance, rising a few hundred feet above a tableland, even if at an elevation of say 15,000 ft., could only be called a hill.[34] But it may be said that any abrupt slope of 2000 ft. or more in vertical height may justly be called a mountain, while abrupt slopes of lesser height may be called hills. Existing classifications, however, do not take account of any difference in kind between mountain and hills, although it is common in the German language to speak of *Hügelland*, *Mittelgebirge* and *Hochgebirge* with a definite significance.

Classification of mountains.

The simple classification employed by Professor James Geikie[35] into mountains of accumulation, mountains of elevation and mountains of circumdenudation, is not considered sufficiently thorough by German geographers, who, following Richthofen, generally adopt a classification dependent on six primary divisions, each of which is subdivided. The terms employed, especially for the subdivisions, cannot be easily translated into other languages, and the English equivalents in the following table are only put forward tentatively:—

Richthofen’s Classification of Mountains[36]

I. *Tektonische Gebirge*—Tectonic mountains.

(*a*) *Bruchgebirge oder Schollengebirge*—Block mountains.

1. *Einseitige Schollengebirge oder Schollenrandgebirge*—Scarp or tilted block mountains.
    (i.) *Tafelscholle*—Table blocks.
    (ii.) *Abrasionsscholle*—Abraded blocks.
    (iii.) *Transgressionsscholle*—Blocks of unconformable strata.
2. *Flexurgebirge*—Flexure mountains.
3. *Horstgebirge*—Symmetrical block mountains.

(*b*) *Faltungsgebirge*—Fold mountains.

1. *Homöomorphe Faltungsgebirge*—Homomorphic fold mountains.
2. *Heteromorphe Faltungsgebirge*—Heteromorphic fold mountains.

II. *Rumpfgebirge oder Abrasionsgebirge*—Trunk or abraded mountains.

III. *Ausbruchsgebirge*—Eruptive mountains.

IV. *Aufschüttungsgebirge*—Mountains of accumulation.

V. *Flachböden*—Plateaux.

(*a*) *Abrasionsplatten*—Abraded plateaux.
(*b*) *Marines Flachland*—Plain of marine erosion.
(*c*) *Schichtungstafelland*—Horizontally stratified tableland.
(*d*) *Übergusstafelland*—Lava plain.
(*e*) *Stromflachland*—River plain.
(*f*) *Flachböden der atmosphärischen Aufschüttung*—Plains of aeolian formation.

VI. *Erosionsgebirge*—Mountains of erosion.

Mountain forms.

From the morphological point of view it is more important to distinguish the associations of forms, such as the *mountain mass* or group of mountains radiating from a centre, with the valleys furrowing their flanks spreading towards every direction; the *mountain chain* or line of heights, forming a long narrow ridge or series of ridges separated by parallel valleys; the *dissected plateau* or highland, divided into mountains of circumdenudation by a system of deeply-cut valleys; and the *isolated peak*, usually a volcanic cone or a hard rock mass left projecting after the softer strata which embedded it have been worn away (Monadnock of Professor Davis).

Distribution of mountains.

The geographical distribution of mountains is intimately associated with the great structural lines of the continents of which they form the culminating region. Lofty lines of fold mountains form the “backbones” of North America in the Rocky Mountains and the west coast systems, of South America in the Cordillera of the Andes, of Europe in the Pyrenees, Alps, Carpathians and Caucasus, and of Asia in the mountains of Asia Minor, converging on the Pamirs and diverging thence in the Himalaya and the vast mountain systems of central and eastern Asia. The remarkable line of volcanoes around the whole coast of the Pacific and along the margin of the Caribbean and Mediterranean seas is one of the most conspicuous features of the globe.

Functions of land forms.

Land waste.

If land forms may be compared to organs, the part they serve in the economy of the earth may, without straining the term, be characterized as functions. The first and simplest function of the land surface is that of guiding loose material to a lower level. The downward pull of gravity suffices to bring about the fall of such material, but the path it will follow and the distance it will travel before coming to rest depend upon the land form. The loose material may, and in an arid region does, consist only of portions of the higher parts of the surface detached by the expansion and contraction produced by heating and cooling due to radiation. Such broken material rolling down a uniform scarp would tend to reduce its steepness by the loss of material in the upper part and by the accumulation of a mound or scree against the lower part of the slope. But where the side is not a uniform scarp, but made up of a series of ridges and valleys, the tendency will be to distribute the detritus in an irregular manner, directing it away from one place and collecting it in great masses in another, so that in time the land form assumes a new appearance. Snow accumulating on the higher portions of the land, when compacted into ice and caused to flow

Glaciers.

downwards by gravity, gives rise, on account of its more coherent character, to continuous glaciers, which mould themselves to the slopes down which they are guided, different ice-streams converging to send forward a greater volume. Gradually coming to occupy definite beds, which are deepened and polished by the friction, they impress a characteristic appearance on the land, which guides them as they traverse it, and, although the ice melts at lower levels, vast quantities of clay and broken stones are brought down and deposited in terminal moraines where the glacier ends.

Rain.

Rain is by far the most important of the inorganic mobile distributions upon which land forms exercise their function of guidance and control. The precipitation of rain from the aqueous vapour of the atmosphere is caused in part by vertical movements of the atmosphere involving heat changes and apparently independent of the surface upon which precipitation occurs; but in greater part it is dictated by the form and altitude of the land surface and the direction of the prevailing winds, which itself is largely influenced by the land. It is on the windward faces of the highest ground, or just beyond the summit of less dominant heights upon the leeward side, that most rain falls, and all that does not evaporate or percolate into the ground is conducted back to the sea by a route which depends only on the form of the land. More mobile and more searching than ice or rock rubbish, the trickling drops are guided by the deepest lines of the hillside in their incipient flow, and as these lines converge, the stream, gaining strength, proceeds in its torrential course to carve its channel deeper and entrench itself in permanent occupation. Thus the stream-bed, from which at first the water might be blown away into a new channel by a gale of wind, ultimately grows to be the strongest line of the landscape. As the main valley deepens, the tributary stream-beds are deepened also, and gradually cut their way headwards, enlarging the area whence they draw their supplies. Thus new land forms are created—valleys of curious complexity, for example—by the "capture" and diversion of 634
the water of one river by another, leading to a change of watershed.[37] The minor tributaries become more numerous and more constant, until the system of torrents has impressed its own individuality on the mountain side. As the river leaves the mountain, ever growing by the accession of tributaries, it ceases, save in flood time, to be a formidable instrument of destruction; the gentler slope of the land surface gives to it only power sufficient to transport small stones, gravel, sand and ultimately mud. Its valley banks are cut back by the erosion of minor tributaries, or by rain-wash if the climate be moist, or left steep and sharp while the river deepens its bed if the climate be arid. The outline of the curve of a valley's sides ultimately depends on the angle of repose of the detritus which covers them, if there has been no subsequent change, such as the passage of a glacier along the valley, which tends to destroy the regularity of the cross-section. The slope of the river bed diminishes until the plain compels the river to move slowly, swinging in *meanders* proportioned to its size, and gradually, controlled by the flattening land, ceasing to transport material, but raising its banks and silting up its bed by the dropped sediment, until, split up and shoaled, its distributaries struggle across its delta to the sea. This is the typical river of which there are infinite varieties, yet every variety would, if time were given, and the land remained unchanged in level relatively to the sea, ultimately approach to the type. Movements of the land either of subsidence or elevation, changes in the land by the action of erosion in cutting back an escarpment or cutting through a col, changes in climate by affecting the rainfall and the volume of water, all tend to throw the river valley out of harmony with the actual condition of its stream. There is nothing more striking in geography than the perfection of the adjustment of a great river system to its valleys when the land has remained stable for a very lengthened period. Before full adjustment has been attained the river bed may be broken in places by waterfalls or interrupted by lakes; after adjustment the bed assumes a permanent outline, the slope diminishing more and more gradually, without a break in its symmetrical descent. Excellent examples of the indecisive drainage of a new land surface, on which the river system has not had time to impress itself, are to be seen in northern Canada and in Finland, where rivers are separated by scarcely perceptible divides, and the numerous lakes frequently belong to more than one river system.

River systems.

Adjustment of rivers to land.

The geographical cycle.

The action of rivers on the land is so important that it has been made the basis of a system of physical geography by Professor W.M. Davis, who classifies land surfaces in terms of the three factors—structure, process and time.[38] Of these time, during which the process is acting on the structure, is the most important. A land may thus be characterized by its position in the "geographical cycle", or cycle of erosion, as young, mature or old, the last term being reached when the base-level of erosion is attained, and the land, however varied its relief may have been in youth or maturity, is reduced to a nearly uniform surface or peneplain. By a re-elevation of a peneplain the rivers of an old land surface may be restored to youthful activity, and resume their shaping action, deepening the old valleys and initiating new ones, starting afresh the whole course of the geographical cycle. It is, however, not the action of the running water on the land, but the function exercised by the land on the running water, that is considered here to be the special province of geography. At every stage of the geographical cycle the land forms, as they exist at that stage, are concerned in guiding the condensation and flow of water in certain definite ways. Thus, for example, in a mountain range at right angles to a prevailing sea-wind, it is the land forms which determine that one side of the range shall be richly watered and deeply dissected by a complete system of valleys, while the other side is dry, indefinite in its valley systems, and sends none of its scanty drainage to the sea. The action of rain, ice and rivers conspires with the movement of land waste to strip the layer of soil from steep slopes as rapidly as it forms, and to cause it to accumulate on the flat valley bottoms, on the graceful flattened cones of alluvial fans at the outlet of the gorges of tributaries, or in the smoothly-spread surface of alluvial plains.

The whole question of the régime of rivers and lakes is sometimes treated under the name hydrography, a name used by some writers in the sense of marine surveying, and by others as synonymous with oceanography. For the study of rivers alone the name potamology[39] has been suggested by Penck, and the subject being of much practical importance has received a good deal of attention.[40]

Lakes and internal drainage.

The study of lakes has also been specialized under the name of limnology (see Lake).[41] The existence of lakes in hollows of the land depends upon the balance between precipitation and evaporation. A stream flowing into a hollow will tend to fill it up, and the water will begin to escape as soon as its level rises high enough to reach the lowest part of the rim. In the case of a large hollow in a very dry climate the rate of evaporation may be sufficient to prevent the water from ever rising to the lip, so that there is no outflow to the sea, and a basin of internal drainage is the result. This is the case, for instance, in the Caspian sea, the Aral and Balkhash lakes, the Tarim basin, the Sahara, inner Australia, the great basin of the United States and the Titicaca basin. These basins of internal drainage are calculated to amount to 22% of the land surface. The percentages of the land surface draining to the different oceans are approximately—Atlantic, 34.3%; Arctic sea, 16.5%; Pacific, 14.4%; Indian Ocean, 12.8%.[42]

Terminology of river systems.

The parts of a river system have not been so clearly defined as is desirable, hence the exaggerated importance popularly attached to “the source” of a river. A well-developed river system has in fact many equally important and widely-separated sources, the most distant from the mouth, the highest, or even that of largest initial volume not being necessarily of greater geographical interest than the rest. The whole of the land which directs drainage towards one river is known as its basin, catchment area or drainage area—sometimes, by an incorrect expression, as its valley or even its watershed. The boundary line between one drainage area and others is rightly termed the watershed, but on account of the ambiguity which has been tolerated it is better to call it water-parting or, as in America, divide. The only other important term which requires to be noted here is *talweg*, a word introduced from the German into French and English, and meaning the deepest line along the valley, which is necessarily occupied by a stream unless the valley is dry.

The functions of land forms extend beyond the control of the circulation of the atmosphere, the hydrosphere and the water which is continually being interchanged between them; they are exercised with increased effect in the higher departments of biogeography and anthropogeography.

Biogeography.

The sum of the organic life on the globe is termed by some geographers the biosphere, and it has been estimated that the whole mass of living substance in existence at one time would cover the surface of the earth to a depth of one-fifth of an inch.[43] The distribution of living organisms is a complex problem, a function of many factors, several of which are yet but little known. They include the biological nature of the organism and its physical environment, the latter involving conditions in which geographical elements, direct or indirect, preponderate. The direct geographical elements are the arrangement of land and sea (continents and islands standing in sharp contrast) and the vertical relief of the globe, which interposes barriers of a less absolute kind between portions of the same land area or oceanic depression. The indirect geographical elements, which, as a rule, act with and intensify the direct, are mainly climatic; the prevailing winds, rainfall, mean and extreme temperatures of every locality depending on the arrangement of land and sea and of land forms. Climate thus guided affects the weathering of rocks, and so determines the kind and arrangement of soil. Different species of organisms come to perfection in different climates; and it may be stated as a general rule that a species, whether of plant or animal, once established at one point, would spread over the whole zone of the climate congenial to it unless some barrier were interposed to its progress. In the case of land and fresh-water organisms the sea is the chief barrier; in the case of marine organisms, the land. Differences in land forms do not exert great influence on the distribution of living creatures directly, but indirectly such land forms as mountain ranges and internal drainage basins are very potent through their action on soil and climate. A snow-capped mountain ridge or an arid desert forms a barrier between different forms of life which is often more effective than an equal breadth of sea. In this way the surface of the land is divided into numerous natural regions, the flora and fauna of each of which include some distinctive species not shared by the others. The distribution of life is discussed in the various articles in this *Encyclopaedia* dealing with biological, botanical and zoological subjects.[44]

The classification of the land surface into areas inhabited by distinctive groups of plants has been attempted by many phyto-geographers, but without resulting in any scheme of general acceptance. The simplest classification is perhaps that of Drude according to climatic zones, subdivided according to continents. This takes account of—(1) the *Arctic-Alpine* zone, including all the vegetation of the region bordering on perpetual snow; (2) the *Boreal* zone, including the temperate lands of North America, Europe and Asia, all of which are substantially alike in botanical character; (3) the *Tropical* zone, divided sharply into (*a*) the tropical zone of the New World, and (*b*) the tropical zone of the Old World, the forms of which differ in a significant degree; (4) the *Austral* zone, comprising all continental land south of the equator, and sharply divided into three regions the floras of which are strikingly distinct—(*a*) South American, (*b*) South African and (*c*) Australian; (5) the *Oceanic*, comprising all oceanic islands, the flora of which consists exclusively of forms whose seeds could be drifted undestroyed by ocean currents or carried by birds. To these might be added the antarctic, which is still very imperfectly known. Many subdivisions and transitional zones have been suggested by different authors. 635

Floral zones.

From the point of view of the economy of the globe this classification by species is perhaps less important than that by mode of life and physiological character in accordance with environment. The following are the chief areas of vegetational activity usually recognized: (1) The ice-deserts of the arctic and antarctic and the highest mountain regions, where there is no vegetation except the lowest forms, like that which causes “red snow.” (2) The tundra or region of intensely cold winters, forbidding tree-growth, where mosses and lichens cover most of the ground when unfrozen, and shrubs occur of species which in other conditions are trees, here stunted to the height of a few inches. A similar zone surrounds the permanent snow on lofty mountains in all latitudes. The tundra passes by imperceptible gradations into the moor, bog and heath of warmer climates. (3) The temperate forests of evergreen or deciduous trees, according to circumstances, which occupy those parts of both temperate zones where rainfall and sunlight are both abundant. (4) The grassy steppes or prairies where the rainfall is diminished and temperatures are extreme, and grass is the prevailing form of vegetation. These pass imperceptibly into—(5) the arid desert, where rainfall is at a minimum, and the only plants are those modified to subsist with the smallest supply of water. (6) The tropical forest, which represents the maximum of plant luxuriance, stimulated by the heaviest rainfall, greatest heat and strongest light. These divisions merge one into the other, and admit of almost indefinite subdivision, while they are subject to great modifications by human interference in clearing and cultivating. Plants exhibit the controlling power of environment to a high degree, and thus vegetation is usually in close adjustment to the bolder geographical features of a region.

Vegetation areas.

The divisions of the earth into faunal regions by Dr P.L. Sclater have been found to hold good for a large number of groups of animals as different in their mode of life as birds and mammals, and they may thus be accepted as based on nature. They are six in number: (1) *Palaearctic*, including Europe, Asia north of the Himalaya, and Africa north of the Sahara; (2) *Ethiopian*, consisting of Africa south of the Atlas range, and Madagascar; (3) *Oriental*, including India, Indo-China and the Malay Archipelago north of Wallace’s line, which runs between Bali and Lombok; (4) *Australian*, including Australia, New Zealand, New Guinea and Polynesia; (5) *Nearctic* or North America, north of Mexico; and (6) *Neotropical* or South America. Each of these divisions is the home of a special fauna, many species of which are confined to it alone; in the Australian region, indeed, practically the whole fauna is peculiar and distinctive, suggesting a prolonged period of complete biological isolation. In some cases, such as the Ethiopian and Neotropical and the Palaearctic and Nearctic regions, the faunas, although distinct, are related, several forms on opposite sides of the Atlantic being analogous, *e.g.* the lion and puma, ostrich and rhea. Where two of the faunal realms meet there is usually, though not always, a mixing of faunas. These facts have led some naturalists to include the Palaearctic and Nearctic regions in one, termed *Holarctic*, and to suggest transitional regions, such as the *Sonoran*, between North and South America, and the *Mediterranean*, between Europe and Africa, or to create sub-regions, such as Madagascar and New Zealand. Oceanic islands have, as a rule, distinctive faunas and floras which resemble, but are not

Faunal realms.

identical with, those of other islands in similar positions.

The study of the evolution of faunas and the comparison of the faunas of distant regions have furnished a trustworthy instrument of pre-historic geographical research, which enables earlier geographical relations of land and sea to be traced out, and the approximate period, or at least the chronological order of the larger changes, to be estimated. In this way, for example, it has been suggested that a land, "Lemuria," once connected Madagascar with the Malay Archipelago, and that a northern extension of the antarctic land once united the three southern continents.

Biological distribution as a means of geographical research.

The distribution of fossils frequently makes it possible to map out approximately the general features of land and sea in long-past geological periods, and so to enable the history of crustal relief to be traced.[45]

While the tendency is for the living forms to come into harmony with their environment and to approach the state of equilibrium by successive adjustments if the environment should happen to change, it is to be observed that the action of organisms themselves often tends to change their environment. Corals and other quick-growing calcareous marine organisms are the most powerful in this respect by creating new land in the ocean. Vegetation of all sorts acts in a similar way, either in forming soil and assisting in breaking up rocks, in filling up shallow lakes, and even, like the mangrove, in reclaiming wide stretches of land from the sea. Plant life, utilizing solar light to combine the inorganic elements of water, soil and air into living substance, is the basis of all animal life. This is not by the supply of food alone, but also by the withdrawal of carbonic acid from the atmosphere, by which vegetation maintains the composition of the air in a state fit for the support of animal life. Man in the primitive stages of culture is scarcely to be distinguished from other animals as regards his subjection to environment, but in the higher grades of culture the conditions of control and reaction become much more complicated, and the department of anthropogeography is devoted to their consideration.

Reaction of organisms on environment.

The first requisites of all human beings are food and protection, in their search for which men are brought into intimate relations with the forms and productions of the earth's surface. The degree of dependence of any people upon environment varies inversely as the degree of culture or civilization, which for this purpose may perhaps be defined as the power of an individual to exercise control over the individual and over the environment for the benefit of the community. The development of culture is to a certain extent a question of race, and although forming one species, the varieties of man differ in almost imperceptible gradations with a complexity defying classification (see Anthropology). Professor Keane groups man round four leading types, which may be named the black, yellow, red and white, or the Ethiopic, Mongolic, American and Caucasic. Each may be subdivided, though not with great exactness, into smaller groups, either according to physical characteristics, of which the form of the head is most important, or according to language.

Anthropogeography.

The black type is found only in tropical or sub-tropical countries, and is usually in a primitive condition of culture, unless educated by contact with people of the white type. They follow the most primitive forms of religion (mainly fetishism), live on products of the woods or of the chase, with the minimum of work, and have only a loose political organization. The red type is peculiar to America, inhabiting every climate from polar to equatorial, and containing representatives of many stages of culture which had apparently developed without the aid or interference of people of any other race until the close of the 15th century. The yellow type is capable of a higher culture, cherishes higher religious beliefs, and inhabits as a rule the temperate zone, although extending to the tropics on one side and to the arctic regions on the other. The white type, originating in the north temperate zone, has spread over the whole world. They have attained the highest culture, profess the purest forms of monotheistic religion, and have brought all the people of the black type and many of those of the yellow under their domination.

Types of man.

The contrast between the yellow and white types has been softened by the remarkable development of the Japanese following the assimilation of western methods.

The actual number of human inhabitants in the world has been calculated as follows:

By Continents.[46]		By Race.[47]	
Asia	875,000,000	White (Caucasic)	770,000,000
Europe	392,000,000	Yellow (Mong.)	540,000,000
Africa	170,000,000	Black (Ethiopic)	175,000,000

America	143,000,000	Red (American)	22,000,000
Australia and Polynesia	7,000,000		—————
	—————	Total	1,507,000,000
Total	1,587,000,000		

In round numbers the population of the world is about 1,600,000,000, and, according to an estimate by Ravenstein,[48] the maximum population which it will be possible for the earth to maintain is 6000 millions, a number which, if the average rate of increase in 1891 continued, would be reached within 200 years.

While highly civilized communities are able to evade many of the restrictions of environment, to overcome the barriers to intercommunication interposed by land or sea, to counteract the adverse influence of climate, and by the development of trade even to inhabit 636
countries which cannot yield a food-supply, the mass of mankind is still completely under the control of those conditions which in the past determined the distribution and the mode of life of the whole human race.

Influence of environment on man.

In tropical forests primitive tribes depend on the collection of wild fruits, and in a minor degree on the chase of wild animals, for their food. Clothing is unnecessary; hence there is little occasion for exercising the mental faculties beyond the sense of perception to avoid enemies, or the inventive arts beyond what is required for the simplest weapons and the most primitive fortifications. When the pursuit of game becomes the chief occupation of a people there is of necessity a higher development of courage, skill, powers of observation and invention; and these qualities are still further enhanced in predatory tribes who take by force the food, clothing and other property prepared or collected by a feebler people. The fruit-eating savage cannot stray beyond his woods which bound his life as the water bounds that of a fish; the hunter is free to live on the margin of forests or in open country, while the robber or warrior from some natural stronghold of the mountains sweeps over the adjacent plains and carries his raids into distant lands. Wide grassy steppes lead to the organization of the people as nomads whose wealth consists in flocks and herds, and their dwellings are tents. The nomad not only domesticates and turns to his own use the gentler and more powerful animals, such as sheep, cattle, horses, camels, but even turns some predatory creatures, like the dog, into a means of defending their natural prey. They hunt the beasts of prey destructive to their flocks, and form armed bands for protection against marauders or for purposes of aggression on weaker sedentary neighbours. On the fertile low grounds along the margins of rivers or in clearings of forests, agricultural communities naturally take their rise, dwelling in villages and cultivating the wild grains, which by careful nurture and selection have been turned into rich cereals. The agriculturist as a rule is rooted to the soil. The land he tills he holds, and acquires a closer connexion with a particular patch of ground than either the hunter or the herdsman. In the temperate zone, where the seasons are sharply contrasted, but follow each other with regularity, foresight and self-denial were fostered, because if men did not exercise these qualities seed-time or harvest might pass into lost opportunities and the tribes would suffer. The more extreme climates of arid regions on the margins of the tropics, by the unpredictable succession of droughts and floods, confound the prevision of uninstructed people, and make prudence and industry qualities too uncertain in their results to be worth cultivating. Thus the civilization of agricultural peoples of the temperate zone grew rapidly, yet in each community a special type arose adapted to the soil, the crop and the climate. On the seashore fishing naturally became a means of livelihood, and dwellers by the sea, in virtue of the dangers to which they are exposed from storm and unseaworthy craft, are stimulated to a higher degree of foresight, quicker observation, prompter decision and more energetic action in emergencies than those who live inland. The building and handling of vessels also, and the utilization of such uncontrollable powers of nature as wind and tide, helped forward mechanical invention. To every type of coast there may be related a special type of occupation and even of character; the deep and gloomy fjord, backed by almost impassable mountains, bred bold mariners whose only outlet for enterprise was seawards towards other lands—the *viks* created the vikings. On the gently sloping margin of the estuary of a great river a view of tranquil inland life was equally presented to the shore-dweller, and the ocean did not present the only prospect of a career. Finally the mountain valley, with its patches of cultivable soil on the alluvial fans of tributary torrents, its narrow pastures on the uplands only left clear of snow in summer, its intensified extremes of climates and its isolation, almost equal to that of an island, has in all countries produced a special type of brave and hardy people, whose utmost effort may bring them comfort, but not wealth, by honest toil, who know little of the outer world, and to whom the natural outlet for ambition is marauding on the fertile plains. The highlander and viking, products of the valleys raised high amid the mountains or half-drowned in the sea, are everywhere of kindred spirit.

It is in some such manner as these that the natural conditions of regions, which must be conformed to by prudence and utilized by labour to yield shelter and food, have led to the growth of peoples differing in their ways of life, thought and speech. The initial differences so produced are confirmed and perpetuated by the same barriers which divide the faunal or floral regions, the sea, mountains, deserts and the like, and much of the course of past history and present politics becomes clear when the combined results of differing race and differing environment are taken into account.[49]

The specialization which accompanies the division of labour has important geographical consequences, for it necessitates communication between communities and the interchange of their products. Trade makes it possible to work mineral resources in localities where food can only be grown with great difficulty and expense, or which are even totally barren and waterless, entirely dependent on supplies from distant sources.

Density of population.

The population which can be permanently supported by a given area of land differs greatly according to the nature of the resources and the requirements of the people. Pastoral communities are always scattered very thinly over large areas; agricultural populations may be almost equally sparse where advanced methods of agriculture and labour-saving machinery are employed; but where a frugal people are situated on a fertile and inexhaustible soil, such as the deltas and river plains of Egypt, India and China, an enormous population may be supported on a small area. In most cases, however, a very dense population can only be maintained in regions where mineral resources have fixed the site of great manufacturing industries. The maximum density of population which a given region can support is very difficult to determine; it depends partly on the race and standard of culture of the people, partly on the nature and origin of the resources on which they depend, partly on the artificial burdens imposed and very largely on the climate. Density of population is measured by the average number of people residing on a unit of area; but in order to compare one part of the world with another the average should, strictly speaking, be taken for regions of equal size or of equal population; and the portions of the country which are permanently uninhabitable ought to be excluded from the calculation.[50] Considering the average density of population within the political limits of countries, the following list is of some value; the figures for a few smaller divisions of large countries are added (in brackets) for comparison:

Average Population on 1 sq. m. (*For 1900 or 1901.*)

Country.	Density of pop.	Country.	Density of pop.
(Saxony)	743*	Ceylon	141**
Belgium	589*	Greece	97
Java	568**	European Turkey	90
(England and Wales)	558	Spain	97
(Bengal)	495**	European Russia	55**
Holland	436	Sweden	30
United Kingdom	344	United States	25
Japan	317	Mexico	18
Italy	293	Norway	18
China proper	270**	Persia	15
German Empire	270	New Zealand	7
Austria	226	Argentina	5
Switzerland	207	Brazil	4.5
France	188	Eastern States of Australia	3
Indian Empire	167**	Dominion of Canada	1.5
Denmark	160**	Siberia	1
Hungary	154**	West Australia	0.2
Portugal	146		

* Almost exclusively industrial.

** Almost exclusively agricultural.

Migration.

The movement of people from one place to another without the immediate intention of returning is known as migration, and according to its origin it may be classed as centrifugal (directed *from* a particular area) and centripetal (directed *towards* a particular area). Centrifugal migration is usually a matter of compulsion; it may be necessitated by natural causes, such as a change of climate leading to the withering of pastures or destruction of agricultural land, to inundation, earthquake, pestilence or to an excess of population over means of support; or to artificial causes, such as the wholesale deportation of a conquered people; or to political or religious persecution. In any case the people are driven out by some adverse change; and when the urgency is great they may require to drive out in turn weaker people who occupy a desirable territory, thus propagating the wave of migration, the direction of which is guided by the forms of the land into inevitable channels. Many of the great historic movements of peoples were doubtless due to the gradual change of geographical or climatic conditions; and the slow desiccation of Central Asia has been plausibly suggested as the real cause of the peopling of modern Europe and of the medieval wars of the Old World, the theatres of which were critical points on the great natural lines of communication between east and west.

In the case of centripetal migrations people flock to some particular place where exceptionally favourable conditions have been found to exist. The rushes to gold-fields and diamond-fields are typical instances; the growth of towns on coal-fields and near other sources of power, and the rapid settlement of such rich agricultural districts as the wheat-lands of the American prairies and great plains are other examples.

There is, however, a tendency for people to remain rooted to the land of their birth, when not compelled or induced by powerful external causes to seek a new home. 637

Thus arises the spirit of patriotism, a product of purely geographical conditions, thereby differing from the sentiment of loyalty, which is of racial origin. Where race and soil conspire to evoke both loyalty and patriotism in a people, the moral qualities of a great and permanent nation are secured. It is noticeable that the patriotic spirit is strongest in those places where people are brought most intimately into relation with the land; dwellers in the mountain or by the sea, and, above all, the people of rugged coasts and mountainous archipelagoes, have always been renowned for love of country, while the inhabitants of fertile plains and trading communities are frequently less strongly attached to their own land.

Political geography.

Amongst nomads the tribe is the unit of government, the political bond is personal, and there is no definite territorial association of the people, who may be loyal but cannot be patriotic. The idea of a country arises only when a nation, either homogeneous or composed of several races, establishes itself in a region the boundaries of which may be defined and defended against aggression from without. Political geography takes account of the partition of the earth amongst organized communities, dealing with the relation of races to regions, and of nations to countries, and considering the conditions of territorial equilibrium and instability.

Boundaries.

The definition of boundaries and their delimitation is one of the most important parts of political geography. Natural boundaries are always the most definite and the strongest, lending themselves most readily to defence against aggression. The sea is the most effective of all, and an island state is recognized as the most stable. Next in importance comes a mountain range, but here there is often difficulty as to the definition of the actual crest-line, and mountain ranges being broad regions, it may happen that a small independent state, like Switzerland or Andorra, occupies the mountain valleys between two or more great countries. Rivers do not form effective international boundaries, although between dependent self-governing communities they are convenient lines of demarcation. A desert, or a belt of country left purposely without inhabitants, like the mark, marches or debatable lands of the middle ages, was once a common means of separating nations which nourished hereditary grievances. The “buffer-state” of modern diplomacy is of the same ineffectual type. A less definite though very practical boundary is that formed by the meeting-line of two languages, or the districts inhabited by two races. The line of fortresses protecting Austria from Italy lies in some places well back from the political boundary, but just inside the linguistic frontier, so as to separate the German and Italian races occupying Austrian territory. Arbitrary lines, either traced from point to point and marked by posts on the ground, or defined as portions of meridians and parallels, are now the most common type of boundaries fixed by treaty. In Europe and Asia frontiers are usually strongly fortified and strictly watched in times of peace as well as during war. In South America strictly defined boundaries are still the exception, and the claims of neighbouring nations have very frequently given rise to war, though now more commonly to arbitration.[51]

Forms of government.

The modes of government amongst civilized peoples have little influence on political geography; some republics are as arbitrary and exacting in their frontier regulations as some absolute monarchies. It is, however, to be noticed that absolute monarchies are confined to the east of Europe and to Asia, Japan being the only established constitutional monarchy east of the Carpathians. Limited monarchies are (with the exception of Japan) peculiar to Europe, and in these the degree of democratic control may be said to diminish as one passes eastwards from the United Kingdom. Republics, although represented in Europe, are the peculiar form of government of America and are unknown in Asia.

The forms of government of colonies present a series of transitional types from the autocratic administration of a governor appointed by the home government to complete democratic self-government. The latter occurs only in the temperate possessions of the British empire, in which there is no great preponderance of a coloured native population. New colonial forms have been developed during the partition of Africa amongst European powers, the sphere of influence being especially worthy of notice. This is a vaguer form of control than a protectorate, and frequently amounts merely to an agreement amongst civilized powers to respect the right of one of their number to exercise government within a certain area, if it should decide to do so at any future time.

The central governments of all civilized countries concerned with external relations are closely similar in their modes of action, but the internal administration may be very varied. In this respect a country is either centralized, like the United Kingdom or France, or federated of distinct self-governing units like Germany (where the units include kingdoms, at least three minor types of monarchies, municipalities and a crown land under a nominated governor), or the United States, where the units are democratic republics. The ultimate cause of the predominant form of federal government may be the geographical diversity of the country, as in the cantons occupying the once isolated mountain valleys of Switzerland, the racial diversity of the people, as in Austria-Hungary, or merely political expediency, as in republics of the American type.

The minor subdivisions into provinces, counties and parishes, or analogous areas, may also be related in many cases to natural features or racial differences perpetuated by historical causes. The territorial divisions and subdivisions often survive the conditions which led to their origin; hence the study of political geography is allied to history as closely as the study of physical geography is allied to geology, and for the same reason.

Towns.

The aggregation of population in towns was at one time mainly brought about by the necessity for defence, a fact indicated by the defensive sites of many old towns. In later times, towns have been more often founded in proximity to valuable mineral resources, and at critical points or nodes on lines of communication. These are places where the mode of travelling or of transport is changed, such as seaports, river ports and railway termini, or natural resting-places, such as a ford, the foot of a steep ascent on a road, the entrance of a valley leading up from a plain into the mountains, or a crossing-place of roads or railways.[52] The existence of a good natural harbour is often sufficient to give origin to a town and to fix one end of a line of land communication.

Lines of communication.

In countries of uniform surface or faint relief, roads and railways may be constructed in any direction without regard to the configuration. In places where the low ground is marshy, roads and railways often follow the ridge-lines of hills, or, as in Finland, the old glacial eskers, which run parallel to the shore. Wherever the relief of the land is pronounced, roads and railways are obliged to occupy the lowest ground winding along the valleys of rivers and through passes in the mountains. In exceptional cases obstructions which it would be impossible or too costly to turn are overcome by a bridge or tunnel, the magnitude of such works increasing with the growth of engineering skill and financial enterprise. Similarly the obstructions offered to water communication by interruption through land or shallows are overcome by cutting canals or dredging out channels. The economy and success of most lines of communication depend on following as far as possible existing natural lines and utilizing existing natural sources of power.[53]

Commercial geography.

Commercial geography may be defined as the description of the earth’s surface with special reference to the discovery, production, transport and exchange of commodities. The transport concerns land routes and sea routes, the latter being the more important. While steam has been said to make a ship independent of wind and tide, it is still true that a long voyage even by steam must be planned so as to encounter the least resistance possible from prevailing winds and permanent currents, and this involves the application of oceanographical and meteorological knowledge. The older navigation by utilizing the power of the wind demands a very intimate knowledge of these conditions, and it is probable that a revival of sailing ships may in the present century vastly increase the importance of the study of maritime meteorology.

The discovery and production of commodities require a knowledge of the distribution of geological formations for mineral products, of the natural distribution, life-conditions and cultivation or breeding of plants and animals and of the labour market. Attention must also be paid to the artificial restrictions of political geography, to the legislative restrictions bearing on labour and trade as imposed in different countries, and, above all, to the incessant fluctuations of the economic conditions of supply and demand and the combinations of capitalists or workers which affect the market.[54] The term "applied geography" has been employed to designate commercial geography, the fact being that every aspect of scientific geography may be applied to practical purposes, including the purposes of trade. But apart from the applied science, there is an aspect of pure geography which concerns the theory of the relation of economics to the surface of the earth.

Conclusion. It will be seen that as each successive aspect of geographical science is considered in its natural sequence the conditions become more numerous, complex, variable and practically important. From the underlying abstract mathematical considerations all through the superimposed physical, biological, anthropological, political and commercial development of the subject runs the determining control exercised by crust-forms acting directly or indirectly on mobile distributions; and this is the essential principle of geography. 638

(H. R. M.)

---

1 A concise sketch of the whole history of geographical method or theory as distinguished from the history of geographical discovery (see later section of this article) is only to be found in the introduction to H. Wagner's *Lehrbuch der Geographie*, vol. i. (Leipzig, 1900), which is in every way the most complete treatise on the principles of geography.

2 *History of Ancient Geography* (Cambridge, 1897), p. 70.

3 See J.L. Myres, "An Attempt to reconstruct the Maps used by Herodotus," *Geographical Journal*, viii. (1896), p. 605.

4 *Geschichte der wissenschaftlichen Erdkunde der Griechen* (Leipzig, 1891), Abt. 3, p. 60.

5 Bunbury's *History of Ancient Geography* (2 vols., London, 1879), Müller's *Geographi Graeci minores* (2 vols., Paris, 1855, 1861) and Berger's *Geschichte der wissenschaftlichen Erdkunde der Griechen* (4 vols., Leipzig, 1887-1893) are standard authorities on the Greek geographers.

6 The period of the early middle ages is dealt with in Beazley's *Dawn of Modern Geography* (London; part i., 1897; part ii., 1901; part iii., 1906); see also Winstedt, *Cosmos Indicopleustes* (1910).

7 From translator's preface to the English version by Mr Dugdale (1733), entitled *A Complete System of General Geography*, revised by Dr Peter Shaw (London, 1756).

8 Printed in *Schriften zur physischen Geographie*, vol. vi. of Schubert's edition of the collected works of Kant (Leipzig, 1839). First published with notes by Rink in 1802.

9 *History of Civilization*, vol. i. (1857).

10 See H.J. Mackinder in *British Association Report* (Ipswich), 1895, p. 738, for a summary of German opinion, which has been expressed by many writers in a somewhat voluminous literature.

11 H. Wagner's year-book, *Geographische Jahrbuch*, published at Gotha, is the best systematic record of the progress of geography in all departments; and Haack's *Geographen Kalender*, also published annually at Gotha, gives complete lists of the geographical societies and geographers of the world.

12 This phrase is old, appearing in one of the earliest English works on geography, William Cuningham's *Cosmographical Glasse conteinyng the pleasant Principles of Cosmographie, Geographie, Hydrographie or Navigation* (London, 1559).

13 See also S. Günther, *Handbuch der mathematischen Geographie* (Stuttgart, 1890).

14 "On the Height of the Land and the Depth of the Ocean," *Scot. Geog. Mag.* iv. (1888), p. 1. Estimates had been made previously by Humboldt, De Lapparent, H. Wagner, and subsequently by Penck and Heiderich, and for the oceans by Karstens.

15 *Petermanns Mitteilungen*, xxv. (1889), p. 17.

16 *Proc. Roy. Soc. Edin.* xvii. (1890) p. 185.

17 *Comptes rendus Acad. Sci.* (Paris, 1890), vol. iii. p. 994.

18 "Areal und mittlere Erhebung der Landflächen sowie der Erdkruste" in Gerland's *Beiträge zur Geophysik*, ii. (1895) p. 667. See also *Nature*, 54 (1896), p. 112.

19 *Petermanns Mitteilungen*, xxxv. (1889) p. 19.

20 The areas of the continental shelf and lowlands are approximately equal, and it is an interesting circumstance that, taken as a whole, the actual coast-line comes just midway on the most nearly level belt of the earth's surface, excepting the ocean floor. The configuration of the continental slope has been treated in detail by Nansen in *Scientific Results of Norwegian North Polar Expedition*, vol. iv. (1904), where full references to the literature of the subject will be found.

21 *British Association Report* (Edinburgh, 1892), p. 699.

22 *Das Antlitz der Erde* (4 vols., Leipzig, 1885, 1888, 1901). Translated under the editorship of E. de Margerie, with much additional matter, as *La Face de la terre*, vols. i. and ii. (Paris, 1897, 1900), and into English by Dr Hertha Sollas as *The Face of the Earth*, vols. i. and ii. (Oxford, 1904, 1906).

23 Élie de Beaumont, *Notice sur les systèmes de montagnes* (3 vols., Paris, 1852).

24 *Vestiges of the Molten Globe* (London, 1875).

25 See J.W. Gregory, "The Plan of the Earth and its Causes," *Geog. Journal*, xiii. (1899) p. 225; Lord Avebury, *ibid.* xv. (1900) p. 46; Marcel Bertrand, "Déformation tétraédrique de la terre et déplacement du pôle," *Comptes rendus Acad. Sci.* (Paris, 1900), vol. cxxx. p. 449; and A. de Lapparent, *ibid.* p. 614.

26 See A.E.H. Love, "Gravitational Stability of the Earth," *Phil. Trans.* ser. A. vol. ccvii. (1907) p. 171.

27 *Rumpf*, in German, the language in which this distinction was first made.

28 *Lehrbuch der Geographie* (Hanover and Leipzig, 1900), Bd. i. S. 245, 249.

29 See, for example, F.G. Hahn's *Insel-Studien* (Leipzig, 1883).

30 See *Geographical Journal*, xxii. (1903) pp. 191-194.

31 The most important works on the classification of land forms are F. von Richthofen, *Führer für Forschungsreisende* (Berlin, 1886); G. de la Noë and E. de Margerie, *Les Formes du terrain* (Paris, 1888); and above all A. Penck, *Morphologie der Erdoberfläche* (2 vols., Stuttgart, 1894). Compare also A. de Lapparent, *Leçons de géographie physique* (2nd ed., Paris, 1898), and W.M. Davis, *Physical Geography* (Boston, 1899).

32 "Geomorphologie als genetische Wissenschaft," in *Report of Sixth International Geog. Congress* (London, 1895), p. 735 (English Abstract, p. 748).

33 On this subject see J. Geikie, *Earth Sculpture* (London, 1898); J.E. Marr, *The Scientific Study of Scenery* (London, 1900); Sir A. Geikie, *The Scenery and Geology of Scotland* (London, 2nd ed., 1887); Lord Avebury (Sir J. Lubbock), *The Scenery of Switzerland* (London, 1896) and *The Scenery of England* (London, 1902).

34 Some geographers distinguish a mountain from a hill by origin; thus Professor Seeley says "a mountain implies elevation and a hill implies denudation, but the external forms of both are often identical." *Report VI. Int. Geog. Congress* (London, 1895), p. 751.

35 "Mountains," in *Scot. Geog. Mag.* ii. (1896) p. 145.

36 *Führer für Forschungsreisende*, pp. 652-685.

37 See, for a summary of river-action, A. Phillipson, *Studien über Wasserscheiden* (Leipzig, 1886); also I.C. Russell, *River Development*, (London, 1898) (published as *The Rivers of North America*, New York, 1898).

38 W.M. Davis, "The Geographical Cycle," *Geog. Journ.* xiv. (1899) p. 484.

39 A. Penck, "Potamology as a Branch of Physical Geography," *Geog. Journ.* x. (1897) p. 619.

40 See, for instance, E. Wisotzki, *Hauptfluss und Nebenfluss* (Stettin, 1889). For practical studies see official reports on the Mississippi, Rhine, Seine, Elbe and other great rivers.

41 F.A. Forel, *Handbuch der Seenkunde: allgemeine Limnologie* (Stuttgart, 1901); F.A. Forel, "La Limnologie, branche de la géographie," *Report VI. Int. Geog. Congress* (London, 1895), p. 593; also *Le Léman* (2 vols., Lausanne, 1892, 1894); H. Lullies, "Studien über Seen," *Jubiläumsschrift der Albertus-Universität* (Königsberg, 1894); and G.R. Credner, "Die Reliktenseen," *Petermanns Mitteilungen*, Ergänzungshefte 86 and 89 (Gotha., 1887, 1888).

42 J. Murray, "Drainage Areas of the Continents," *Scot. Geog. Mag.* ii. (1886) p. 548.

43 Wagner, *Lehrbuch der Geographie* (1900), i. 586.

44 For details, see A.R. Wallace, *Geographical Distribution of Animals and Island Life*; A. Heilprin, *Geographical and Geological Distribution of Animals* (1887); O. Drude, *Handbuch der Pflanzengeographie*; A. Engler, *Entwickelungsgeschichte der Pflanzenwelt*; also Beddard, *Zoogeography* (Cambridge, 1895); and Sclater, *The Geography of Mammals* (London, 1899).

45 See particularly A. de Lapparent, *Traité de géologie* (4th ed., Paris, 1900).

46 Estimate for 1900. H. Wagner, *Lehrbuch der Geographie*, i. P. 658.

47 Estimate for year not stated. A.H. Keane in *International Geography*, p. 108.

48 In *Proc. R. G. S.* xiii. (1891) p. 27.

49 On the influence of land on people see Shaler, *Nature and Man in America* (New York and London, 1892); and Ellen C. Semple's *American History and its Geographic Conditions* (Boston,

1903).

50 See maps of density of population in Bartholomew's great large-scale atlases, *Atlas of Scotland* and *Atlas of England*.

51 For the history of territorial changes in Europe, see Freeman, *Historical Geography of Europe*, edited by Bury (Oxford), 1903; and for the official definition of existing boundaries, see Hertslet, *The Map of Europe by Treaty* (4 vols., London, 1875, 1891); *The Map of Africa by Treaty* (3 vols., London, 1896). Also Lord Curzon's Oxford address on *Frontiers* (1907).

52 For numerous special instances of the determining causes of town sites, see G.G. Chisholm, "On the Distribution of Towns and Villages in England," *Geographical Journal* (1897), ix. 76, x. 511.

53 The whole subject of anthropogeography is treated in a masterly way by F. Ratzel in his *Anthropogeographie* (Stuttgart, vol. i. 2nd ed., 1899, vol. ii. 1891), and in his *Politische Geographie* (Leipzig, 1897). The special question of the reaction of man on his environment is handled by G.P. Marsh in *Man and Nature, or Physical Geography as modified by Human Action* (London, 1864).

54 For commercial geography see G.G. Chisholm, *Manual of Commercial Geography* (1890).

GEOID (from Gr. γῆ, the earth), an imaginary surface employed by geodesists which has the property that every element of it is perpendicular to the plumb-line where that line cuts it. Compared with the "spheroid of reference" the surface of the geoid is in general depressed over the oceans and raised over the great land masses. (See Earth, Figure of the.)

GEOK-TEPE, a former fortress of the Turkomans, in Russian Transcaspia, in the oasis of Akhal-tekke, on the Transcaspian railway, 28 m. N.W. of Askabad. It consisted of a walled enclosure 1¾ m. in circuit, the wall being 18 ft. high and 20 to 30 ft. thick. In December 1880 the place was attacked by 6000 Russians under General Skobelev, and after a siege of twenty-three days was carried by storm, although the defenders numbered 25,000. A monument and a small museum commemorate the event.

GEOLOGY (from Gr. γῆ, the earth, and λόγος, science), the science which investigates the physical history of the earth. Its object is to trace the structural progress of our planet from the earliest beginnings of its separate existence, through its various stages of growth, down to the present condition of things. It seeks to determine the manner in which the evolution of the earth's great surface features has been effected. It unravels the complicated processes by which each continent has been built up. It follows, even into detail, the varied sculpture of mountain and valley, crag and ravine. Nor does it confine itself merely to changes in the inorganic world. Geology shows that the present races of plants and animals are the descendants of other and very different races which once peopled the earth. It teaches that there has been a progressive development of the inhabitants, as well as one of the globe on which they have dwelt; that each successive period in the earth's history, since the introduction of living things, has been marked by characteristic types of the animal and vegetable kingdoms; and that, however imperfectly the remains of these organisms have been preserved or may be deciphered, materials exist for a history of life upon the planet. The geographical distribution of existing faunas and floras is often made clear and intelligible by geological evidence; and in the same way light is thrown upon some of the remoter phases in the history of man himself. A subject so comprehensive as this must require a wide and varied basis of evidence. It is one of the characteristics of geology to gather evidence from sources which at first sight seem far removed from its scope, and to seek aid from almost every other leading branch of science. Thus, in dealing with the earliest conditions of the planet, the geologist must fully avail himself of the labours of the astronomer. Whatever is ascertainable by telescope, spectroscope or chemical analysis, regarding the constitution of other heavenly bodies, has a geological bearing. The experiments of the physicist, undertaken to determine conditions of matter and of energy, may sometimes be taken as the starting-points of geological investigation. The work of the chemical laboratory forms the foundation of a vast and increasing mass of geological inquiry. To the botanist, the zoologist, even to the unscientific, if observant, traveller by land or sea, the geologist turns for information and assistance.

But while thus culling freely from the dominions of other sciences, geology claims as its peculiar territory the rocky framework of the globe. In the materials composing that framework, their composition and arrangement, the processes of their formation, the changes which they have undergone, and the terrestrial revolutions to which they bear witness, lie the main data of geological history. It is the task of the geologist to group these elements in such a way that they may be made to yield up their evidence as to the march of events in the evolution of the planet. He finds that they have in large measure arranged themselves in chronological sequence,—the oldest lying at the bottom and the newest at the top. Relics of an ancient sea-floor are overlain by traces of a vanished land-surface; these are in turn covered by the deposits of a former lake, above which once more appear proofs of the return of the sea. Among these rocky records lie the lavas and ashes of long-extinct volcanoes. The ripple left upon the shore, the cracks formed by the sun’s heat upon the muddy bottom of a dried-up pool, the very imprint of the drops of a passing rainshower, have all been accurately preserved, and yield their evidence as to geographical conditions often widely different from those which exist where such markings are now found.

But it is mainly by the remains of plants and animals imbedded in the rocks that the geologist is guided in unravelling the chronological succession of geological changes. He has found that a certain order of appearance characterizes these organic remains, that each great group of rocks is marked by its own special types of life, and that these types can be recognized, and the rocks in which they occur can be correlated even in distant countries, and where no other means of comparison would be possible. At one moment he has to deal with the bones of some large mammal scattered through a deposit of superficial gravel, at another time with the minute foraminifers and ostracods of an upraised sea-bottom. Corals and crinoids crowded and crushed into a massive limestone where they lived and died, ferns and terrestrial plants matted together into a bed of coal where they originally grew, the scattered shells of a submarine sand-bank, the snails and lizards which lived and died within a hollow-tree, the insects which have been imprisoned within the exuding resin of old forests, the footprints of birds and quadrupeds, the trails of worms left upon former shores—these, and innumerable other pieces of evidence, enable the geologist to realize in some measure what the faunas and floras of successive periods have been, and what geographical changes the site of every land has undergone.

It is evident that to deal successfully with these varied materials, a considerable acquaintance with different branches of science is needful. Especially necessary is a tolerably wide knowledge of the processes now at work in changing the surface of the earth, and of at least those forms of plant and animal life whose remains are apt to be preserved in geological deposits, or which in their structure and habitat enable us to realize what their forerunners were. It has often been insisted that the present is the key to the past; and in a wide sense this assertion is eminently true. Only in proportion as we understand the present, where everything is open on all sides to the fullest investigation, can we expect to decipher the past, where so much is obscure, imperfectly preserved or not preserved at all. A study of the existing economy of nature ought thus to be the foundation of the geologist’s training.

While, however, the present condition of things is thus employed, we must obviously be on our guard against the danger of unconsciously assuming that the phase of nature’s operations which we now witness has been the same in all past time, that geological changes have always or generally taken place in former ages in the manner and on the scale which we behold to-day, and that at the present time all the great geological processes, which have produced changes in the past eras of the earth’s history, are still existent and active. As a working hypothesis we may suppose that the nature of geological processes has remained constant from the beginning; but we cannot postulate that the action of these processes has never varied in energy. The few centuries wherein man has been observing nature obviously form much too brief an interval by which to measure the intensity of geological action in all past time. For aught we can tell the present is an era of quietude and slow change, compared with some of the eras which have preceded it. Nor perhaps can we be quite sure that, when we have explored every geological process now in progress, we have exhausted all the causes of change which, even in comparatively recent times, have been at work.

In dealing with the geological record, as the accessible solid part of the globe is called, we cannot too vividly realize that at the best it forms but an imperfect chronicle. Geological history cannot be compiled from a full and continuous series of documents. From the very nature of its origin the record is necessarily fragmentary, and it has been further mutilated and obscured by the revolutions of successive ages. And even where the chronicle of events is continuous, it is of very unequal value in different places. In one case, for example, it may present us with an unbroken succession of deposits many thousands of feet in thickness, from which, however, only a few meagre facts as to geological history can be gleaned. In another instance it brings before us, within the compass of a few yards, the evidence of a most varied and complicated series of changes in physical geography, as well as an abundant and interesting suite of organic remains. These and other characteristics of the geological record become more apparent and intelligible as we proceed in the study of the science. 639

Classification.—For systematic treatment the subject may be conveniently arranged in the following parts:—

1. *The Historical Development of Geological Science.*—Here a brief outline will be given of the gradual growth of geological conceptions from the days of the Greeks and Romans down to modern times, tracing the separate progress of the more important branches of inquiry and noting some of the stages which in each case have led up to the present condition of the science.

2. *The Cosmical Aspects of Geology.*—This section embraces the evidence supplied by astronomy and physics regarding the form and motions of the earth, the composition of the planets and sun, and the probable history of the solar system. The subjects dealt with under this head are chiefly treated in separate articles.

3. *Geognosy.*—An inquiry into the materials of the earth’s substance. This division, which deals with the parts of the earth, its envelopes of air and water, its solid crust and the probable condition of its interior, especially treats of the more important minerals of the crust, and the chief rocks of which that crust is built up. Geognosy thus lays a foundation of knowledge regarding the nature of the materials constituting the mass of the globe, and prepares the way for an investigation of the processes by which these materials are produced and altered.

4. *Dynamical Geology* studies the nature and working of the various geological processes whereby the rocks of the earth’s crust are formed and metamorphosed, and by which changes are effected upon the distribution of sea and land, and upon the forms of terrestrial surfaces. Such an inquiry necessitates a careful examination of the existing geological economy of nature, and forms a fitting introduction to an inquiry into the geological changes of former periods.

5. *Geotectonic or Structural Geology* has for its object the architecture of the earth’s crust. It embraces an inquiry into the manner in which the various materials composing this crust have been arranged. It shows that some have been formed in beds or strata of sediment on the floor of the sea, that others have been built up by the slow aggregation of organic forms, that others have been poured out in a molten condition or in showers of loose dust from subterranean sources. It further reveals that, though originally laid down in almost horizontal beds, the rocks have subsequently been crumpled, contorted and dislocated, that they have been incessantly worn down, and have often been depressed and buried beneath later accumulations.

6. *Palaeontological Geology.*—This branch of the subject, starting from the evidence supplied by the organic forms which are found preserved in the crust of the earth, includes such questions as the relations between extinct and living types, the laws which appear to have governed the distribution of life in time and in space, the relative importance of different genera of animals in geological inquiry, the nature and use of the evidence from organic remains regarding former conditions of physical geography. Some of these problems belong also to zoology and botany, and are more fully discussed in the articles Palaeontology and Palaeobotany.

7. *Stratigraphical Geology.*—This section might be called geological history. It works out the chronological succession of the great formations of the earth’s crust, and endeavours to

trace the sequence of events of which they contain the record. More particularly, it determines the order of succession of the various plants and animals which in past time have peopled the earth, and thus ascertains what has been the grand march of life upon this planet.

8. *Physiographical Geology*, proceeding from the basis of fact laid down by stratigraphical geology regarding former geographical changes, embraces an inquiry into the origin and history of the features of the earth’s surface—continental ridges and ocean basins, plains, valleys and mountains. It explains the causes on which local differences of scenery depend, and shows under what very different circumstances, and at what widely separated intervals, the hills and mountains, even of a single country, have been produced.

Most of the detail embraced in these several sections is relegated to separate articles, to which references are here inserted. The following pages thus deal mainly with the general principles and historical development of the science:—

Part I.—Historical Development

Geological Ideas among the Greeks and Romans.—Many geological phenomena present themselves in so striking a form that they could hardly fail to impress the imagination of the earliest and rudest races of mankind. Such incidents as earthquakes and volcanic eruptions, destructive storms on land and sea, disastrous floods and landslips suddenly strewing valleys with ruin, must have awakened the terror of those who witnessed them. Prominent features of landscape, such as mountain-chains with their snows, clouds and thunderstorms, dark river-chasms that seem purposely cleft open in order to give passage to the torrents that rush through them, crags with their impressive array of pinnacles and recesses must have appealed of old, as they still do, to the awe and wonder of those who for the first time behold them. Again, banks of sea-shells in far inland districts would, in course of time, arrest the attention of the more intelligent and reflective observers, and raise in their minds some kind of surmise as to how such shells could ever have come there. These and other conspicuous geological problems found their earliest solution in legends and myths, wherein the more striking terrestrial features and the elemental forces of nature were represented to be the manifestation of the power of unseen supernatural beings.

The basin of the Mediterranean Sea was especially well adapted, from its physical conditions, to be the birth-place of such fables. It is a region frequently shaken by earthquakes, and contains two distinct centres of volcanic activity, one in the Aegean Sea and one in Italy. It is bounded on the north by a long succession of lofty snow-capped mountain-ranges, whence copious rivers, often swollen by heavy rains or melted snows, carry the drainage into the sea. On the south it boasts the Nile, once so full of mystery; likewise wide tracts of arid desert with their dreaded dust storms. The Mediterranean itself, though an inland sea, is subject to gales, which, on exposed coasts, raise breakers quite large enough to give a vivid impression of the power of ocean waves. The countries that surround this great sheet of water display in many places widely-spread deposits full of sea shells, like those that still live in the neighbouring bays and gulfs. Such a region was not only well fitted to supply subjects for mythology, but also to furnish, on every side, materials which, in their interest and suggestiveness, would appeal to the reason of observant men.

It was natural, therefore, that the early philosophers of Greece should have noted some of these geological features, and should have sought for other explanations of them than those to be found in the popular myths. The opinions entertained in antiquity on these subjects may be conveniently grouped under two heads: (1) Geological processes now in operation, and (2) geological changes in the past.

1. *Contemporary Processes.*—The geological processes of the present time are partly at work underground and partly on the surface of the earth. The former, from their frequently disastrous character, received much attention from Greek and Roman authors. Aristotle, in his *Meteorics*, cites the speculations of several of his predecessors which he rejects in favour of his own opinion to the effect that earthquakes are due to the generation of wind within the earth, under the influence of the warmth of the sun and the internal heat. Wind, being the lightest and most rapidly moving body, is the cause of motion in other bodies, and fire, united with wind, becomes flame, which is endowed with great rapidity of motion. Aristotle looked upon earthquakes and volcanic eruptions as closely connected with each other, the discharge of hot materials to the surface being the result of a severe earthquake, when finally the wind rushes out with violence, and sometimes buries the surrounding country under sparks and cinders, as had happened at Lipari. These crude conceptions of the nature of volcanic action, and the cause of earthquakes, continued to prevail for many centuries.

Earthquakes and volcanoes.

They are repeated by Lucretius, who, however, following Anaximenes, includes as one of the causes of earthquakes the fall of mountainous masses of rock undermined by time, and the consequent propagation of gigantic tremors far and wide through the earth. Strabo, having travelled through the volcanic districts of Italy, was able to recognize that Vesuvius had once been an active volcano, although no eruption had taken place from it within human memory. He continued to hold the belief that volcanic energy arose from the movement of subterranean wind. He believed that the district around the Strait of Messina, which had formerly suffered from destructive earthquakes, was seldom visited by them after the volcanic vents of that region had been opened, so as to provide an escape for the subterranean fire, wind, water and burning masses. He cites in his *Geography* a number of examples of widespread as well as local sinkings of land, and alludes also to the uprise of the sea-bottom. He likewise regards some islands as having been thrown up by volcanic agency, and others as torn from the mainland by such convulsions as earthquakes.

The most detailed account of earthquake phenomena which has come down to us from antiquity is that of Seneca in his *Quaestiones Naturales*. This philosopher had been much interested in the accounts given him by survivors and witnesses of the earthquake which convulsed the district of Naples in February A.D. 63. He distinguished several distinct movements of the ground: 1st, the up and down motion (*succussio*); 2nd, the oscillatory motion (*inclinatio*); and probably a third, that of trembling or vibration. While admitting that some earthquakes may arise from the collapse of the walls of subterranean cavities, he adhered to the old idea, held by the most numerous and important previous writers, that these commotions are caused mainly by the movements of wind imprisoned within the earth. As to the origin of volcanic outbursts he supposed that the subterranean wind in struggling for an outlet, and whirling through the chasms and passages, meets with great store of sulphur and other combustible substances, which by mere friction are set on fire. The elder Pliny reiterates the commonly accepted opinion as to the efficacy of wind underground. In discussing the phenomena of earthquakes he remarks that towns with many culverts and houses with cellars suffer less than others, and that at Naples those houses are most shaken which stand on hard ground. It thus appears that with regard to subterranean geological operations, no advance was made during the time of the Greeks and Romans as to the theoretical explanation of these phenomena; but a considerable body of facts was collected, especially as to the effects of earthquakes and the occurrence of volcanic eruptions.

Action of rivers.

The superficial processes of geology, being much less striking than those of subterranean energy, naturally attracted less attention in antiquity. The operations of rivers, however, which so intimately affect a human population, were watched with more or less care. Herodotus, struck by the amount of alluvial silt brought down annually by the Nile and spread over the flat inundated land, inferred that "Egypt is the gift of the river." Aristotle, in discussing some of the features of rivers, displays considerable acquaintance with the various drainage-systems on the north side of the Mediterranean basin. He refers to the mountains as condensers of the atmospheric moisture, and shows that the largest rivers rise among the loftiest high grounds. He shows how sensibly the alluvial deposits carried down to the sea increase the breadth of the land, and cites some parts of the shores of the Black Sea, where, in sixty years, the rivers had brought down such a quantity of material that the vessels then in use required to be of much smaller draught than previously, the water shallowing so much that the marshy ground would, in course of time, become dry land. Strabo supplies further interesting information as to the work of rivers in making their alluvial plains and in pushing their deltas seaward. He remarks that these deltas are prevented from advancing farther outward by the ebb and flow of the tides.

Occurrences of fossils.

2. *Past Processes.*—The abundant well-preserved marine shells exposed among the upraised Tertiary and post-Tertiary deposits in the countries bordering the Mediterranean are not infrequently alluded to in Greek and Latin literature. Xenophanes of Colophon (614 B.C.) noticed the occurrence of shells and other marine productions inland among the mountains, and inferred from them that the land had risen out of the sea. A similar conclusion was drawn by Xanthus the Lydian (464 B.C.) from shells like scallops and cockles, which were found far from the sea in Armenia and Lower Phrygia. Herodotus, Eratosthenes, Strato and Strabo noted the vast quantities of fossil shells in different parts of Egypt, together with beds of salt, as evidence that the sea had once spread over the country. But by far the most philosophical opinions on the past mutations of the earth's surface are those expressed by Aristotle in the treatise already cited. Reviewing the evidence of these changes, he recognized that the sea now covers tracts that were once dry land, and that land will one day reappear where there is now sea. These alternations are to be regarded as following each other in a certain order and periodicity. But they are apt to escape our notice because they require successive periods of time, which, compared with our brief existence, are of enormous duration, and because they are brought about so imperceptibly that we fail to detect them in progress. In a

celebrated passage in his *Metamorphoses*, Ovid puts into the mouth of the philosopher Pythagoras an account of what was probably regarded as the Pythagorean view of the subject in the Augustan age. It affirms the interchange of land and sea, the erosion of valleys by descending rivers, the washing down of mountains into the sea, the disappearance of the rivers and the submergence of land by earthquake movements, the separation of some islands from, and the union of others with, the mainland, the uprise of hills by volcanic action, the rise and extinction of burning mountains. There was a time before Etna began to glow, and the time is coming when the mountain will cease to burn.

From this brief sketch it will be seen that while the ancients had accumulated a good deal of information regarding the occurrence of geological changes, their interpretations of the phenomena were to a considerable extent mere fanciful speculation. They had acquired only a most imperfect conception of the nature and operation of the geological processes; and though many writers realized that the surface of the earth has not always been, and will not always remain, as it is now, they had no glimpse of the vast succession of changes of that surface which have been revealed by geology. They built hypotheses on the slenderest basis of fact, and did not realize the necessity of testing or verifying them.

Progress of Geological Conceptions in the Middle Ages.—During the centuries that succeeded the fall of the Western empire little progress was made in natural science. The schoolmen in the monasteries and other seminaries were content to take their science from the literature of Greece and Rome. The Arabs, however, not only collected and translated that literature, but in some departments made original observations themselves. To one of the most illustrious of their number, Avicenna, the translator of Aristotle, a treatise has been ascribed, in which singularly modern ideas are expressed regarding mountains, some of which are there stated to have been produced by an uplifting of the ground, while others have been left prominent, owing to the wearing away of the softer rocks around them. In either case, it is confessed that the process would demand long tracts of time for its completion.

After the revival of learning the ancient problem presented by fossil shells imbedded in the rocks of the interior of many countries received renewed attention. But the conditions for its solution were no longer what they had been in the days of the philosophers of antiquity. Men were not now free to adopt and teach any doctrine they pleased on the subject. The Christian church had meanwhile arisen to power all over Europe, and adjudged as heretics all who ventured to impugn any of her dogmas. She taught that the land and the sea had been separated on the third day of creation, before the appearance of any animal life, which was not created until the fifth day. To assert that the dry land is made up in great part of rocks that were formed in the sea, and are crowded with the remains of animals, was plainly to impugn the veracity of the Bible. Again, it had come to be the orthodox belief that only somewhere about 6000 years had elapsed since the time of Adam and Eve. If any thoughtful observer, impressed with the overwhelming force of the evidence that the fossiliferous formations of the earth's crust must have taken long periods of time for their accumulation, ventured to give public expression to his conviction, he ran considerable risk of being proceeded against as a heretic. It was needful, therefore, to find some explanation of the facts of nature, which would not run counter to the ecclesiastical system of the day. Various such interpretations were proposed, doubtless in an honest endeavour at reconciliation. Three of these deserve special notice: (1) Many able observers and diligent collectors of fossils persuaded themselves that these objects never belonged to organisms of any kind, but should be regarded as mere "freaks of nature," having no more connexion with any once living creature than the frost patterns on a window. They were styled "formed" or "figured" stones, "lapides sui generis," and were asserted to be due to some inorganic imitative process within the earth or to the influence of the stars. (2) Observers who could not resist the evidence of their senses that the fossil shells once belonged to living animals, and who, at the same time, felt the necessity of accounting for the presence of marine organisms in the rocks of which the dry land is largely built up, sought a way out of the difficulty by invoking the Deluge of Noah. Here was a catastrophe which, they said, extended over the whole globe, and by which the entire dry land was submerged even up to the tops of the high hills. True, it only lasted one hundred and fifty days, but so little were the facts then appreciated that no difficulty seems to have been generally felt in crowding the accumulation of the thousands of feet of fossiliferous formations into that brief space of time. (3) Some more intelligent men in Italy, recognizing that these interpretations could not be upheld, fell back upon the idea that the rocks in which fossil shells are imbedded might have been heaped up by repeated and vigorous eruptions from volcanic centres. Certain modern eruptions in the Aegean Sea and in the Bay of Naples had drawn attention to the rapidity with which hills of considerable size could be piled around an active crater. It was argued that if Monte Nuovo near Naples could have been accumulated to a height of nearly 500 ft. in two days, there seemed to be no reason against believing that, during the time of the Flood, and in the course of the centuries that have elapsed since that event, the whole of the fossiliferous rocks might have been deposited. Unfortunately for this hypothesis it ignored the fact that these rocks do not consist of volcanic materials.

So long as the fundamental question remained in dispute as to the true character and history of the stratified portion of the earth's crust containing organic remains, geology as a science could not begin its existence. The diluvialists (those who relied on the hypothesis of the Flood) held the field during the 16th, 17th and a great part of the 18th century. They were looked on as the champions of orthodoxy; and, on that account, they doubtless wielded much more influence than would have been gained by them from the force of their arguments. Yet during those ages there were not wanting occasional observers who did good service in combating the prevalent misconceptions, and in preparing the way for the ultimate triumph of truth. It was more especially in Italy, where many of the more striking phenomena of geology are conspicuously displayed, that the early pioneers of the science arose, and that for several generations the most marked progress was made towards placing the investigations of the past history of the earth upon a basis of careful observation and scientific deduction. One of the first of these leaders was Leonardo da Vinci (1452-1519), who, besides his achievements in painting, sculpture, architecture and engineering, contributed some notable observations regarding the great problem of the origin of fossil shells. He ridiculed the notion that these objects could have been formed by the influence of the stars, and maintained that they had once belonged to living organisms, and therefore that what is now land was formerly covered by the sea. Girolamo Fracastorio (1483-1553) claimed that the shells could never have been left by the Flood, which was a mere temporary inundation, but that they proved the mountains, in which they occur, to have been successively uplifted out of the sea. On the other hand, even an accomplished anatomist like Gabriello Falloppio (1523-1562) found it easier to believe that the bones of elephants, teeth of sharks, shells and other fossils were mere earthy inorganic concretions, than that the waters of Noah's Flood could ever nave reached as far as Italy.

Leonardo da Vinci; Fracastorio; Falloppio.

By much the most important member of this early band of Italian writers was undoubtedly Nicolas Steno (1631-1687), who, though born in Copenhagen, ultimately settled in Florence. Having made a European reputation as an anatomist, his attention was drawn to geological problems by finding that the rocks of the north of Italy contained what appeared to be sharks' teeth closely resembling those of a dog-fish, of which he had published the anatomy. Cautiously at first, for fear of offending orthodox opinions, but afterwards more boldly, he proclaimed his conviction that those objects had once been part of living animals, and that they threw light on some of the past history of the earth. He published in 1669 a small tract, *De solido intra solidum naturaliter contento*, in which he developed the ideas he had formed of this history from an attentive study of the rocks. He showed that the stratified formations of the hills and valleys consist of such materials as would be laid down in the form of sediment in turbid water; that where they contain marine productions this water is proved to have been the sea; that diversities in their composition point to commingling of currents, carrying different kinds of sediment of which the heaviest would first sink to the bottom. He made original and important observations on stratification, and laid down some of the fundamental axioms in stratigraphy. He reasoned that as the original position of strata was approximately horizontal, when they are found to be steeply inclined or vertical, or bent into arches, they have been disrupted by subterranean exhalations, or by the falling in of the roofs of underground cavernous spaces. It is to this alteration of the original position of the strata that the inequalities of the earth's surface, such as mountains, are to be ascribed, though some have been formed by the outburst of fire, ashes and stones from inside the earth. Another effect of the dislocation has been to provide fissures, which serve as outlets for springs. Steno's anatomical training peculiarly fitted him for dealing authoritatively with the question of the nature and origin of the fossils contained in the rocks. He had no hesitation in affirming that, even if no shells had ever been found living in the sea, the internal structure of these fossils would demonstrate that they once formed parts of living animals. And not only shells, but teeth, bones and skeletons of many kinds of fishes had been quarried out of the rocks, while some of the strata had skulls, horns and teeth of land-animals. Illustrating his general principles by a sketch of what he supposed to have been the past history of Tuscany, he added a series of diagrams which show how clearly he had conceived the essential elements of stratigraphy. He thought he could perceive the records of six successive phases in the evolution of the framework of that country, and was inclined to believe that a similar chronological sequence would be found all over the world. He anticipated the objections that would be brought against his views on account of the insuperable difficulty in granting the length of time that would be required for all the geographical vicissitudes which his interpretation required. He thought that many of the fossils must be as old as the time of the general deluge, but he was careful not to indulge in any speculation as to the antiquity of the earth.

Nicolas Steno.

To the Italian school, as especially typified in Steno, must be assigned the honour of having thus begun to lay firmly and truly the first foundation stones of the modern science of geology. The same school included Antonio Vallisneri (1661-1730), who surpassed his predecessors in his wider and more exact knowledge of the fossiliferous rocks that form the backbone of the Italian peninsula, which he contended were formed during a wide and prolonged submergence of the region, altogether different from the brief deluge of Noah. There was likewise Lazzaro Moro (1687-1740), who did good service against the diluvialists, but the fundamental feature of his system of nature lay in the preponderant part which, unaware of the great difference between volcanic materials and ordinary sediment, he assigned to volcanic action in the production of the sedimentary rocks of the earth's crust. He supposed that in the beginning the globe was completely surrounded with water, beneath which the solid earth lay as a smooth ball. On the third day of creation, however, vast fires were kindled inside the globe, whereby the smooth surface of stone was broken up, and portions of it, appearing above the water, formed the earliest land. From that time onward, volcanic eruptions succeeded each other, not only on the emerged land, but on the sea-floor, over which the ejected material spread in an ever augmenting thickness of sedimentary strata. In this way Moro carried the history of the stratified rocks beyond the time of the Flood back to the Creation, which was supposed to have been some 1600 years earlier; and he brought it down to the present day, when fresh sedimentary deposits are continually accumulating. He thus incurred no censure from the ecclesiastical guardians of the faith, and he succeeded in attracting increased public attention to the problems of geology. The influence of his teaching, however, was subsequently in great part due to the Carmelite friar Generelli, who published an eloquent exposition of Moro's views.

Lazzaro Moro.

The Cosmogonists and Theories of the Earth.—While in Italy substantial progress was made in collecting information regarding the fossiliferous formations of that country, and in forming conclusions concerning them based upon more or less accurate observations, the tendency to mere fanciful speculation, which could not be wholly repressed in any country, reached a remarkable extravagance in England. In proportion as materials were yet lacking from which to construct a history of the evolution of our planet in accordance with the teaching of the church, imagination supplied the place of ascertained fact, and there appeared during the last twenty years of the 18th century a group of English cosmogonists, who, by the sensational character of their speculations, aroused general attention both in Britain and on the continent. It may be doubted, however, whether the effect of their writings was not to hinder the advance of true science by diverting men from the observation of nature into barren controversy over unrealities. It is not needful here to do more than mention the names of Thomas Burnet, whose *Sacred Theory of the Earth* appeared in 1681, and William Whiston, whose New Theory of the Earth was published in 1696. Hardly less fanciful than these writers, though his practical acquaintance with rocks and fossils was infinitely greater, was John Woodward, whose *Essay towards a Natural History of the Earth* dates from 1695. More important as a contribution to science was the catalogue of the large collection of fossils, which he had made from the rocks of England and which he bequeathed to the university of Cambridge. This catalogue appeared in 1728-1729 with the title of *An attempt towards a Natural History of the Fossils of England*.

A striking contrast to these cosmogonists is furnished by another group, which arose in France and Germany, and gave to the world the first rational ideas concerning the probable primeval evolution of our globe. The earliest of these pioneers was the illustrious philosopher René Descartes (1596-1650). He propounded a scheme of cosmical development in which he represented the earth, like the other planets, to have been originally a mass of glowing material like the sun, and to have gradually cooled on the outside, while still retaining an incandescent, self-luminous nucleus. Yet with this noble conception, which modern science has accepted, Descartes could not shake himself free from the time-honoured error in regard to the origin of volcanic action. He thought that certain exhalations within the earth condense into oil, which, when in violent motion, enters into the subterranean cavities, where it passes into a kind of smoke. This smoke is from time to time ignited by a spark of fire and, pressing violently against its containing walls, gives rise to earthquakes. If the flame breaks through to the surface at the top of a mountain, it may escape with enormous energy, hurling forth much earth mingled with sulphur or bitumen, and thus producing a volcano. The mountain might burn for a long time until at last its store of fuel in the shape of sulphur or bitumen would be exhausted. Not only did the philosopher refrain from availing himself of the high internal temperature of the globe as the source of volcanic energy, he even did not make use of it as the cause of the ignition of his supposed internal fuel, but speculated on the kindling of the subterranean fires by the spirits or gases setting fire to the exhalations, or by the fall of masses of rock and the sparks produced by their friction or percussion.

Descartes.

The ideas of Descartes regarding planetary evolution were enlarged and made more definite by Wilhelm Gottfried Leibnitz (1646-1716), whose teaching has largely influenced all subsequent speculation on the subject. In his great tract, the *Protogaea* (published in 1749, thirty-three years after his death), he traced the probable passage of our earth from an original condition of incandescent vapour into that of a smooth molten globe, which, by continuous cooling, acquired an external solid crust and rugose surface. He thought that the more ancient rocks, such as granite and gneiss, might be portions of the earliest outer crust; and that as the external solidification advanced, immense subterranean cavities were left which were filled with air and water. By the collapse of the roofs of these caverns, valleys might be originated at the surface, while the solid intervening walls would remain in place and form mountains. By the disruption of the crust, enormous bodies of water were launched over the surface of the earth, which swept vast quantities of sediment together, and thus gave rise to sedimentary deposits. After many vicissitudes of this kind, the terrestrial forces calmed down, and a more stable condition of things was established.

Leibnitz.

An important feature in the cosmogony of Leibnitz is the prominent place which he assigned to organic remains in the stratified rocks of the crust. Ridiculing the foolish attempts to account for the presence of these objects by calling them “sports of nature,” he showed that they are to be regarded as historical monuments; and he adduced a number of instances wherein successive platforms of strata, containing organic remains, bear witness to a series of advances and retreats of the sea. He recognized that some of the fossils appeared to have nothing like them in the living world of to-day, but some analogous forms might yet be found, he thought, in still unexplored parts of the earth; and even if no living representatives should ever be discovered, many types of animals might have undergone transformation during the great changes which had affected the surface of the earth. In spite of his clear realization of the vast store of potential energy residing within the highly heated interior of the earth, Leibnitz continued to regard volcanic action as due to the combustion of inflammable substances enclosed within the terrestrial crust, such as stone-coal, naphtha and sulphur.

Appealing to a much wider public than Descartes or Leibnitz, and basing his speculations on a wider acquaintance with the organic and inorganic realms of nature, G.L.L. de Buffon (1707-1788) was undoubtedly one of the most influential forces that in Europe guided the growth of geological ideas during the 18th century. He published in 1749 a *Theory of the Earth*, in which he adopted views similar to those of Descartes and Leibnitz as to planetary evolution; but though he realized the importance of fossils as records of former conditions of the earth’s surface, he accounted for them by supposing that they had been deposited from a universal ocean, a large part of which had subsequently been engulfed into caverns in the interior of the globe. Thirty years later, after having laboured with skill and enthusiasm in all branches of natural history, he published another work, his famous *Époques de la nature* (1778), which is specially remarkable as the first attempt to deal with the history of the earth in a chronological manner, and to compute, on a basis of experiment, the antiquity of the several stages of this history. His experiments were made with globes of cast iron, and could not have yielded results of any value for his purpose; but in so far as his calculations were not mere random guesses but had some kind of foundation on experiment, they deserve respectful recognition. He divided the history of our earth into six periods of unequal duration, the whole comprising a period of some 70,000 or 75,000 years. He supposed that the stage of incandescence, before the globe had consolidated to the centre, lasted 2936 years, and that about 35,000 years elapsed before the surface had cooled sufficiently to be touched, and therefore to be capable of supporting living things. Terrestrial animal life, however, was not introduced until 55,000 or 60,000 years after the beginning of the world or about 15,000 years before our time. Looking into the future, he foresaw that, by continued refrigeration, our globe will eventually become colder than ice, and this fair face of nature, with its manifold varieties of plant and animal life, will perish after having existed for 132,000 years.

Buffon.

Buffon’s conception of the operation of the geological agents did not become broader or more accurate in the interval between the appearance of his two treatises. He still continued to believe in the lowering of the ocean by subsidence into vast subterranean cavities, with a consequent emergence of land. He still looked on volcanoes as due to the burning of “pyritous and combustible stones,” though he now called in the co-operation of electricity. He calculated that the first volcanoes could not arise until some 50,000 years after the beginning of the world, by which time a sufficient extent of dense vegetation had been buried in the earth to supply them with fuel. He appears to have had but an imperfect acquaintance with the literature of his own time. At least there can be little doubt that had he availed himself of the labours of his own countryman, Jean Etienne Guettard (1715-1786), of Giovanni Arduíno (1714-1795) in Italy, and of Johann Gottlob Lehmann (d. 1767) and George Christian Füchsel (1722-1773) in Germany, he would have been able to give to his “epochs” a more definite succession of events and a greater correspondence with the facts of nature.

Among the writers of the 18th century, who formed philosophical conceptions of the system of processes by which the life of our earth as a habitable globe is carried on, a foremost place must be assigned to James Hutton (1726-1797). Educated for the medical profession, he studied at Edinburgh and at Paris, and took his doctor’s degree at Leiden. But having inherited a small landed property in Berwickshire, he took to agriculture, and after putting his land into excellent order, let his farm and betook himself to Edinburgh, there to gratify the scientific tastes which he had developed early in life. He had been more especially led to study minerals and rocks, and to meditate on the problems which they suggest as to the constitution and history of the earth. His journeys in Britain and on the continent of Europe had furnished him with material for reflection; and he had gradually evolved a system or theory in which all the scattered facts could be arranged so as to show their mutual dependence and their place in the orderly mechanism of the world. He used to discuss his views with one or two of his friends, but refrained from publishing them to the world until, on the foundation of the Royal Society of Edinburgh, he communicated an outline of his doctrine to that learned body in 1785. Some years later he expanded this first essay into a larger work in two volumes, which were published in 1795 with the title of *Theory of the Earth, with Proofs and Illustrations*.

James Hutton.

Hutton’s teaching has exercised a profound influence on modern geology. This influence, however, has arisen less from his own writings than from the account of his doctrines given by his friend John Playfair in the classic work entitled *Illustrations of the Huttonian Theory*, published in 1802. Hutton wrote in so prolix and obscure a style as rather to repel than attract readers. Playfair, on the other hand, expressed himself in such clear and graceful language as to command general attention, and to gain wide acceptance for his master’s views. Unlike the older cosmogonists, Hutton refrained from trying to explain the origin of things, and from speculations as to what might possibly have been the early history of our globe. He determined from the outset to interpret the past by what can be seen to be the present order of nature; and he refused to admit the operation of causes which cannot be shown to be part of the actual terrestrial system. Like other observers who had preceded him, he recognized in the various rocks composing the dry land evidence of former geographical conditions very different from those which now prevail. He saw that the vast majority of rocks consist of hardened sediments and must have been deposited in the sea. He could distinguish among them an older or Primary series, and a younger or Secondary series; and did not dispute the existence of a Tertiary series claimed by Peter Simon Pallas (1741-1811). He believed that these various aqueous accumulations had been consolidated by subterranean heat, that the oldest and lowest rocks had suffered most from this action, that into these more deep-seated masses subsequent veins and larger bodies of molten matter were injected from below, and thus that what was originally loose detritus eventually became changed in such crystalline schists as are now found in mountain-chains. In the course of these terrestrial revolutions sedimentary strata, originally more or less nearly horizontal, have been pushed upward, dislocated, crumpled, placed on end, and even elevated to form ranges of lofty mountains. Hutton looked upon these disturbances as due to the expansive power of subterranean heat; but he did not attempt to sketch the mechanism of the process, and he expressly declined to offer any conjecture as to how the land so elevated remains in that position. He thought that the interior of our planet may “be a fluid mass, melted, but unchanged by the action of heat”; and, far from connecting volcanoes with the combustion of inflammable substances, as had been the prevalent belief for so many centuries, he looked upon them as a beneficent provision of “spiracles to the subterranean furnace, in order to prevent the unnecessary elevation of land and fatal effects of earthquakes.”

John Playfair.

A distinguishing feature of the Huttonian philosophy is to be seen in the breadth of its conceptions regarding the geological operations continually in progress on the surface of the globe. Hutton saw that the land is undergoing a ceaseless process of degradation, through the influence of the air, frost, rain, rivers and the sea, and that in course of time, if no countervailing agency should intervene, the whole of the dry land will be washed away into the sea. But he also perceived that this universal erosion is not everywhere carried on at the same rate; that it is specially active along the channels of torrents and rivers, and that, owing to this difference these channels are gradually deepened and widened, until the complicated valley-system of a country is carved out. He recognized that the detritus worn away from the land must be spread out over the floor of the sea, so as to form there strata similar to those that compose most of the dry land. As he could detect in the structure of land convincing evidence that former sea floors had been elevated to form the continents and islands of to-day, he could look forward to future ages, when the same subterranean agency which had raised up the present land would again be employed to uplift the bed of the existing ocean, thus to renew the surface of our earth as a habitable globe, and to start a fresh cycle of erosion and deposition.

Though Hutton was not unaware that organic remains abound in many of the stratified rocks, he left them out of consideration in the elaboration of his theory. It was otherwise with one of his French contemporaries, the illustrious J.B. Lamarck (1744-1829), who, after having attained great eminence as a botanist, turned to zoology when he was nearly fifty years of age, and before long rose to even greater distinction in that department of science. His share in the classification and description of the mollusca and in founding invertebrate palaeontology, his theory of organic evolution and his philosophical treatment of many biological questions have been tardily recognized, but his contributions to geology have been less generally acknowledged. When he accepted the "professorship of zoology; of insects, of worms and of microscopic animals" at the Museum of Natural History, Paris, in 1793, he at once entered with characteristic ardour and capacity into the new field of research then opened to him. In dealing with the mollusca he considered not merely the living but also the extinct forms, especially the abundant, varied and well-preserved genera and species furnished by the Tertiary deposits of the Paris basin, of which he published descriptions and plates that proved of essential service in the stratigraphical work of Cuvier and Alexandre Brongniart (1770-1847). His labours among these relics of ancient seas and lakes led him to ponder over the past history of the globe, and as he was seldom dilatory in making known the opinions he had formed, he communicated some of his conclusions to the National Institute in 1799. These, including a further elaboration of his views, he published in 1802 in a small volume entitled Hydrogéologie.

Lamarck.

This treatise, though it did not reach a second edition and has never been reprinted, deserves an honourable place in geological literature. Its object, the author states, was to present some important and novel considerations, which he thought should form the basis of a true theory of the earth. He entirely agreed with the doctrine of the subaerial degradation of the land and the erosion of valleys by running water. Not even Playfair could have stated this doctrine more emphatically, and it is worthy of notice that Playfair's *Illustrations of the Huttonian Theory* appeared in the same year with Lamarck's book. The French naturalist, however, carried his conclusions so far as to take no account of any great movements of the terrestrial crust, which might have produced or modified the main physical features of the surface of the globe. He thought that all mountains, except such as were thrown up by volcanic agency or local accidents, have been cut out of plains, the original surfaces of which are indicated by the crests and summits of these elevations.

Lamarck, in reflecting upon the wide diffusion of fossil shells and the great height above the sea at which they are found, conceived the extraordinary idea that the ocean basin has been scoured out by the sea, and that, by an impulse communicated to the waters through the influence chiefly of the moon, the sea is slowly eating away the eastern margins of the continents, and throwing up detritus on their western coasts, and is thus gradually shifting its basin round the globe. He would not admit the operation of cataclysms; but insisted as strongly as Hutton on the continuity of natural processes, and on the necessity of explaining former changes of the earth's surface by causes which can still be seen to be in operation. As might be anticipated from his previous studies, he brought living things and their remains into the forefront of his theory of the earth. He looked upon fossils as one of the chief means of comprehending the revolutions which the surface of the earth has undergone; and in his little volume he again and again dwells on the vast antiquity to which these revolutions bear witness. He acutely argues, from the condition of fossil shells, that they must have lived and died where their remains are now found.

In the last part of his treatise Lamarck advances some peculiar opinions in physics and chemistry, which he had broached eighteen years before, but which had met with no acceptance among the scientific men of his time. He believed that the tendency of all compound substances is to decay, and thereby to be resolved into their component constituents. Yet he saw that the visible crust of the earth consists almost wholly of compound bodies. He therefore set himself to solve the problem thus presented. Perceiving that the biological action of living organisms is constantly forming combinations of matter, which would never have otherwise come into existence, he proceeded to draw the extraordinary conclusion that the action of plant and animal life (the *Pouvoir de la vie*) upon the inorganic world is so universal and so potent, that the rocks and minerals which form the outer part of the earth's crust are all, without exception, the result of the operations of once living bodies. Though this sweeping deduction must be allowed to detract from the value of Lamarck's work, there can be no doubt that he realized, more fully than any one had done before him, the efficacy of plants and animals as agents of geological change.

The last notable contributor to the cosmological literature of geology was another illustrious Frenchman, the comparative anatomist Cuvier (1769-1832). He was contemporary with Lamarck, but of a very different type of mind. The brilliance of his speculations, and the

Cuvier. charm with which he expounded them, early gained for him a prominent place in the society of Paris. He too was drawn by his zoological studies to investigate fossil organic remains, and to consider the former conditions of the earth's surface, of which they are memorials. It was among the vertebrate organisms of the Paris basin that he found his chief material, and from them that he prepared the memoirs which led to him being regarded as the founder of vertebrate palaeontology. But beyond their biological interest, they awakened in him a keen desire to ascertain the character and sequence of the geographical revolutions to which they bear witness. He approached the subject from an opposite and less philosophical point of view than that of Lamarck, coming to it with certain preconceived notions, which affected all his subsequent writings. While Lamarck was by instinct an evolutionist, who sought to trace in the history of the past the operation of the same natural processes as are still at work, Cuvier, on the other hand, was a catastrophist, who invoked a succession of vast cataclysms to account for the interruptions in the continuity of the geological record.

In a preliminary *Discourse* prefixed to his *Recherches sur les ossemens fossiles* (1821) Cuvier gave an outline of what he conceived to have been the past history of our globe, so far as he had been able to comprehend it from his investigations of the Tertiary formations of France. He believed that in that history evidence can be recognized of the occurrence of many sudden and disastrous revolutions, which, to judge from their effects on the animal life of the time, must have exceeded in violence anything we can conceive at the present day, and must have been brought about by other agencies than those which are now in operation. Yet, in spite of these catastrophes, he saw that there has been an upward progress in the animal forms inhabiting the globe, until the series ended in the advent of man. He could not, however, find any evidence that one species has been developed from another, for in that case there should have been traces of intermediate forms among the stratified formations, where he affirmed that they had never been found. A prominent position in the *Discourse* is given to a strenuous argument to disprove the alleged antiquity of some nations, and to show that the last great catastrophe occurred not more than some 5000 or 6000 years ago. Cuvier thus linked himself with those who in previous generations had contended for the efficacy of the Deluge. But his researches among fossil animals had given him a far wider outlook into the geological past, and had opened up to him a succession of deeply interesting problems in the history of life upon the earth, which, though he had not himself material for their solution, he could foresee would be cleared up in the future.

Gradual Shaping of Geology into a Distinct Branch of Science.—It will be seen from the foregoing historical sketch that it was only after the lapse of long centuries, and from the labours of many successive generations of observers and writers, that what we now know as the science of geology came to be recognized as a distinct department of natural knowledge, founded upon careful and extended study of the structure of the earth, and upon observation of the natural processes, which are now at work in changing the earth's surface. The term "geology,"[1] descriptive of this branch of the investigation of nature, was not proposed until the last quarter of the 18th century by Jean André De Luc (1727-1817) and Horace Benedict De Saussure (1740-1749). But the science was then in a markedly half-formed condition, theoretical speculation still in large part supplying the place of deductions from a detailed examination of actual fact. In 1807 a few enterprising spirits founded the Geological Society of London for the special purpose of counteracting the prevalent tendency and confining their intention "to investigate the mineral structure of the earth." The cosmogonists and framers of Theories of the Earth were succeeded by other schools of thought. The Catastrophists saw in the composition of the crust of the earth distinct evidence that the forces of nature were once much more stupendous in their operation than they now are, and that they had from time to time devastated the earth's surface; extirpating the races of plants and animals, and preparing the ground for new creations of organized life. Then came the Uniformitarians, who, pushing the doctrines of Hutton to an extreme which he did not propose, saw no evidence that the activity of the various geological causes has ever seriously differed from what it is at present. They were inclined to disbelieve that the stratified formations of the earth's crust furnish conclusive evidence of a gradual progression, from simple types of life in the oldest strata to the most highly developed forms in the youngest; and saw no reason why remains of the higher vertebrates should not be met with among the Palaeozoic formations. Sir Charles Lyell (1797-1875) was the great leader of this school. His admirably clear and philosophical presentations of geological facts which, with unwearied industry, he collected from the writings of observers in all parts of the world, impressed his views upon the whole English-speaking world, and gave to geological science a coherence and interest which largely accelerated its progress. In his later years, however, he frankly accepted the views of Darwin in regard to the progressive character of the geological record.

The youngest of the schools of geological thought is that of the Evolutionists. Pointing to the whole body of evidence from inorganic and organic nature, they maintain that the history of our planet has been one of continual and unbroken development from the earliest cosmical beginnings down to the present time, and that the crust of the earth contains an abundant, though incomplete, record of the successive stages through which the plant and animal kingdoms have reached their existing organization. The publication of Darwin's *Origin of Species* in 1859, in which evolution was made the key to the history of the animal and vegetable kingdoms, produced an extraordinary revolution in geological opinion. The older schools of thought rapidly died out, and evolution became the recognized creed of geologists all over the world. 644

Development of Opinion regarding Igneous Rocks.—So long as the idea prevailed that volcanoes are caused by the combustion of inflammable substances underground, there could be no rational conception of volcanic action and its products. Even so late as the middle of the 18th century, as above remarked, such a good observer as Lazzaro Moro drew so little distinction between volcanic and other rocks that he could believe the fossiliferous formations to have been mainly formed of materials ejected from eruptive vents. After his time the notion continued to prevail that all the rocks which form the dry land were laid down under water. Even streams of lava, which were seen to flow from an active crater, were regarded only as portions of sedimentary or other rocks, which had been melted by the fervent heat of the burning inflammable materials that had been kindled underground. In spite of the speculations of Descartes and Leibnitz, it was not yet generally comprehended that there exists beneath the terrestrial crust a molten magma, which, from time to time, has been injected into that crust, and has pierced through it, so as to escape at the surface with all the energy of an active volcano. What we now recognize to be memorials of these former injections and propulsions were all confounded with the rocks of unquestionably aqueous origin. The last great teacher by whom these antiquated doctrines were formulated into a system and promulgated to the world was Abraham Gottlob Werner (1749-1815), the most illustrious German mineralogist and geognost of the second half of the 18th century. While still under twenty-six years of age, he was appointed teacher of mining and mineralogy at the Mining Academy of Freiberg in Saxony—a post which he continued to fill up to the end of his life. Possessed of great enthusiasm for his subject, clear, methodical and eloquent in his exposition of it, he soon drew around him men from all parts of the world, who repaired to study under the great oracle of what he called geognosy (Gr. γῆ, the earth, γνῶσις, knowledge) or earth-knowledge. Reviving doctrines that had been current long before his time, he taught that the globe was once completely surrounded with an ocean, from which the rocks of the earth's crust were deposited as chemical precipitates, in a certain definite order over the whole planet. Among these "universal formations" of aqueous origin were included many rocks, which have long been recognized to have been once molten, and to have risen from below into the upper parts of the terrestrial crust. Werner, following the old tradition, looked upon volcanoes as modern features in the history of the planet, which could not have come into existence until a sufficient amount of vegetation had been buried to furnish fuel for their maintenance. Hence he attached but little importance to them, and did not include in his system of rocks any division of volcanic or igneous materials. From the predominant part assigned by him to the sea in the accumulation of the materials of the visible part of the earth, Werner and his school were known as "Neptunists."

Werner.

But many years before the Saxon professor began to teach, clear evidence had been produced from central France that basalt, one of the rocks claimed by him as a chemical precipitate and a universal formation, is a lava which has been poured out in a molten state at various widely separated periods of time and at many different places. So far back as 1752 J.E. Guettard (1715-1786) had shown that the basaltic rocks of Auvergne are true lavas, which have flowed out in streams from groups of once active cones. Eleven years later the observation was confirmed and greatly extended by Nicholas Desmarest (1725-1815), who, during a long course of years, worked out and mapped the complicated volcanic records of that interesting region, and demonstrated to all who were willing impartially to examine the evidence the true volcanic nature of basalt. These views found acceptance from some observers, but they were vehemently opposed by the followers of Werner, who, by the force of his genius, made his theoretical conceptions predominate all over Europe. The controversy as to the origin of basalt was waged with great vigour during the later decades of the 18th century. Desmarest took no part in it. He had accumulated such conclusive proof of the correctness of his deductions, and had so fully expounded the clearness of the evidence in their favour furnished by the region of Auvergne, that, when any one came to consult him on the subject, he contented himself with giving the advice to "go and see." While the debate was in progress on the continent, the subject was approached from a new and independent point of view by Hutton in Scotland. This illustrious philosopher, as already stated, realized the importance of the internal heat of the globe in consolidating the sedimentary rocks, and believed that molten material from the earth's interior has been protruded from below into the overlying crust. Some of the material thus injected could be recognized, he thought, in

Origin of basalt.

granite and in the various dark massive rocks which, known in Scotland under the name of "whinstone," were afterwards called "Trap," and are now grouped under various names, such as basalt, dolerite and diorite. So important a share did Hutton thus assign to the internal heat in the geological evolution of the planet, that he and those who adopted the same opinions were styled "Plutonists," or, especially where they concerned themselves with the volcanic origin of basalt, "Vulcanists." The geological world was thus divided into two hostile camps, that of the Neptunists or Wernerians, and that of the Plutonists, Vulcanists or Huttonians.

After many years of futile controversy the first serious weakening of the position of the dominant Neptunist school arose from the defection of some of the most prominent of Werner's pupils. In particular Jean François D'Aubuisson de Voisins (1769-1819), who had written a treatise on the aqueous origin of the basalts of Saxony, went afterwards to Auvergne, where he was speedily a convert to the views expounded by Desmarest as to the volcanic nature of basalt. Having thus to relinquish one of the fundamental articles of the Freiberg faith, he was subsequently led to modify his adherence to others until, as he himself confessed, his views came almost wholly to agree with those of Hutton. Not less complete, and even more important, was the conversion of the great Leopold von Buch (1774-1853). He, too, was trained by Werner himself, and proved to be the most illustrious pupil of the Saxon professor. Full of admiration for the Neptunism in which he had been reared, he, in his earliest separate work, maintained the aqueous origin of basalt, and contrasted the wide field opened up to the spirit of observation by his master's teaching with the narrower outlook offered by "the volcanic theory." But a little further acquaintance with the facts of nature led Von Buch also to abandon his earlier prepossessions. It was a personal visit to the volcanic region of Auvergne that first opened his eyes, and led him to recant what he had believed and written about basalt. But the abandonment of so essential a portion of the Wernerian creed prepared the way for further relinquishments. When a few years later he went to Norway and found to his astonishment that granite, which he had been taught to regard as the oldest chemical precipitate from the universal ocean, could there be seen to have broken through and metamorphosed fossiliferous limestones, and to have sent veins into them, his faith in Werner's order of the succession of the rocks in the earth's crust received a further momentous shock. While one after another of the Freiberg doctrines crumbled away before him, he was now able to interrogate nature on a wider field than the narrow limits of Saxony, and he was thus gradually led to embrace the tenets of the opposite school. His commanding position, as the most accomplished geologist on the continent, gave great importance to his recantation of the Neptunist creed. His defection indeed was the severest blow that this creed had yet sustained. It may be said to have rung the knell of Wernerianism, which thereafter rapidly declined in influence, while Plutonism came steadily to the front, where it has ever since remained.

Although Desmarest had traced in Auvergne a long succession of volcanic eruptions, of which the oldest went back to a remote period of time, and although he had shown that this succession, coupled with the records of contemporaneous denudation, might be used in defining epochs of geological history, it was not until many years after his day that volcanic action came to be recognized as a normal part of the mechanism of our globe, which had been in operation from the remotest past, and which had left numerous records among the rocks of the terrestrial crust. During the progress of the controversy between the two great opposing factions in the later portion of the 18th and the first three decades of the 19th century, those who espoused the Vulcanist cause were intent on proving that certain rocks, which are intercalated among the stratified formations and which were claimed by the Neptunists as obviously formed by water, are nevertheless of truly igneous origin. These observers fixed their eyes on the evidence that the material of such rocks, instead of having been deposited from aqueous solution, had once been actually molten, and had in that condition been thrust between the strata, had enveloped portions of them, and had indurated or otherwise altered them. They spoke of these masses as "unerupted lavas"; and undoubtedly in innumerable instances they were right. But their zeal to establish an intrusive origin led them to overlook the proofs that some intercalated sheets of igneous material had not been injected into the strata, but had been poured out at the surface as truly volcanic discharges, and therefore belonged to the ancient periods represented by the strata between which they are interposed. It may readily be supposed that any proofs of the contemporaneous intercalation of such sheets would be eagerly seized upon by the Neptunists in favour of their aqueous theory. The influence of the ancient belief that "burning mountains" could only rise from the combustion of subterranean inflammable materials extended even into the ranks of the Vulcanists, so far at least as to lead to a general acquiescence in the assumption that volcanoes appeared to belong to a late phase in the history of the planet. It was not until after considerable progress had been made in determining the palaeontological distinctions and order of succession of the stratified formations of the earth's crust that it became possible to trace among these formations a succession of volcanic episodes which were contemporaneous with them. In no part of the world has an ampler record of such episodes been preserved than in the British Isles. It was natural, therefore, that the subject should there receive most attention. As far back as 1820 Ami Boué (1794-1881) showed that the Old Red Sandstone of Scotland includes a great series of volcanic rocks, and that other rocks of volcanic origin are associated with the Carboniferous formations. H.T. de la Beche (1796-1855) afterwards traced proofs of contemporaneous eruptions among the Devonian rocks of the south-west of England. Adam Sedgwick (1785-1873) showed, first in the Lake District, and afterwards in North Wales, the presence of abundant volcanic sheets among the oldest divisions of the Palaeozoic series; while Roderick Impey Murchison (1792-1871) made similar discoveries among the Lower Silurian rocks. From the time of these pioneers the volcanic history of the country has been worked out by many observers until it is now known with a fulness as yet unattained in any other region. 645

Growth of Opinion regarding Earthquakes.—We have seen how crude were the conceptions of the ancients regarding the causes of volcanic action, and that they connected volcanoes and earthquakes as results of the commotion of wind imprisoned within subterranean caverns and passages. One of the earliest treatises, in which the phenomena of terrestrial movements were discussed in the spirit of modern science, was the posthumous collection of papers by Robert Hooke (1635-1703), entitled *Lectures and Discourses of Earthquakes and Subterranean Eruptions*, where the probable agency of earthquakes in upheaving and depressing land is fully considered, but without any definite pronouncement as to the author's conception of its origin. Hooke still associated earthquakes with volcanic action, and connected both with what he called "the general congregation of sulphurous subterraneous vapours." He conceived that some kind of "fermentation" takes place within the earth, and that the materials which catch fire and give rise to eruptions or earthquakes are analogous to those that constitute gunpowder. The first essay wherein earthquakes are treated from the modern point of view as the results of a shock that sends waves through the crust of the earth was written by the Rev. John Michell, and communicated to the Royal Society in the year 1760. Still under the old misconception that volcanoes are due to the combustion of inflammable materials, which he thought might be set on fire by the spontaneous combustion of pyritous strata, he supposed that, by the sudden access of large bodies of water to these subterranean fires, vapour is produced in such quantity and with such force as to give rise to the shock. From the centre of origin of this shock waves, he thought, are propagated through the earth, which are largest at the start and gradually diminish as they travel outwards. By drawing lines at different places in the direction of the track of these waves, he believed that the place of common intersection of these lines would be nearly the centre of the disturbance. In this way he showed that the great Lisbon earthquake of 1755 had its focus under the Atlantic, somewhere between the latitudes of Lisbon and Oporto, and he estimated that the depth at which it originated could not be much less than 1 m., and probably did not exceed 3 m. Michell, however, misconceived the character of the waves which he described, seeing that he believed them to be due to the actual propagation of the vapour itself underneath the surface of the earth. A century had almost passed after the date of his essay before modern scientific methods of observation and the use of recording instruments began to be applied to the study of earthquake phenomena. In 1846 Robert Mallet (1810-1881) published an important paper "On the Dynamics of Earthquakes" in the *Transactions of the Royal Irish Academy*. From that time onward he continued to devote his energies to the investigation, studying the effects of the Calabrian earthquake of 1857, experimenting on the transmission of waves of shock through various materials, caused by exploding charges of gunpowder, and collecting all the information to be obtained on the subject. His writings, and especially his work in two volumes on *The First Principles of Observational Seismology*, must be regarded as having laid the foundations of this branch of modern geology (see Earthquake; Seismometer).

History of the Evolution of Stratigraphical Geology.—Men had long been familiar with the evidence that the present dry land once lay under the sea, before they began to realize that the rocks, of which the land consists, contain a record of many alternations of land and sea, and relics of a long succession of plants and animals from early and simple types up to the manifold and complex forms of to-day. In countries where coal-mining had been prosecuted for generations, it had been recognized that the rocks consist of strata superposed on each other in a definite order, which was found to extend over the whole of a district. As far back as 1719 John Strachey drew attention to this fact in a communication published in the *Philosophical Transactions*. John Michell (1760), in the paper on earthquakes already cited, showed that he had acquired a clear understanding of the order of succession among stratified formations, and perceived that to disturbances of the terrestrial crust must be ascribed the fact that the lower or older and more inclined strata form the mountains, while the younger and more horizontal strata are spread over the plains.

In Italy G. Arduíno (1713-1795) classified the rocks in the north of the peninsula as

Primitive, Secondary, Tertiary and Volcanic. A similar threefold order was announced for the Harz and Erzgebirge by J.G. Lehmann in 1756. He recognized in that region an ancient series of rocks in inclined or vertical strata, which rise to the tops of the hills and descend to an unknown depth into the interior. These masses, he thought, were contemporaneous with the making of the world. Next came the Flötzgebirge, consisting of younger sediments, disposed in flat or gently inclined sheets which overlie the first and more disturbed series, and are full of petrified remains of plants and animals. Lastly he included the mountains which have from time to time been formed by local accidents. Still more advanced were the conceptions of G.C. Füchsel, who in the year 1762 published in Latin *A History of the Earth and the Sea, based on a History of the Mountains of Thuringia*; and in 1773, in German, a *Sketch of the most Ancient History of the Earth and Man*. In these works he described the stratigraphical relations and general characters of the various geological formations in his little principality; and taking them as indicative of a general order of succession, he traced what he believed to have been a series of revolutions through which the earth has passed. In interpreting this geological history, he laid great stress on the evidence of the fossils contained in the rocks. He recognized that the various formations differ from each other in their enclosed organic remains, and that from these differences the existence of former sea-bottoms and land surfaces can be determined.

The labours of these pioneers paved the way for the advent of Werner. Though the system evolved by this teacher claimed to discard theory and to be established on a basis of observed facts, it rested on a succession of hypotheses, for which no better foundation could be shown than the belief of their author in their validity. Starting from the extremely limited stratigraphical range displayed in the geological structure of Saxony, he took it as a type for the rest of the globe, persuading himself and impressing upon his followers that the rocks of that small kingdom were to be taken as examples of his “universal formations.” The oldest portion of the series, classed by him as “Primitive,” consisted of rocks which he maintained had been deposited from chemical solution. Yet they included granite, gneiss, basalt, porphyry and serpentine, which, even in his own day, were by many observers correctly regarded as of igneous origin. A later group of rocks, to which he gave the name of “Transition,” comprised, in his belief, partly chemical, partly mechanical sediments, and contained the earliest fossil organic remains. A third group, for which he reserved Lehmann’s name “Flötz,” was made up chiefly of mechanical detritus, while youngest of all came the “Alluvial” series of loams, clays, sands, gravels and peat. It was by the gradual subsidence of the ocean that, as he believed, the general mass of the dry land emerged, the first-formed rocks being left standing up, sometimes on end, to form the mountains, while those of later date, less steeply inclined, occupied successively lower levels down to the flat alluvial accumulations of the plains. Neither Werner, nor any of his followers, ventured to account for what became of the water as the sea-level subsided, though, in despite of their antipathy to anything like speculation, they could not help suggesting, as an answer to the cogent arguments of their opponents, that “one of the celestial bodies which sometimes approach near to the earth may have been able to withdraw a portion of our atmosphere and of our ocean.” Nor was any attempt made to explain the extraordinary nature of the supposed chemical precipitates of the universal ocean. The progress of inquiry even in Werner’s lifetime disproved some of the fundamental portions of his system. Many of the chemical precipitates were shown to be masses that had been erupted in a molten state from below. His order of succession was found not to hold good; and though he tried to readjust his sequence and to introduce into it modifications to suit new facts, its inherent artificiality led to its speedy decline after his death. It must be conceded, however, that the stress which he laid upon the fact that the rocks of the earth’s crust were deposited in a definite order had an important influence in directing attention to this subject, and in preparing the way for a more natural system, based not on mere mineralogical characters, but having regard to the organic remains, which were now being gathered in ever-increasing numbers and variety from stratified formations of many different ages and from all parts of the globe.

It was in France and in England that the foundations of stratigraphy, based upon a knowledge of organic remains, were first successfully laid. Abbé J.L. Giraud-Soulavie (1752-1813), in his *Histoire naturelle de la France méridionale*, which appeared in seven volumes, subdivided the limestones of Vivarais into five ages, each marked by a distinct assemblage of shells. In the lowest strata, representing the first age, none of the fossils were believed by him to have any living representatives, and he called these rocks “Primordial.” In the next group a mingling of living with extinct forms was observable. The third age was marked by the presence of shells of still existing species. The strata of the fourth series were characterized by carbonaceous shales or slates, containing remains of primordial vegetation, and perhaps equivalents of the first three calcareous series. The fifth age was marked by recent deposits containing remains of terrestrial vegetation and of land animals. It is remarkable that these sagacious conclusions should have been formed and published at a time when the geologists of the Continent were engaged in the controversy about the origin of basalt, or in disputes about the character and stratigraphical position of the supposed universal formations, and when the interest and importance of fossil organic remains still remained unrecognized by the vast majority of the combatants.

The rocks of the Paris basin display so clearly an orderly arrangement, and are so distinguished for the variety and perfect preservation of their enclosed organic remains, that 646
they could not fail to attract the early notice of observers. J. É. Guettard, G.F. Rouelle (1703-1770), N. Desmarest, A.L. Lavoisier (1743-1794) and others made observations in this interesting district. But it was reserved for Cuvier (1769-1832) and A. Brongniart (1770-1847) to work out the detailed succession of the Tertiary formations, and to show how each of these is characterized by its own peculiar assemblage of organic remains. The later progress of investigation has slightly corrected and greatly amplified the tabular arrangement established by these authors in 1808, but the broad outlines of the Tertiary stratigraphy of the Paris basin remain still as Cuvier and Brongniart left them. The most important subsequent change in the classification of the Tertiary formations was made by Sir Charles Lyell, who, conceiving in 1828 the idea of a classification of these rocks by reference to their relative proportions of living and extinct species of shells, established, in collaboration with G.P. Deshayes, the now universally accepted divisions Eocene, Miocene and Pliocene.

Long before Cuvier and Brongniart published an account of their researches, another observer had been at work among the Secondary formations of the west of England, and had independently discovered that the component members of these formations were each distinguished by a peculiar group of organic remains; and that this distinction could be used to discriminate them over all the region through which he had traced them. The remarkable man who arrived at this far-reaching generalization was William Smith (1769-1839), a land surveyor who, in the prosecution of his professional business, found opportunities of traversing a great part of England, and of putting his deductions to the test. As the result of these journeys he accumulated materials enough to enable him to produce a geological map of the country, on which the distribution and succession of the rocks were for the first time delineated. Smith’s labours laid the foundation of stratigraphical geology in England and he was styled even in his lifetime the “Father of English geology.” From his day onward the significance of fossil organic remains gained rapidly increasing recognition. Thus in England the outlines traced by him among the Secondary and Tertiary formations were admirably filled in by Thomas Webster (1773-1844); while the Cretaceous series was worked out in still greater detail in the classic memoirs of William Henry Fitton (1780-1861).

There was one stratigraphical domain, however, into which William Smith did not enter. He traced his sequence of rocks down into the Coal Measures, but contented himself with only a vague reference to what lay underneath that formation. Though some of these underlying rocks had in various countries yielded abundant fossils, they had generally suffered so much from terrestrial disturbances, and their order of succession was consequently often so much obscured throughout western Europe, that they remained but little known for many years after the stratigraphy of the Secondary and Tertiary series had been established. At last in 1831 Murchison began to attack this *terra incognita* on the borders of South Wales, working into it from the Old Red Sandstone, the stratigraphical position of which was well known. In a few years he succeeded in demonstrating the existence of a succession of formations, each distinguished by its own peculiar assemblage of organic remains which were distinct from those in any of the overlying strata. To these formations he gave the name of Silurian (*q.v.*). From the key which his researches supplied, it was possible to recognize in other countries the same order of formations and the same sequence of fossils, so that, in the course of a few years, representatives of the Silurian system were found far and wide over the globe. While Murchison was thus engaged, Sedgwick devoted himself to the more difficult task of unravelling the complicated structure of North Wales. He eventually made out the order of the several formations there, with their vast intercalations of volcanic material. He named them the Cambrian system (*q.v.*), and found them to contain fossils, which, however, lay for some time unexamined by him. He at first believed, as Murchison also did, that his rocks were all older than any part of the Silurian series. It was eventually discovered that a portion of them was equivalent to the lower part of that series. The oldest of Sedgwick’s groups, containing distinctive fossils, retain the name Cambrian, and are of high interest, as they enclose the remains of the earliest faunas which are yet well known. Sedgwick and Murchison rendered yet another signal service to stratigraphical geology by establishing, in 1839, on a basis of palaeontological evidence supplied by W. Lonsdale, the independence of the Devonian system (*q.v.*).

For many years the rocks below the oldest fossiliferous deposits received comparatively little attention. They were vaguely described as the “crystalline schists” and were often referred to as parts of the primeval crust in which no chronology was to be looked for. W.E.

Logan (1798-1875) led the way, in Canada, by establishing there several vast series of rocks, partly of crystalline schists and gneisses (Laurentian) and partly of slates and conglomerates (Huronian). Later observers, both in Canada and the United States, have greatly increased our knowledge of these rocks, and have shown their structure to be much more complex than was at first supposed (see Archean System).

During the latter half of the 19th century the most important development of stratigraphical geology was the detailed working out and application of the principle of zonal classification to the fossiliferous formations—that is, the determination of the sequence and distribution of organic remains in these formations, and the arrangement of the strata into zones, each of which is distinguished by a peculiar assemblage of fossil species (see under Part VI.). The zones are usually named after one especially characteristic species. This system of classification was begun in Germany with reference to the members of the Jurassic system (*q.v.*) by A. Oppel (1856-1858) and F.A. von Quenstedt (1858), and it has since been extended through the other Mesozoic formations. It has even been found to be applicable to the Palaeozoic rocks, which are now subdivided into palaeontological zones. In the Silurian system, for example, the graptolites have been shown by C. Lapworth to furnish a useful basis for zonal subdivisions. The lowest fossiliferous horizon in the Cambrian rocks of Europe and North America is known as the *Olenellus* zone, from the prominence in it of that genus of trilobite.

Another conspicuous feature in the progress of stratigraphy during the second half of the 19th century was displayed by the rise and rapid development of what is known as Glacial geology. The various deposits of “drift” spread over northern Europe, and the boulders scattered across the surface of the plains had long attracted notice, and had even found a place in popular legend and superstition. When men began to examine them with a view to ascertain their origin, they were naturally regarded as evidences of the Noachian deluge. The first observer who drew attention to the smoothed and striated surfaces of rock that underlie the Drifts was Hutton’s friend, Sir James Hall, who studied them in the lowlands of Scotland and referred them to the action of great debacles of water, which, in the course of some ancient terrestrial convulsion, had been launched across the face of the country. Playfair, however, pointed out that the most potent geological agents for the transportation of large blocks of stone are the glaciers. But no one was then bold enough to connect the travelled boulders with glaciers on the plains of Germany and of Britain. Yet the transporting agency of ice was invoked in explanation of their diffusion. It came to be the prevalent belief among the geologists of the first half of the 19th century, that the fall of temperature, indicated by the gradual increase in the number of northern species of shells in the English Crag deposits, reached its climax during the time of the Drift, and that much of the north and centre of Europe was then submerged beneath a sea, across which floating icebergs and floes transported the materials of the Drift and dropped the scattered boulders. As the phenomena are well developed around the Alps, it was necessary to suppose that the submergence involved the lowlands of the Continent up to the foot of that mountain chain—a geographical change so stupendous as to demand much more evidence than was adduced in its support. At last Louis Agassiz (1807-1873), who had varied his palaeontological studies at Neuchâtel by excursions into the Alps, was so much struck by the proofs of the former far greater extension of the Swiss glaciers, that he pursued the investigation and satisfied himself that the ice had formerly extended from the Alpine valleys right across the great plain of Switzerland, and had transported huge boulders from the central mountains to the flanks of the Jura. In the year 1840 he visited Britain and soon found evidence of similar conditions there. He showed that it was not by submergence in a sea cumbered with floating ice, but by the former presence of vast glaciers or sheets of ice that the Drift and erratic blocks had been distributed. The idea thus propounded by him did not at once command complete approval, though traces of ancient glaciers in Scotland and Wales were soon detected by native geologists, particularly by W. Buckland, Lyell, J.D. Forbes and Charles Maclaren. Robert Chambers (1802-1871) did good service in gathering additional evidence from Scotland and Norway in favour of Agassiz’s views, which steadily gained adherents until, after some quarter of a century, they were adopted by the great majority of geologists in Britain, and subsequently in other countries. Since that time the literature of geology has been swollen by a vast number of contributions in which the history of the Glacial period, and its records both in the Old and New World, have been fully discussed.

Rise and Progress of Palaeontological Geology.—As this branch of the science deals with the evidence furnished by fossil organic remains as to former geographical conditions, it early attracted observers who, in the superficial beds of marine shells found at some distance from the coast, saw proofs of the former submergence of the land under the sea. But the occurrence of fossils embedded in the heart of the solid rocks of the mountains offered much greater difficulties of explanation, and further progress was consequently slow. Especially baneful was the belief that these objects were mere sports of nature, and had no connexion with any once living organisms. So long as the true organic origin of the fossil plants and animals contained in the rocks was in dispute, it was hardly possible that much advance could be made in their systematic study, or in the geological deductions to be drawn from them. One good result of the controversy, however, was to be seen in the large collections of these "formed stones" that were gathered together in the cabinets and museums of the 17th and 18th centuries. The accumulation and comparison of these objects naturally led to the production of treatises in which they were described and not unfrequently illustrated by good engravings. Switzerland was more particularly noted for the number and merit of its works of this kind, such as that of K.N. Lang (*Historia lapidum figuratorum Helvetiae*, 1708) and those of Johann Jacob Scheuchzer (1672-1733). In England, also, illustrated treatises were published both by men who looked on fossils as mere freaks of nature, and by those who regarded them as proofs of Noah's flood. Of the former type were the works of Martin Lister (1638-1712) and Robert Plot (*Natural History of Oxfordshire*, 1677). The Celtic scholar Edward Llwyd (1660-1709) wrote a Latin treatise containing good plates of a thousand fossils in the Ashmolean Museum, Oxford, and J. Woodward, in 1728-1729, published his *Natural History of the Fossils of England*, already mentioned, wherein he described his own extensive collection, which he bequeathed to the University of Cambridge, where it is still carefully preserved. The most voluminous and important of all these works, however, appeared at a later date at Nuremberg. It was begun by G.W. Knorr (1705-1761), who himself engraved for it a series of plates, which for beauty and accuracy have seldom been surpassed. After his death the work was continued by J.E.I. Walch (1725-1778), and ultimately consisted of four massive folio volumes and nearly 300 plates under the title of *Lapides diluvii universalis testes*. Although the authors supposed their fossils to be relics of Noah's flood, their work must be acknowledged to mark a distinct onward stage in the palaeontological department of geology.

It was in France that palaeontological geology began to be cultivated in a scientific spirit. The potter Bernard Palissy, as far back as 1580, had dwelt on the importance of fossil shells as monuments of revolutions of the earth's surface; but the observer who first undertook the detailed study of the subject was Jean Etienne Guettard, who began in 1751 to publish his descriptions of fossils in the form of memoirs presented to the Academy of Sciences of Paris. To him they were not only of deep interest as monuments of former types of existence, but they had an especial value as records of the changes which the country had undergone from sea to land and from land to sea. More especially noteworthy was a monograph by him which appeared in 1765 bearing the title "On the accidents that have befallen Fossil Shells compared with those which are found to happen to shells now living in the Sea." In this treatise he showed that the fossils have been encrusted with barnacles and serpulae, have been bored into by other organisms, and have often been rounded or broken before final entombment; and he inferred that these fossils must have lived and died on the sea-floor under similar conditions to those which obtain on the sea-floor to-day. His argument was the most triumphant that had ever been brought against the doctrine of *lusus naturae*, and that of the efficacy of Noah's flood—doctrines which still held their ground in Guettard's day. When Soulavie, Cuvier and Brongniart in France, and William Smith in England, showed that the rock formations of the earth's crust could be arranged in chronological order, and could be recognized far and wide by means of their enclosed organic remains, the vast significance of these remains in geological research was speedily realized, and palaeontological geology at once entered on a new and enlarged phase of development. But apart from their value as chronological monuments, and as witnesses of former conditions of geography, fossils presented in themselves a wide field of investigation as types of life that had formerly existed, but had now passed away. It was in France that this subject first took definite shape as an important branch of science. The mollusca of the Tertiary deposits of the Paris basin became, in the hands of Lamarck, the basis on which invertebrate palaeontology was founded. The same series of strata furnished to Cuvier the remains of extinct land animals, of which, by critical study of their fragmentary bones and skeletons, he worked out restorations that may be looked on as the starting-point of vertebrate palaeontology. These brilliant researches, rousing widespread interest in such studies, showed how great a flood of light could be thrown on the past history of the earth and its inhabitants. But the full significance of these extinct types of life could not be understood so long as the doctrine of the immutability of species, so strenuously upheld by Cuvier, maintained its sway among naturalists. Lamarck, as far back as the year 1800, had begun to propound his theory of evolution and the transformation of species; but his views, strongly opposed by Cuvier and the great body of naturalists of the day, fell into neglect. Not until after the publication in 1859 of the *Origin of Species* by Charles Darwin were the barriers of old prejudice in this matter finally broken down. The possibility of tracing the ancestry of living forms back into the remotest ages was then perceived; the time-honoured fiction that the stratified formations record a series of catastrophes and re-creations was finally dissipated; and the earth's crust was seen to contain a noble, though imperfect, record of the grand evolution of organic types of which our planet has been the theatre.

Development of Petrographical Geology.—Theophrastus, the favourite pupil of Aristotle, wrote a treatise *On Stones*, which has come down to our own day, and may be regarded as the earliest work on petrography. At a subsequent period Pliny, in his *Natural History*, collected all that was known in his day regarding the occurrence and uses of minerals and rocks. But neither of these works is of great scientific importance, though containing much interesting information. Minerals from their beauty and value attracted notice before much attention was paid to rocks, and their study gave rise to the science of mineralogy long before geology came into existence. When rocks began to be more particularly scrutinized, it was chiefly from the side of their usefulness for building and other economic purposes. The occurrence of marine shells in many of them had early attracted attention to them. But their varieties of composition and origin did not become the subject of serious study until after Linnaeus and J.G. Wallerius in the 18th century had made a beginning. The first important contribution to this department of the science was that of Werner, who in 1786 published a classification and description of rocks in which he arranged them in two divisions, simple and compound, and further distinguished them by various external characters and by their relative age. The publication of this scheme may be said to mark the beginning of scientific petrography. Werner's system, however, had the serious defect that the chronological order in which he grouped the rocks, and the hypothesis by which he accounted for them as chemical precipitates from the original ocean, were both alike contrary to nature. It was hardly possible indeed that much progress could be made in this branch of geology until chemistry and mineralogy had made greater advances; and especially until it was possible to ascertain the intimate chemical and mineralogical composition, and the minute structure of rocks. The study, however, continued to be pursued in Germany, where the influence of Werner's enthusiasm still led men to enter the petrographical rather than the palaeontological domain. The resources of modern chemistry were pressed into the service, and analyses were made and multiplied to such a degree that it seemed as if the ultimate chemical constitution of every type of rock had now been thoroughly revealed. The condition of the science in the middle of the 19th century was well shown by J.L.A. Roth, who in 1861 collected about 1000 trustworthy analyses which up to that time had been made. But though the chemical elements of the rocks had been fairly well determined, the manner in which they were combined in the compound rocks could for the most part be only more or less plausibly conjectured. As far back as 1831 an account was published of a process devised by William Nicol of Edinburgh, whereby sections of fossil wood could be cut, mounted on glass, and reduced to such a degree of transparency as to be easily examined under a microscope. Henry Sorby, of Sheffield, having seen Nicol's preparations, perceived how admirably adapted the process was for the study of the minute structure and composition of rocks. In 1858 he published in the *Quarterly Journal of the Geological Society* a paper "On the Microscopical Structure of Crystals." This essay led to a complete revolution of petrographical methods and gave a vast impetus to the study of rocks. Petrology entered upon a new and wider field of investigation. Not only were the mineralogical constituents of the rocks detected, but minute structures were revealed which shed new light on the origin and history of these mineral masses, and opened up new paths in theoretical geology. In the hands of H. Vogelsang, F. Zirkel, H. Rosenbusch, and a host of other workers in all civilized countries, the literature of this department of the science has grown to a remarkable extent. Armed with the powerful aid of modern optical instruments, geologists are now able with far more prospect of success to resume the experiments begun a century before by de Saussure and Hall. G.A. Daubrée, C. Friedel, E. Sarasin, F. Fouqué and A. Michel Lévy in France, C. Doelter y Cisterich and E. Hussak of Gratz, J. Morozewicz of Warsaw and others, have greatly advanced our knowledge by their synthetical analyses, and there is every reason to hope that further advances will be made in this field of research.

Rise of Physiographical Geology.—Until stratigraphical geology had advanced so far as to show of what a vast succession of rocks the crust of the earth is built up, by what a long and complicated series of revolutions these rocks have come to assume their present positions, and how enormous has been the lapse of time which all these changes represent, it was not possible to make a scientific study of the surface features of our globe. From ancient times it had been known that many parts of the land had once been under the sea; but down even to the beginning of the 19th century the vaguest conceptions continued to prevail as to the operations concerned in the submergence and elevation of land, and as to the processes whereby the present outlines of terrestrial topography were determined. We have seen, for instance, that according to the teaching of Werner the oldest rocks were first precipitated from solution in the universal ocean to form the mountains, that the vertical position of their strata was original, that as the waters subsided successive formations were deposited and laid bare, and that finally the superfluous portion of the ocean was whisked away into space by some unexplained co-operation of another planetary body. Desmarest, in his investigation of the volcanic history of Auvergne, was the first observer to perceive by what a long process of sculpture the present configuration of the land has been brought about. He showed conclusively that the valleys have been carved out by the streams that flow in them, and that while they have sunk deeper and deeper into the framework of the land, the spaces of ground between them have been left as intervening ridges and hills. De Saussure learnt a similar lesson from his studies of the Alps, and Hutton and Playfair made it a cardinal feature in their theory of the earth. Nevertheless the idea encountered so much opposition that it made but little way until after the middle of the 19th century. Geologists preferred to believe in convulsions of nature, whereby valleys were opened and mountains were upheaved. That the main features of the land, such as the great mountain-chains, had been produced by gigantic plication of the terrestrial crust was now generally admitted, and also that minor fractures and folds had probably initiated many of the valleys. But those who realized most vividly the momentous results achieved by ages of subaerial denudation perceived that, as Hutton showed, even without the aid of underground agency, the mere flow of water in streams across a mass of land must in course of time carve out just such a system of valleys as may anywhere be seen. It was J.B. Jukes who, in 1862, first revived the Huttonian doctrine, and showed how completely it explained the drainage-lines in the south of Ireland. Other writers followed in quick succession until, in a few years, the doctrine came to be widely recognized as one of the established principles of modern geology. Much help was derived from the admirable illustrations of land-sculpture and river-erosion supplied from the Western Territories and States of the American Union. 648

Another branch of physiographical geology which could only come into existence after most of the other departments of the science had made large progress, deals with the evolution of the framework of each country and of the several continents and oceans of the globe. It is now possible, with more or less confidence, to trace backward the history of every terrestrial area, to see how sea and land have there succeeded each other, how rivers and lakes have come and gone, how the crust of the earth has been ridged up at widely separated intervals, each movement determining some line of mountains or plains, how the boundaries of the oceans have shifted again and again in the past, and thus how, after so prolonged a series of revolutions, the present topography of each country, and of the globe as a whole, has been produced. In the prosecution of this subject maps have been constructed to show what is conjectured to have been the distribution of sea and land during the various geological periods in different parts of the world, and thus to indicate the successive stages through which the architecture of the land has been gradually evolved. The most noteworthy contribution to this department of the science is the *Antlitz der Erde* of Professor Suess of Vienna. This important and suggestive work has been translated into French and English.

Part II.—Cosmical Aspects

Before geology had attained to the position of an inductive science, it was customary to begin investigations into the history of the earth by propounding or adopting some more or less fanciful hypothesis in explanation of the origin of our planet, or even of the universe. Such preliminary notions were looked upon as essential to a right understanding of the manner in which the materials of the globe had been put together. One of the distinguishing features of Hutton’s Theory of the Earth consisted in his protest that it is no part of the province of geology to discuss the origin of things. He taught that in the materials from which geological evidence is to be compiled there can be found “no traces of a beginning, no prospect of an end.” In England, mainly to the influence of the school which he founded, and to the subsequent rise of the Geological Society of London, which resolved to collect facts instead of fighting over hypotheses, is due the disappearance of the crude and unscientific cosmologies by which the writings of the earlier geologists were distinguished.

But there can now be little doubt that in the reaction against those visionary and often grotesque speculations, geologists were carried too far in an opposite direction. In allowing themselves to believe that geology had nothing to do with questions of cosmogony, they gradually grew up in the conviction that such questions could never be other than mere speculation, interesting or amusing as a theme for the employment of the fancy, but hardly coming within the domain of sober and inductive science. Nor would they soon have been awakened out of this belief by anything in their own science. It is still true that in the data with which they are accustomed to deal, as comprising the sum of geological evidence, there can be found no trace of a beginning, though the evidence furnished by the terrestrial crust shows a general evolution of organic forms from some starting-point which cannot be seen. The oldest rocks which have been discovered on any part of the globe have probably been derived from other rocks older than themselves. Geology by itself has not yet revealed, and is little likely ever to reveal, a trace of the first solid crust of our globe. If, then, geological history is to be compiled from direct evidence furnished by the rocks of the earth, it cannot begin at the beginning of things, but must be content to date its first chapter from the earliest period of which any record has been preserved among the rocks.

Nevertheless, though geology in its usual restricted sense has been, and must ever be, unable to reveal the earliest history of our planet, it no longer ignores, as mere speculation, what is attempted in this subject by its sister sciences. Astronomy, physics and chemistry have in late years all contributed to cast light on the earlier stages of the earth's existence, previous to the beginning of what is commonly regarded as geological history. But whatever extends our knowledge of the former conditions of our globe may be legitimately claimed as part of the domain of geology. If this branch of inquiry, therefore, is to continue worthy of its name as the science of the earth, it must take cognizance of these recent contributions from other sciences. It must no longer be content to begin its annals with the records of the oldest rocks, but must endeavour to grope its way through the ages which preceded the formation of any rocks. Thanks to the results achieved with the telescope, the spectroscope and the chemical laboratory, the story of these earliest ages of our earth is every year becoming more definite and intelligible.

Up to the present time no definite light has been thrown by physics on the origin and earliest condition of our globe. The famous nebular theory (*q.v.*) of Kant and Laplace sketched the supposed evolution of the solar system from a gaseous nebula, slowly rotating round a more condensed central portion of its mass, which eventually became the sun. As a consequence of increased rapidity of rotation resulting from cooling and contraction, the nebula acquired a more and more lenticular form, until at last it threw off from its equatorial protuberance a ring of matter. Subsequently the same process was repeated, and other similar rings successively separated from the parent mass. Each ring went through a corresponding series of changes until it ultimately became a planet, with or without one or more attendant satellites. The intimate relationship of our earth to the sun and the other planets was, in this way, shown. But there are some serious physical difficulties in the way of the acceptance of the nebular hypothesis. Another explanation is given by the meteoritic hypothesis, according to which, out of the swarms of meteorites with which the regions of space are crowded, the sun and planets have been formed by gradual accretion.

According to these theoretical views we should expect to find a general uniformity of composition in the constituent matter of the solar system. For many years the only available evidence on this point was derived from the meteorites (*q.v.*) which so constantly fall from outer space upon the surface of the earth. These bodies were found to consist of elements, all of which had been recognized as entering into the constitution of the earth. But the discoveries of spectroscopic research have made known a far more widely serviceable method of investigation, which can be applied even to the luminous stars and nebulae that lie far beyond the bounds of the solar system. By this method information has been obtained regarding the constitution of the sun, and many of our terrestrial metals, such as iron, nickel and magnesium, have been ascertained to exist in the form of incandescent vapour in the solar atmosphere. The present condition of the sun probably represents one of the phases through which stars and planets pass in their progress towards becoming cool and dark bodies in space. If our globe was at first, like its parent sun, an incandescent mass of probably gaseous matter, occupying much more space than it now fills, we can conceive that it has ever since been cooling and contracting until it has reached its present form and dimensions, and that it still retains a high internal temperature. Its oblately spheroidal form is such as would be assumed by a rotating mass of matter in the transition from a vaporous and self-luminous or liquid condition to one of cool and dark solidity. But it has been claimed that even a solid spherical globe might develop, under the influence of protracted rotation, such a shape as the earth at present possesses.

The observed increase of temperature downwards in our planet has hitherto been 649
generally accepted as a relic and proof of an original high temperature and mobility of substance. Recently, however, the validity of this proof has been challenged on the ground that the ascertained amount of radium in the rocks of the outer crust is more than sufficient to account for the observed downward increase of temperature. Too little, however, is known of the history and properties of what is called radium to afford a satisfactory ground on which to discard what has been, and still remains, the prevalent belief on this subject.

An important epoch in the geological history of the earth was marked by the separation of the moon from its mass (see Tide). Whether the severance arose from the rupture of a surrounding ring or the gradual condensation of matter in such a ring, or from the ejection of a single mass of matter from the rapidly rotating planet, it has been shown that our satellite was only a few thousand miles from the earth's surface, since when it has retreated to its present distance of 240,000 m. Hence the influence of the moon's attraction, and all the geological effects to which it gives rise, attained their maximum far back in the development of the globe, and have been slowly diminishing throughout geological history.

The sun by virtue of its vast size has not yet passed out of the condition of glowing gas, and still continues to radiate heat beyond the farthest planet of the solar system. The earth, however, being so small a body in comparison, would cool down much more quickly. Underneath its hot atmosphere a crust would conceivably begin to form over its molten surface, though the interior might still possess a high temperature and, owing to the feeble conducting power of rocks, would remain intensely hot for a protracted series of ages.

Full information regarding the form and size of the earth, and its relations to the other planetary members of the solar system, will be found in the articles PLANET and SOLAR SYSTEM. For the purposes of geological inquiry the reader will bear in mind that the equatorial diameter of our globe is estimated to be about 7925 m., and the polar diameter about 7899 m.; the difference between these two sums representing the amount of flattening at the poles (about 26½ m.). The planet has been compared in shape to an orange, but it resembles an orange which has been somewhat squeezed, for its equatorial circumference is not a regular circle but an ellipse, of which the major axis lies in long. 8° 15′ W.—on a meridian which cuts the north-west corner of America, passing through Portugal and Ireland, and the north-east corner of Asia in the opposite hemisphere.

The rotation of the earth on its axis exerts an important influence on the movements of the atmosphere, and thereby affects the geological operations connected with these movements. The influence of rotation is most marked in the great aerial circulation between the poles and the equator. Currents of air, which set out in a meridional direction from high latitudes towards the equator, come from regions where the velocity due to rotation is small to where it is greater, and they consequently fall behind. Thus, in the northern hemisphere a north wind, as it moves away from its northern source of origin, is gradually deflected more and more towards the west and becomes a north-east current; while in the opposite hemisphere a wind making from high southern latitudes towards the equator becomes, from the same cause, a south-east current. Where, on the other hand, the air moves from the equatorial to the polar regions its higher velocity of rotation carries it eastward, so that on the south side of the equator it becomes a north-west current and on the north side a south-west current. It is to this cause that the easting and westing of the great atmospheric currents are to be attributed, as is familiarly exemplified in the trade winds.

The atmospheric circulation thus deflected influences the circulation of the ocean. The winds which persistently blow from the north-east on the north side of the equator, and from the south-east on the south side, drive the superficial waters onwards, and give rise to converging oceanic currents which unite to form the great westerly equatorial current.

A more direct effect of terrestrial rotation has been claimed in the case of rivers which flow in a meridional direction. It has been asserted that those, which in the northern hemisphere flow from north to south, like the Volga, by continually passing into regions where the velocity of rotation is increasingly greater, are thrown more against their western than their eastern banks, while those whose general course is in an opposite direction, like the Irtisch and Yenesei, press more upon their eastern sides. There cannot be any doubt that the tendency of the streams must be in the directions indicated. But when the comparatively slow current and constantly meandering course of most rivers are taken into consideration, it may be doubted whether the influence of rotation is of much practical account so far as river-erosion is concerned.

One of the cosmical relations of our planet which has been more especially prominent in geological speculations relates to the position of the earth’s axis of rotation. Abundant evidence has now been obtained to prove that at a comparatively late geological period a rich flora, resembling that of warm climates at the present day, existed in high latitudes even within less than 9° of the north pole, where, with an extremely low temperature and darkness lasting for half of the year, no such vegetation could possibly now exist. It has accordingly been maintained by many geologists that the axis of rotation must have shifted, and that when the remarkable Arctic assemblage of fossil plants lived the region of their growth must have lain in latitudes much nearer to the equator of the time.

The possibility of any serious displacement of the rotational axis since a very early period in the earth’s history has been strenuously denied by astronomers, and their arguments have been generally, but somewhat reluctantly, accepted by geologists, who find themselves confronted with a problem which has hitherto seemed insoluble. That the axis is not rigidly stable, however, has been postulated by some physicists, and has now been demonstrated by actual observation and measurement. It is admitted that by the movement of large bodies of water the air over the surface of the globe, and more particularly by the accumulation of vast masses of snow and ice in different regions, the position of the axis might be to some extent shifted; more serious effects might follow from widespread upheavals or depressions of the surface of the lithosphere. On the assumption of the extreme rigidity of the earth’s interior, however, the general result of mathematical calculation is to negative the supposition that in any of these ways within the period represented by what is known as the “geological record,” that is, since the time of the oldest known sedimentary formations, the rotational axis has ever been so seriously displaced as to account for such stupendous geological events as the spread of a luxuriant vegetation far up into polar latitudes. If, however, the inside of the globe possesses a great plasticity than has been allowed, the shifting of the axis might not be impossible, even to such an extent as would satisfy the geological requirements. This question is one on which the last word has not been said, and regarding which judgment must remain in suspense.

In recent years fresh information bearing on the minor devagations of the pole has been obtained from a series of several thousand careful observations made in Europe and North America. It has thus been ascertained that the pole wanders with a curiously irregular but somewhat spiral movement, within an amplitude of between 40 and 50 ft., and completes its erratic circuit in about 428 days. It was not supposed that its movement had any geological interest, but Dr John Milne has recently pointed out that the times of sharpest curvature in the path of the pole coincide with the occurrence of large earthquakes, and has suggested that, although it can hardly be assumed that this coincidence shows any direct connexion between earthquake frequency and changes in the position of the earth’s axis, both effects may not improbably arise from the same redistribution of surface material by ocean currents and meteorological causes.

If for any reason the earth’s centre of gravity were sensibly displaced, momentous geological changes would necessarily ensue. That the centre of gravity does not coincide with the centre of figure of the globe, but lies to the south of it, has long been known. This 650
greater aggregation of dense material in the southern hemisphere probably dates from the early ages of the earth’s consolidation, and it is difficult to believe that any readjustment of the distribution of this material in the earth’s interior is now possible. But certain rearrangements of the hydrosphere on the surface of the globe may, from time to time, cause a shifting of the centre of gravity, which will affect the level of the ocean. The accumulation of enormous masses of ice around the pole will give rise to such a displacement, and will thus increase the body of oceanic water in the glaciated hemisphere. Various calculations have been made of the effect of the transference of the ice-cap from one pole to the other, a revolution which may possibly have occurred more than once in the past history of the globe. James Croll estimated that if the mass of ice in the southern hemisphere be assumed to be 1000 ft. thick down to lat. 60°, its removal to the opposite hemisphere would raise the level of the sea 80 ft. at the north pole, while the Rev. Osmond Fisher made the rise as much as 409 ft. The melting of the ice would still further raise the sea-level by the addition of so large a volume of water to the ocean. To what extent superficial changes of this kind have operated in geological history remains an unsolved problem, but their probable occurrence in the past has to be recognized as one of the factors that must be considered in tracing the revolutions of the earth’s surface.

The Age of the Earth.—Intimately connected with the relations of our globe to the sun and the other members of the solar system is the question of the planet’s antiquity—a subject of great geological importance, regarding which much discussion has taken place since the middle of the 19th century. Though an account of this discussion necessarily involves allusion to departments of geology which are more appropriately referred to in later parts of this article, it may perhaps be most conveniently included here.

Geologists were for many years in the habit of believing that no limit could be assigned to the antiquity of the planet, and that they were at liberty to make unlimited drafts on the ages of the past. In 1862 and subsequent years, however, Lord Kelvin (then Sir William Thomson) pointed out that these demands were opposed to known physical facts, and that the amount of time required for geological history was not only limited, but must have been comprised within a comparatively narrow compass. His argument rested on three kinds of evidence: (1) the internal heat and rate of cooling of the earth; (2) the tidal retardation of the earth’s rotation; and (3) the origin and age of the sun’s heat.

1. Applying Fourier’s theory of thermal conductivity, Lord Kelvin contended that in the known rate of increase of temperature downward and beneath the surface, and the rate of loss of heat from the earth, we have a limit to the antiquity of the planet. He showed, from the data available at the time, that the superficial consolidation of the globe could not have occurred less than 20 million years ago, or the underground heat would have been greater

than it is; nor more than 400 million years ago, otherwise the underground temperature would have shown no sensible increase downwards. He admitted that very wide limits were necessary. In subsequently discussing the subject, he inclined rather towards the lower than the higher antiquity, but concluded that the limit, from a consideration of all the evidence, must be placed within some such period of past time as 100 millions of years.

2. The argument from tidal retardation proceeds on the admitted fact that, owing to the friction of the tide-wave, the rotation of the earth is retarded, and is, therefore, much slower now than it must have been at one time. Lord Kelvin affirmed that had the globe become solid some 10,000 million years ago, or indeed any high antiquity beyond 100 million years, the centrifugal force due to the more rapid rotation must have given the planet a very much greater polar flattening than it actually possesses. He admitted, however, that, though 100 million years ago that force must have been about 3% greater than now, yet “nothing we know regarding the figure of the earth, and the disposition of land and water, would justify us in saying that a body consolidated when there was more centrifugal force by 3% than now, might not now be in all respects like the earth, so far as we know it at present.”

3. The third argument, based upon the age of the sun’s heat, is confessedly less to be relied on than the two previous ones. It proceeds upon calculations as to the amount of heat which would be available by the falling together of masses from space, which gave rise by their impact to our sun. The vagueness of the data on which this argument rests may be inferred from the fact that in one passage P.G. Tait placed the limit of time during which the sun has been illuminating the earth as, “on the very highest computation, not more than about 15 or 20 millions of years”; while, in another sentence of the same volume, he admitted that, “by calculations in which there is no possibility of large error, this hypothesis [of the origin of the sun’s heat by the falling together of masses of matter] is thoroughly competent to explain 100 millions of years’ solar radiation at the present rate, perhaps more.” In more recently reviewing his argument, Lord Kelvin expressed himself in favour of more strictly limiting geological time than he had at first been disposed to do. He insists that the time “was more than 20 and less than 40 millions of years and probably much nearer 20 than 40.” Geologists appear to have reluctantly brought themselves to believe that perhaps, after all, 100 millions of years might suffice for the evolution of geological history. But when the time was cut down to 15 or 20 millions they protested that such a restricted period was insufficient for that evolution, and though they did not offer any effective criticism of the arguments of the physicists they felt convinced that there must be some flaw in the premises on which these arguments were based.

By degrees, however, there have arisen among the physicists themselves grave doubts as to the validity of the physical evidence on which the limitation of the earth’s age has been founded, and at the same time greater appreciation has been shown of the signification and strength of the geological proofs of the high antiquity of our planet. In an address from the chair of the Mathematical Section of the British Association in 1886, Professor (afterwards Sir) George Darwin reviewed the controversy, and pronounced the following deliberate judgment in regard to it: “In considering these three arguments I have adduced some reasons against the validity of the first [tidal friction], and have endeavoured to show that there are elements of uncertainty surrounding the second [secular cooling of the earth]; nevertheless, they undoubtedly constitute a contribution of the first importance to physical geology. Whilst, then, we may protest against the precision with which Professor Tait seeks to deduce results from them, we are fully justified in following Sir William Thomson, who says that ‘the existing state of things on the earth, life on the earth—all geological history showing continuity of life—must be limited within some such period of past time as 100 million years’.” Lord Kelvin has never dealt with the geological and palaeontological objections against the limitation of geological time to a few millions of years. But Professor Darwin, in the address just cited, uttered the memorable warning: “At present our knowledge of a definite limit to geological time has so little precision that we should do wrong summarily to reject theories which appear to demand longer periods of time than those which now appear allowable.” In his presidential address to the British Association at Cape Town in 1905 he returned to the subject, remarking that the argument derived from the increase of underground temperature “seems to be entirely destroyed” by the discovery of the properties of radium. He thinks that “it does not seem extravagant to suppose that 500 to 1000 million years may have elapsed since the birth of the moon.” He has “always believed that the geologists were more nearly correct than the physicists, notwithstanding the fact that appearances were so strongly against them,” and he concludes thus: “It appears, then, that the physical argument is not susceptible of a greater degree of certainty than that of the geologists, and the scale of geological time remains in great measure unknown” (see also Tide, chap. viii.).

In an address to the mathematical section of the American Association for the Advancement of Science in 1889, the vice-president of the section, R.S. Woodward, thus expressed himself with regard to the physical arguments brought forward by Lord Kelvin and Professor Tait in limitation of geological time: “Having been at some pains to look into this matter, I feel bound to state that, although the hypothesis appears to be the best which can be formulated at present, the odds are against its correctness. Its weak links are the unverified assumptions of an initial uniform temperature and a constant diffusivity. Very likely these are approximations, but of what order we cannot decide. Furthermore, if we accept the hypothesis, the odds appear to be against the present attainment of trustworthy numerical results, since the data for calculation, obtained mostly from observations on continental areas, are far too meagre to give satisfactory average values for the entire mass of the earth.”

Still more emphatic is the protest made from the physical side by Professor John Perry. He has attacked each of the three lines of argument of Lord Kelvin, and has impugned the validity of the conclusions drawn from them. The argument from tidal retardation he dismisses as fallacious, following in this contention the previous criticism of the Rev. Maxwell Close and Sir George Darwin. In dealing with the argument based on the secular cooling of the earth, he holds it to be perfectly allowable to assume a much higher conductivity for the interior of the globe, and that such a reasonable assumption would enable us greatly to increase our estimate of the earth’s antiquity. As for the third argument, from the age of the sun’s heat, he points out that the sun may have been repeatedly fed by a supply of meteorites from outside, while the earth may have been protected from radiation, and been able to retain much of its heat by being enveloped in a dense atmosphere. Remarking that “almost anything is possible as to the present internal state of the earth,” he concludes thus: “To sum up, we can find no published record of any lower maximum age of life on the earth, as calculated by physicists, than 400 millions of years. From the three physical arguments Lord Kelvin’s higher limits are 1000, 400 and 500 million years. I have shown that we have reasons for believing that the age, from all these, may be very considerably underestimated. It is to be observed that if we exclude everything but the arguments from mere physics, the *probable* age of life on the earth is much less than any of the above estimates; but if the palaeontologists have good reasons for demanding much greater times, I see nothing from the physicists’ point of view which denies them four times the greatest of these estimates.”

A fresh line of argument against Lord Kelvin’s limitation of the antiquity of our globe has recently been started by the remarkable discoveries in radio-activity. From the ascertained properties of radium it appears to be possible that our estimates of solar heat, as derived from the theory of gravitation, may have to be augmented ten or twenty times; that stores of radium and similar bodies within the earth may have indefinitely deferred the establishment of the present temperature gradient from the surface inward; that consequently the earth may have remained for long ages at a temperature not greatly different from that which it now possesses, and hence that the times during which our globe has supported animal and vegetable life may be very much longer than that allowed in the estimates previously made by physicists from other data (see Radioactivity).

The arguments from the geological side against the physical contention that would limit the age of our globe to some 10 or 20 millions of years are mainly based on the observed rates of geological and biological changes at the present time upon land and sea, and on the nature, physical history and organic contents of the stratified crust of the earth. Unfortunately, actual numerical data are not obtainable in many departments of geological activity, and even where they can be procured they do not yet rest on a sufficiently wide collection of accurate and co-ordinated observations. But in some branches of dynamical geology, material exists for, at least, a preliminary computation of the rate of change. This is more especially the case in respect of the wide domain of denudation. The observational records of the action of the sea, of springs, rivers and glaciers are becoming gradually fuller and more trustworthy. A method of making use of these records for estimating the rate of denudation of the land has been devised. Taking the Mississippi as a general type of river action, it has been shown that the amount of material conveyed by this stream into the sea in one year is equivalent to the lowering of the general surface of the drainage basin of the river by $\frac{1}{6000}$ of a foot. This would amount to one foot in 6000 years and 1000 ft. in 6 million years. So that at the present rate of waste in the Mississippi basin a whole continent might be worn away in a few millions of years.

It is evident that as deposition and denudation are simultaneous processes, the ascertainment of the rate at which solid material is removed from the surface of the land supplies some necessary information for estimating the rate at which new sedimentary formations are being accumulated on the floor of the sea, and for a computation of the length of time that would be required at the present rate of change for the deposition of all the stratified rocks that enter into the composition of the crust of our globe. If the thickness of these rocks be assumed to be 100,000 ft., and if we could suppose them to have been laid down over as wide an area as that of the drainage basins from the waste of which they were derived, then at the present rate of denudation their accumulation would require some 600 millions of years. But, as Dr A.R. Wallace has justly pointed out, the tract of sea-floor over which the material derived from the waste of the terrestrial surface is laid down is at present much less than that from which this material is worn away. We have no means, however, of determining what may have been the ratio between the two areas in past time. Certainly ancient marine sedimentary rocks cover at the present day a much more extensive area than that in which they are now being elaborated. If we take the ratio postulated by Dr Wallace—1 to 19—the 100,000 ft. of sedimentary strata would require 31 millions of years for their accumulation. It is quite possible, however, that this ratio may be much too high. There are reasons for believing that the proportion of coast-line to land area has been diminishing during geological time; in other words, that in early times the land was more insular and is now more continental. So that the 31 millions of years may be much less than the period that would be required, even on the supposition of continuous uninterrupted denudation and sedimentation, during the whole of the time represented by the stratified formations.

But no one who has made himself familiar with the actual composition of these formations and the detailed structure of the terrestrial crust can fail to recognize how vague, imperfect and misleading are the data on which such computations are founded. It requires no prolonged acquaintance with the earth’s crust to impress upon the mind that one all-important element is omitted, and indeed can hardly be allowed for from want of sufficiently precise data, but the neglect of which must needs seriously impair the value of all numerical calculations made without it. The assumption that the stratified formations can be treated as if they consisted of a continuous unbroken sequence of sediments, indicating a vast and uninterrupted process of waste and deposition, is one that is belied on every hand by the actual structure of these formations. It can only give us a minimum of the time required; for, instead of an unbroken series, the sedimentary formations are full of “unconformabilities”—gaps in the sequence of the chronological records—as if whole chapters and groups of chapters had been torn out of a historical work. It can often be shown that these breaks of continuity must have been of vast duration, and actually exceeded in chronological importance thick groups of strata lying below and above them (see Part VI.). Moreover, even among the uninterrupted strata, where no such unconformabilities exist, but where the sediments follow each other in apparently uninterrupted sequence, and might be thought to 652
have been deposited continuously at the same general rate, and without the intervention of any pause, it can be demonstrated that sometimes an inch or two of sediment might, on certain horizons, represent the deposit of an enormously longer period than a hundred or a thousand times the same amount of sediment on other horizons. A prolonged study of these questions leads to a profound conviction that in many parts of the geological record the time represented by sedimentary deposits may be vastly less than the time which is not so represented.

It has often been objected that the present rate of geological change ought not to be taken as a measure of the rate in past time, because the total sum of terrestrial energy has been steadily diminishing, and geological processes must consequently have been more vigorous in former ages than they are now. Geologists do not pretend to assert that there has been no variation or diminution in the activities of the various processes which they have to study. What they do insist on is that the present rate of change is the only one which we can watch and measure, and which will thus supply a statistical basis for any computations on the subject. But it has been dogmatically affirmed that because terrestrial energy has been diminishing therefore all kinds of geological work must have been more vigorously and more rapidly carried on in former times than now; that there were far more abundant and more stupendous volcanoes, more frequent and more destructive earthquakes, more gigantic upheavals and subsidences, more powerful oceanic waves and tides, more violent atmospheric disturbances with heavier rainfall and more active denudation.

It is easy to make these assertions, and they look plausible; but, after all, they rest on nothing stronger than assumption. They can be tested by an appeal to the crust of the earth, in which the geological history of our planet has been so fully recorded. Had such portentous manifestations of geological activity ever been the normal condition of things since the beginning of that history, there ought to be a record of them in the rocks. But no evidence for them has been found there, though it has been diligently sought for in all quarters of the globe. We may confidently assert that while geological changes may quite possibly have taken place on a gigantic scale in the earliest ages of the earth’s existence, of which no geological record remains, there is no proof that they have ever done so since the time when the very oldest of the stratified formations were deposited. There is no need to maintain that they have always been conducted precisely on the same scale as now, or to deny that they may have gradually become less vigorous as the general sum of terrestrial energy has diminished. But we may unhesitatingly affirm that no actual evidence of any such progressive diminution of activity has been adduced from the geological record in the crust of the earth: that, on the contrary, no appearances have been detected there which necessarily demand the assumption of those more powerful operations postulated by physicists, or which are not satisfactorily explicable by reference to the existing scale of nature’s processes.

That this conclusion is warranted even with regard to the innate energy of the globe itself will be seen if we institute a comparison between the more ancient and the more recent manifestations of that energy. Take, for example, the proofs of gigantic plication, fracture and displacement within the terrestrial crust. These, as they have affected the most ancient rocks of Europe, have been worked out in great detail in the north-west of Scotland. But they are not essentially different from or on a greater scale than those which have been proved to have affected the Alps, and to have involved strata of so recent a date as the older Tertiary formations. On the contrary, it may be doubted whether any denuded core of an ancient mountain-chain reveals traces of such stupendous disturbances of the crust as those which have given rise to the younger mountain-chains of the globe. It may, indeed, quite well have been the rule that instead of diminishing in intensity of effect, the consequences of terrestrial contraction have increased in magnitude, the augmenting thickness of the crust offering greater resistance to the stresses, and giving rise to vaster plications, faults, thrust-planes and metamorphism, as this growing resistance had to be overcome.

The assertion that volcanic action must have been more violent and more persistent in ancient times than it is now has assuredly no geological evidence in its support. It is quite true that there are vastly more remains of former volcanoes scattered over the surface of the globe than there are active craters now, and that traces of copious eruptions of volcanic material can be followed back into some of the oldest parts of the geological record. But we have no proof that ever at any one time in geological history there have been more or larger or more vigorous volcanoes than those of recent periods. It may be said that the absence of such proof ought not to invalidate the assertion until a far wider area of the earth’s surface has been geologically studied. But most assuredly, as far as geological investigation has yet gone, there is an overwhelming body of evidence to show that from the earliest epochs in geological history, as registered in the stratified rocks, volcanic action has manifested itself very much as it does now, but on a less rather than on a greater scale. Nowhere can this subject be more exhaustively studied than in the British Isles, where a remarkably complete series of volcanic eruptions has been chronicled ranging from the earliest Palaeozoic down to older Tertiary time. The result of a prolonged study of British volcanic geology has demonstrated that, even to minute points of detail, there has been a singular uniformity in the phenomena from beginning to end. The oldest lavas and ashes differ in no essential respect from the youngest. Nor have they been erupted more copiously or more frequently. Many successive volcanic periods have followed each other after prolonged intervals of repose, each displaying the same general sequence of phenomena and similar evidence of gradual diminution and extinction. The youngest, instead of being the feeblest, were the most extensive outbursts in the whole of this prolonged series.

If now we turn for evidence of the alleged greater activity of all the epigene or superficial forces, and especially for proofs of more rapid denudation and deposition on the earth’s surface, we search for it in vain among the stratified formations of the terrestrial crust. Had the oldest of these rocks been accumulated in a time of great atmospheric perturbation, of torrential rains, colossal tides and violent storms, we might surely expect to find among the sediments some proof of such disturbed meteorological and geographical conditions. We should look, on the one hand, for tumultuous accumulations of coarse unworn detritus, rapidly swept by rains, floods and waves from land to sea, and on the other hand, for an absence of any evidence of the tranquil and continuous deposit of such fine laminated silt as could only settle in quiet water. But an appeal to the geological record is made in vain for any such proofs. The oldest sediments, like the youngest, reveal the operation only of such agents and such rates of activity as are still to be witnessed in the accumulation of the same kind of deposits. If, for instance, we search the most ancient thick sedimentary formation in Britain—the Torridon Sandstone of north-west Scotland, which is older than the oldest

fossiliferous deposits—we meet with nothing which might not be found in any Palaeozoic, Mesozoic or Cainozoic group of similar sediments. We see an accumulation, at least 8000 or 10,000 ft. thick, of consolidated sand, gravel and mud, such as may be gathering now on the floor of any large mountain-girdled lake. The conglomerates of this ancient series are not pell-mell heaps of angular detritus, violently swept away from the land and huddled promiscuously on the sea-floor. They are, in general, built up of pebbles that have been worn smooth, rounded and polished by prolonged attrition in running water, and they follow each other on successive platforms with intervening layers of finer sediment. The sandstones are composed of well water-worn sand, some of which has been laid down so tranquilly that its component grains have been separated out in layers according to their specific gravity, in such manner that they now present dark laminae in which particles of magnetic iron, zircon and other heavy minerals have been sifted out together, just as iron-sand may be seen 653
gathered into thin sheets on sandy beaches at the present day. Again, the same series of primeval sediments includes intercalations of fine silt, which has been deposited as regularly and intermittently there as it has been among the most recent formations. These bands of shale have been diligently searched for fossils, as yet without success; but they may eventually disclose organic remains older than any hitherto found in Europe.

We now come to the consideration of the palaeontological evidence as to the value of geological time. Here the conclusions derived from a study of the structure of the sedimentary formations are vastly strengthened and extended. In the first place, the organization of the most ancient plants and animals furnishes no indication that they had to contend with any greater violence of storm, flood, wave or ocean-current than is familiar to their modern descendants. The oldest trees, shrubs, ferns and club-mosses display no special structures that suggest a difference in the general conditions of their environment. The most ancient crinoids, sponges, crustaceans, arachnids and molluscs were as delicately constructed as those of to-day, and their remains are often found in such perfect preservation as to show that neither during their lifetime nor after their death were they subject to any greater violence of the elements than their living representatives now experience. Of much more cogency, however, is the evidence supplied by the grand upward succession of organic forms, from the most ancient stratified rocks up to the present day. No biologist now doubts for a moment that this marvellous succession is the result of a gradual process of evolution from lower to higher types of organization. There may be differences of opinion as to the causes which have governed this process and the order of the steps through which it has advanced, but no one who is conversant with the facts will now venture to deny that it has taken place, and that, on any possible explanation of its progress, it must have demanded an enormous lapse of time. In the Cambrian or oldest fossiliferous formations there is already a large and varied fauna, in which the leading groups of invertebrate life are represented. On no tenable hypothesis can these be regarded as the first organisms that came into being on our planet. They must have had a long ancestry, and as Darwin first maintained, the time required for their evolution may have been "as long as, or probably far longer than, the whole interval from the Silurian [Cambrian] age to the present day." The records of these earliest eras of organic development have unfortunately not survived the geological revolutions of the past; at least, they have not yet been recovered. But it cannot be doubted that they once existed and registered their testimony to the prodigious lapse of time prior to the deposition of the most ancient fossiliferous formations which have escaped destruction.

The impressive character of the evidence furnished by the sequence of organic forms throughout the great series of fossiliferous strata can hardly be fully realized without a detailed and careful study of the subject. Professor E.B. Poulton, in an address to the zoological section of the British Association at the Liverpool Meeting in 1896, showed how overwhelming are the demands which this evidence makes for long periods of time, and how impossible it is of comprehension unless these demands be conceded. The history of life upon the earth, though it will probably always be surrounded with great and even insuperable difficulties, becomes broadly comprehensible in its general progress when sufficient time is granted for the evolution which it records; but it remains unintelligible on any other conditions.

Taken then as a whole, the body of evidence, geological and palaeontological, in favour of the high antiquity of our globe is so great, so manifold, and based on such an ever-increasing breadth of observation and reflection, that it may be confidently appealed to in answer to the physical arguments which would seek to limit that antiquity to ten or twenty millions of years. In the present state of science it is out of our power to state positively what must be the lowest limit of the age of the earth. But we cannot assume it to be much less, and it may possibly have been much more, than the 100 millions of years which Lord

Kelvin was at one time willing to concede.[2]

Part III.—Geognosy. The Investigation of the Nature and Composition of the Materials of which the Earth Consists

This division of the science is devoted to a description of the parts of the earth—of the atmosphere and ocean that surround the planet, and more especially of the solid materials that underlie these envelopes and extend downwards to an unknown distance into the interior. These various constituents of the globe are here considered as forms of matter capable of being analysed, and arranged according to their composition and the place they take in the general composition of the globe.

Viewed in the simplest way the earth may be regarded as made up of three distinct parts, each of which ever since an early period of planetary history has been the theatre of important geological operations. (1) An envelope of air, termed the *atmosphere*, which surrounds the whole globe; (2) A lower and less extensive envelope of water, known as the *hydrosphere* (Gr. ὕδωρ, water) which, constituting the oceans and seas, covers nearly three-fourths of the underlying solid surface of the planet; (3) A globe, called the *lithosphere* (Gr. λίθος, stone), the external part of which, consisting of solid stone, forms the *crust*, while underneath, and forming the vast mass of the interior, lies the *nucleus*, regarding the true constitution of which we are still ignorant.

1. *The Atmosphere.*—The general characters of the atmosphere are described in separate articles (see especially Atmosphere; Meteorology). Only its relations to geology have here to be considered. As this gaseous envelope encircles the whole globe it is the most universally present and active of all the agents of geological change. Its efficacy in this respect arises partly from its composition, and the chemical reactions which it effects upon the surface of the land, partly from its great variations in temperature and moisture, and partly from its movements.

Many speculations have been made regarding the chemical composition of the atmosphere during former geological periods. There can indeed be little doubt that it must originally have differed greatly from its present condition. If the whole mass of the planet originally existed in a gaseous state, there would be practically no atmosphere. The present outer envelope of air may be considered to be the surviving relic of this condition, after all the other constituents have been incorporated into the hydrosphere and lithosphere. The oxygen, which now forms fully a half of the outer crust of the earth, was doubtless originally, whether free or in combination, part of the atmosphere. So, too, the vast beds of coal found all over the world, in geological formations of many different ages, represent so much carbonic acid once present in the air. The chlorides and other salts in the sea may likewise partly represent materials carried down out of the atmosphere in the primitive condensation of the aqueous vapour, though they have been continually increased ever since by contributions from the drainage of the land. It has often been suggested that, during the Carboniferous period, the atmosphere must have been warmer and more charged with aqueous vapour and carbon dioxide than at the present day, to admit of so luxuriant a flora as that from which the coal-seams were formed. There seems, however, to be at present no method of arriving at any certainty on this subject. Lastly, the amount of carbonic acid absorbed in the weathering of rocks at the surface, and the consequent production of carbonates, represents an enormous abstraction of this gas.

As at present constituted, the atmosphere is regarded as a mechanical mixture of nearly four volumes of nitrogen and one of oxygen, together with an average of 3.5 parts of carbon dioxide in every 10,000 parts of air, and minute quantities of various other gases and solid particles. Of the vapours contained in it by far the most important is that of water which, although always present, varies greatly in amount according to variations in temperature. By condensation the water vapour appears in visible form as dew, mist, cloud, rain, hail, snow and ice, and in these forms includes and carries down some of the other vapours, gases and solid particles present in the air. The circulation of water from the atmosphere to the land, from the land to the sea, and again from the sea to the land, forms the great geological process whereby the habitable condition of the planet is maintained and the surface of the land is sculptured (Part IV.). 654

2. *The Hydrosphere.*—The water envelope covers nearly three-fourths of the surface of the earth, and forms the various oceans and seas which, though for convenience of reference distinguished by separate names, are all linked together in one great body. The physical characters of this vast envelope are discussed in separate articles (see Ocean and Oceanography). Viewed from the geological standpoint, the features of the sea that specially

deserve attention are first the composition of its waters, and secondly its movements.

Sea-water is distinguished from that of ordinary lakes and rivers by its greater specific gravity and its saline taste. Its average density is about 1.026, but it varies even within the same ocean, being least where large quantities of fresh water are added from rain or melting snow and ice, and greatest where evaporation is most active. That sea-water is heavier than fresh arises from the greater proportion of salts which it contains in solution. These salts constitute about three and a half parts in every hundred of water. They consist mainly of chlorides of sodium and magnesium, the sulphates of magnesium, calcium and potassium, with minuter quantities of magnesium bromide and calcium carbonate. Still smaller proportions of other substances have been detected, gold for example having been found in the proportion of 1 part in 15,180,000.

That many of the salts have existed in the sea from the time of its first condensation out of the primeval atmosphere appears to be probable. It is manifest, however, that, whatever may have been the original composition of the oceans, they have for a vast section of geological time been constantly receiving mineral matter in solution from the land. Every spring, brook and river removes various salts from the rocks over which it moves, and these substances, thus dissolved, eventually find their way into the sea. Consequently sea-water ought to contain more or less traceable proportions of every substance which the terrestrial waters can remove from the land, in short, of probably every element present in the outer shell of the globe, for there seems to be no constituent of this earth which may not, under certain circumstances, be held in solution in water. Moreover, unless there be some counteracting process to remove these mineral ingredients, the ocean water ought to be growing, insensibly perhaps, but still assuredly, saltier, for the supply of saline matter from the land is incessant.

To the geologist the presence of mineral solutions in sea-water is a fact of much importance, for it explains the origin of a considerable part of the stratified rocks of the earth’s crust. By evaporation the water has given rise to deposits of rock-salt, gypsum and other materials. The lime contained in solution, whether as sulphate or carbonate, has been extracted by many tribes of marine animals, which have thus built up out of their remains vast masses of solid limestone, of which many mountain-chains largely consist.

Another important geological feature of the sea is to be seen in the fact that its basins form the great receptacles for the detritus worn away from the land. Besides the limestones, the visible parts of the terrestrial crust are, in large measure, composed of sedimentary rocks which were originally laid down on the sea-bottom. Moreover, by its various movements, the sea occupies a prominent place among the epigene or superficial agents which produce geological changes on the surface of the globe.

3. *The Lithosphere.*—Beneath the gaseous and liquid envelopes lies the solid part of the planet, which is conveniently regarded as consisting of two parts,—(*a*) the crust, and (*b*) the interior or nucleus.

The crust.

It was for a long time a prevalent belief that the interior of the globe is a molten mass round which an outer shell has gradually formed through cooling. Hence the term “crust” was applied to this external solid envelope, which was variously computed to be 10, 20, or more miles in thickness. The portion of this crust accessible to human observation was seen to afford abundant evidence of vast plications and corrugations of its substance, which were regarded as only explicable on the supposition of a thin solid collapsible shell floating on a denser liquid interior. When, however, physical arguments were adduced to show the great rigidity of the earth as a whole, the idea of a thin crust enclosing a molten nucleus was reluctantly abandoned by geologists, who found the problem of the earth’s interior to be incapable of solution by any evidence which their science could produce. They continued, however, to use the term “crust” as a convenient word to denote the cool outer layer of the earth’s mass, the structure and history of which form the main subjects of geological investigation. More recently, however, various lines of research have concurred in suggesting that, whatever may be the condition of the interior, its substance must differ greatly from that of the outer shell, and that there may be more reason than appeared for the retention of the name of crust. Observations on earthquake motion by Dr John Milne and others, show that the rate and character of the waves transmitted through the interior of the earth differ in a marked degree from those propagated along the crust. This difference indicates that rocky material, such as we know at the surface, may extend inwards for some 30 m., below which the earth’s interior rapidly becomes fairly homogeneous and possesses a high rigidity. From measurements of the force of gravity in India by Colonel S.G. Burrard, it has been inferred that the variations in density of the outer parts of the earth do not descend farther than 30 or 40 m., which might be assumed to be the limit of the thickness of the crust. Recent researches in regard to the radio-active substances present in rocks suggest that the crust is not more than 50 m. thick, and that the interior differs from it in possessing little or no radio-active material.

Though we cannot hope ever to have direct acquaintance with more than the mere outside skin of our planet, we may be led to infer the irregular distribution of materials within the crust from the present distribution of land and water, and the observed differences in the amount of deflection of the plumb-line near the sea and near mountain-chains. The fact that the southern hemisphere is almost wholly covered with water appears explicable only on the assumption of an excess of density in the mass of that portion of the planet. The existence of such a vast sheet of water as that of the Pacific Ocean is to be accounted for, as Archdeacon J.H. Pratt pointed out, by the presence of "some excess of matter in the solid parts of the earth between the Pacific Ocean and the earth's centre, which retains the water in its place, otherwise the ocean would flow away to the other parts of the earth." A deflection of the plumb-line towards the sea, which has in a number of cases been observed, indicates that "the density of the crust beneath the mountains must be less than that below the plains, and still less than that below the ocean-bed." Apart therefore from the depression of the earth's surface in which the oceans lie, we must regard the internal density, whether of crust or nucleus, to be somewhat irregularly arranged, there being an excess of heavy materials in the water hemisphere, and beneath the ocean-beds, as compared with the continental masses.

The interior.

In our ignorance regarding the chemical constitution of the nucleus of our planet, an argument has sometimes been based upon the known fact that the specific gravity of the globe as a whole is about double that of the crust. This has been held by some writers to prove that the interior must consist of much heavier material and is therefore probably metallic. But the effect of pressure ought to make the density of the nucleus much higher, even if the interior consisted of matter no heavier than the crust. That the total density of the planet does not greatly exceed its observed amount seems only explicable on the supposition that some antagonistic force counteracts the effects of pressure. The only force we can suppose capable of so acting is heat. But comparatively little is yet known regarding the compression of gases, liquids and solids under such vast pressures as must exist within the nucleus.

That the interior of the earth possesses a high temperature is inferred from the evidence of various sources. (1) Volcanoes, which are openings that constantly, or intermittently, give out hot vapours and molten lava from reservoirs beneath the crust. Besides active volcanoes, it is known that former eruptive vents have been abundantly and widely distributed over the 655
globe from the earliest geological periods down to our own day. (2) Hot springs are found in many parts of the globe, with temperatures varying up to the boiling point of water. (3) From mines, tunnels and deep borings into the earth it has been ascertained that in all quarters of the globe below the superficial zone of invariable temperature, there is a progressive increase of heat towards the interior. The rate of this increase varies, being influenced, among other causes, by the varying conductivity of the rocks. But the average appears to be about 1° Fahr. for every 50 or 60 ft. of descent, as far down as observations have extended. Though the increase may not advance in the same proportion at great depths, the inference has been confidently drawn that the temperature of the nucleus must be exceedingly high.

The probable condition of the earth's interior has been a fruitful source of speculation ever since geology came into existence; but no general agreement has been arrived at on the subject. Three chief hypotheses have been propounded: (1) that the nucleus is a molten mass enclosed within a solid shell; (2) that, save in local vesicular spaces which may be filled with molten or gaseous material, the globe is solid and rigid to the centre; (3) that the great body of the nucleus consists of incandescent vapours and gases, especially vaporous iron, which under the gigantic pressure within the earth are so compressed as to confer practical rigidity on the globe as a whole, and that outside this main part of the nucleus the gases pass into a shell of molten magma, which, in turn, shades off outwards into the comparatively thin, cool solidified crust. Recent seismological observations have led to the inference that the outer crust, some 30 to 45 m. thick, must rapidly merge into a fairly homogeneous nucleus which, whatever be its constitution, transmits undulatory movements through its substance with uniform velocity and is believed to possess a high rigidity.

The origin of the earth's high internal temperature has been variously accounted for. Most usually it has been assumed to be the residue of the original "tracts of fluent heat" out of which the planet shaped itself into a globe. According to another supposition the effects of the gradual gravitational compression of the earth's mass have been the main source of the high temperature. Recent researches in radio-activity, to which reference has already been made, have indicated another possible source of the internal heat in the presence of radium in the rocks of the crust. This substance has been detected in all igneous rocks, especially among the granites, in quantity sufficient, according to the Hon. R.J. Strutt, to account for the observed temperature-gradient in the crust, and to indicate that this crust cannot be more than 45 m. thick, otherwise the outflow of heat would be greater than the amount actually ascertained. Inside this external crust containing radio-active substances, it is supposed, as already stated, that the nucleus consists of some totally different matter containing little or no radium.

Constitution of the Earth's Crust.—As the crust of the earth contains the "geological record," or stony chronicle from which geology interprets the history of our globe, it forms the main subject of study to the geologist. The materials of which this crust consists are known as minerals and rocks. From many chemical analyses, which have been made of these materials, the general chemical constitution of, at least, the accessible portion of the crust has been satisfactorily ascertained. This information becomes of much importance in speculations regarding the early history of the globe. Of the elements known to the chemist the great majority form but a small proportion of the composition of the crust, which is mainly built up of about twenty of them. Of these by far the most important are the non-metallic elements oxygen and silicon. The former forms about 47% and the latter rather more than 28% of the original crust, so that these two elements make up about three-fourths of the whole. Next after them come the metals aluminium (8.16%), iron (4.64), calcium (3.50), magnesium (2.62), sodium (2.63), and potassium (2.35). The other twelve elements included in the twenty vary in amount from a proportion of 0.41% in the case of titanium, to not more than 0.01% of chlorine, fluorine, chromium, nickel and lithium. The other fifty or more elements exist in such minute proportions in the crust that, probably, not one of them amounts to as much as 0.01%, though they include the useful metals, except iron. Taking the crust, and the external envelopes of the ocean and the air, we thus perceive that these outer parts of our planet consist of more than three-fourths of non-metals and less than one-fourth of metals.

The combinations of the elements which are of most importance in the constitution of the terrestrial crust consist of oxides. From the mean of a large number of analyses of the rocks of the lower or primitive portion of the crust, it has been ascertained that silica (SiO_2) forms almost 60% and alumina (Al_2O_3) upwards of 15% of the whole. The other combinations in order of importance are lime (CaO) 4.90%, magnesia (MgO) 4.36, soda (Na_2O) 3.55, ferrous oxide (FeO) 3.52, potash (K_2O) 2.80, ferric oxide (Fe_2O_3) 2.63, water (H_2O) 1.52, titanium oxide (TiO_2) 0.60, phosphoric acid (P_2O_5) 0.22; the other combinations of elements thus form less than 1% of the crust.

These different combinations of the elements enter into further combinations with each other so as to produce the wide assortment of simple minerals (see Mineralogy). Thus, silica and alumina are combined to form the aluminous silicates, which enter so largely into the composition of the crust of the earth. The silicates of magnesia, potash and soda constitute other important families of minerals. A mass of material composed of one, but more usually of more than one mineral, is known as a *rock*. Under this term geologists are accustomed to class not only solid stone, such as granite and limestone, but also less coherent materials such as clay, peat and even loose sand. The accessible portion of the earth's crust consists of various kinds of rocks, which differ from each other in structure, composition and origin, and are therefore susceptible of diverse classifications according to the point of view from which they are considered. The details of this subject will be found in the article Petrology.

Classification of Rocks.—Various systems of classification of rocks have been proposed, but none of them is wholly satisfactory. The most useful arrangement for most purposes of the geologist is one based on the broad differences between them in regard to their mode of origin. From this point of view they may be ranged in three divisions:

1. In the first place, a large number of rocks may be described as original or underived, for it is not possible to trace them back to any earlier source. They belong to the primitive constitution of the planet, and, as they have all come up from below through the crust, they serve to show the nature of the material which lies immediately below the outer parts of that crust. They include the numerous varieties of lava, which have been poured out in a molten state from volcanic vents, also a great series of other rocks which, though they may never have been erupted to the surface, have been forced upward in a melted condition into the other rocks of the crust and have solidified there. From their mode of origin this great class of rocks has been called "igneous" or "eruptive." As they generally show no definite internal structure save such as may result from joints, they have been termed "massive" or "unstratified," to distinguish them from those of the second division which are strongly marked out by the presence of a stratified structure. The igneous rocks present a

considerable range of composition. For the most part they consist mainly of aluminous silicates, some of them being highly acid compounds with 75% or more of silica. But they also include highly basic varieties wherein the proportion of silica sinks to 40%, and where magnesia greatly predominates over alumina. The textures of igneous rocks likewise comprise a wide series of varieties. On the one hand, some are completely vitreous, like obsidian, which is a natural glass. From this extreme every gradation may be traced through gradual increase of the products of devitrification, until the mass may become completely crystalline. Again, some crystalline igneous rocks are so fine in grain as not to show their component crystals save under the microscope, while in others the texture is so coarse as to present the component minerals in separate crystals an inch or more in length. These differences indicate that, at first, the materials of the rock may have been as completely molten as artificial glass, and that the crystalline condition has been subsequently developed by cooling, and the separation of the chemical constituents into definite crystalline minerals. Many of the characters of igneous rocks have been reproduced experimentally by fusing together their minerals, or the constituents of their minerals, in the proper proportion. But it has not yet been found possible to imitate the structure of such rocks as granite. Doubtless these rocks consolidated with extreme slowness at great depths below the surface, under vast pressures and probably in the presence of water or water-vapour—conditions which cannot be adequately imitated in a laboratory.

Though the igneous rocks occupy extensive areas in some countries, they nevertheless cover a much smaller part of the whole surface of the land than is taken up by the second division or stratified rocks. But they increase in quantity downwards and probably extend continuously round the globe below the other rocks. This important series brings before us the relations of the molten magma within the earth to the overlying crust and to the outer surface. On the one hand, it includes the oldest and most deep-seated extravasations of that magma, which have been brought to light by ruptures and upheavals of the crust and prolonged denudation. On the other, it presents to our study the varied outpourings of molten and fragmentary materials in the discharges of modern and ancient volcanoes. 656
Between these two extremes of position and age, we find that the crust has been, as it were, riddled with injections of the magma from below. These features will be further noticed in Part V. of this article.

2. The “sedimentary” or “stratified rocks” form by much the larger part of the dry land of the globe, and they are prolonged to an unknown distance from the shores under the bed of the sea. They include those masses of mineral matter which, unlike the igneous rocks, can be traced back to a definite origin on the surface of the earth. Three distinct types may be recognized among them: (*a*) By far the largest proportion of them consists of different kinds of sediment derived from the disintegration of pre-existing rocks. In this “fragmental” group are placed all the varieties of shingle, gravel, sand, clay and mud, whether these materials remain in a loose incoherent condition, or have been compacted into solid stone. (*b*) Another group consists of materials that have been deposited by chemical precipitation from solution in water. The white sinter laid down by calcareous springs is a familiar example on a small scale. Beds of rock-salt, gypsum and dolomite have, in some regions, been accumulated to a thickness of many thousand feet, by successive precipitations of the salt contained in the water of inland seas. (*c*) An abundant and highly important series of sedimentary formations has been formed from the remains of plants and animals. Such accumulations may arise either from the transport and deposit of these remains, as in the case of sheets of drift-wood, and banks of drifted sea-shells, or from the growth and decay of the organisms on the spot, as happens in peat bogs and in coral-reefs.

As the sedimentary rocks have for the most part been laid down under water, and more especially on the sea-floor, they are often spoken of as “aqueous,” in contradistinction to the igneous rocks. Some of them, however, are accumulated by the drifting action of wind upon loose materials, and are known as “aeolian” formations. Familiar instances of such wind-formed deposits are the sand-dunes along many parts of the sea coast. Much more extensive in area are the sands of the great deserts in the arid regions of the globe.

It is from the sedimentary rocks that the main portion of geological history is derived. They have been deposited one over another in successive strata from a remote period in the development of the globe down to the present time. From this arrangement they have been termed “stratified,” in contrast to the unstratified or igneous series. They have preserved memorials of the geographical revolutions which the surface of the earth has undergone; and above all, in the abundant fossils which they have enclosed, they furnish a momentous record of the various tribes of plants and animals which have successively flourished on land and sea. Their investigation is thus the most important task which devolves upon the geologist.

3. In the third place comes a series of rocks which are not now in their original condition, but have undergone such alteration as to have acquired new characters that more or less

conceal their first structures. Some of them can be readily recognized as altered igneous masses; others are as manifestly of sedimentary origin; while of many it is difficult to decide what may have been their pristine character. To this series the term "metamorphic" has been applied. Its members are specially distinguished by a prevailing fissile, or schistose, structure which they did not at first possess, and which differs from anything found in unaltered igneous or sedimentary rocks. This fissility is combined with a more or less pronounced crystalline structure. These changes are believed to be the result of movements within the crust of the earth, whereby the most solid rocks were crushed and sheared, while, at the same time, under the influence of a high temperature and the presence of water, they underwent internal chemical reactions, which led to a rearrangement and recomposition of their mineral constituents and the production of a crystalline structure (see METAMORPHISM).

Among the less altered metamorphic rocks of sedimentary origin, the successive laminae of deposit of the original sediment can be easily observed; but they are also traversed by a new set of divisional planes, along which they split across the original bedding. Together with this superinduced cleavage there have been developed in them minute hairs, scales and rudimentary crystals. Further stages of alteration are marked by the increase of micaceous scales, garnets and other minerals, especially along the planes of cleavage, until the whole rock becomes crystalline, and displays its chief component minerals in successive discontinuous folia which merge into each other, and are often crumpled and puckered. Massive igneous rocks can be observed to have undergone intense crushing and cleavage, and to have ultimately assumed a crystalline foliated character. Rocks which present this aspect are known as schists (*q.v.*). They range from the finest silky slates, or phyllites, up to the coarsest gneisses, which in hand-specimens can hardly be distinguished from granites. There is indeed every reason to believe that such gneisses were probably originally true granites, and that their foliation and recrystallization have been the result of metamorphism.

The schists are more especially to be found in the heart of mountain-chains, and in regions where the lowest and oldest parts of the earth's crust have, in the course of geological revolutions, been exposed to the light of day. They have been claimed by some writers to be part of the original or primitive surface of our globe that first consolidated on the molten nucleus. But the progress of investigation all over the world has shown that this supposition cannot be sustained. The oldest known rocks present none of the characters of molten material that has cooled and hardened in the air, like the various forms of recent lava. On the contrary, they possess many of the features characteristic of bodies of eruptive material that have been injected into the crust at some depth underground, and are now visible at the surface, owing to the removal by denudation of the rocks under which they consolidated. In their less foliated portions they can be recognized as true eruptive rocks. In many places gneisses that possess a thoroughly typical foliation have been found to pierce ancient sedimentary formations as intrusive bosses and veins.

PART IV.—DYNAMICAL GEOLOGY

This section of the science includes the investigation of those processes of change which are at present in progress upon the earth, whereby modifications are made on the structure and composition of the crust, on the relations between the interior and the surface, as shown by volcanoes, earthquakes and other terrestrial disturbances, on the distribution of oceans and continents, on the outlines of the land, on the form and depth of the sea-bottom, on climate, and on the races of plants and animals by which the earth is tenanted. It brings before us, in short, the whole range of activities which it is the province of geology to study, and leads us to precise notions regarding their relations to each other and the results which they achieve. A knowledge of this branch of the subject is thus the essential groundwork of a true and fruitful acquaintance with the principles of geology, seeing that it necessitates a study of the present order of nature, and thus provides a key for the interpretation of the past.

The whole range of operations included within the scope of inquiry in this branch of the science may be regarded as a vast cycle of change, into which we may break at any point, and round which we may travel, only to find ourselves brought back to our starting-point. It is a matter of comparatively small moment at what part of the cycle we begin our inquiries. We shall always find that the changes we see in action have resulted from some that preceded, and give place to others which follow them.

At an early time in the earth's history, anterior to any of the periods of which a record remains in the visible rocks, the chief sources of geological action probably lay within the earth itself. If, as is generally supposed, the planet still retained a great store of its initial heat, it was doubtless the theatre of great chemical changes, giving rise, perhaps, to manifestations of volcanic energy somewhat like those which have so marvellously roughened the surface of the moon. As the outer layers of the globe cooled, and the disturbances due to internal heat and chemical action became less marked, the conditions would arise in which the materials for geological history were accumulated. The influence of the sun, which must always have operated, would then stand out more clearly, giving rise to that wide circle of superficial changes wherein variations of temperature and the circulation of air and water over the surface of the earth come into play.

In the pursuit of his inquiries into the past history and into the present *régime* of the earth, the geologist must needs keep his mind ever open to the reception of evidence for kinds and especially for degrees of action which he had not before imagined. Human experience has been too short to allow him to assume that all the causes and modes of geological change have been definitively ascertained. On the earth itself there may remain for future discovery evidence of former operations by heat, magnetism, chemical change or otherwise, which may explain many of the phenomena with which geology has to deal. Of the influences, so many and profound, which the sun exerts upon our planet, we can as yet only perceive a little. Nor can we tell what other cosmical influences may have lent their aid in the evolution of geological changes.

Much useful information regarding many geological processes has been obtained from experimental research in laboratories and elsewhere, and much more may be confidently looked for from future extensions of this method of inquiry. The early experiments of Sir 657
James Hall, already noticed, formed the starting-point for numerous subsequent researches, which have elucidated many points in the origin and history of rocks. It is true that we cannot hope to imitate those operations of nature which demand enormous pressures and excessively high temperatures combined with a long lapse of time. But experience has shown that in regard to a large number of processes, it is possible to imitate nature’s working with sufficient accuracy to enable us to understand them, and so to modify and control the results as to obtain a satisfactory solution of some geological problems.

In the present state of our knowledge, all the geological energy upon and within the earth must ultimately be traced back to the primeval energy of the parent nebula or sun. There is, however, a certain propriety and convenience in distinguishing between that part of it which is due to the survival of some of the original energy of the planet and that part which arises from the present supply of energy received day by day from the sun. In the former case we have to deal with the interior of the earth, and its reaction upon the surface; in the latter, we deal with the surface of the earth and to some extent with its reaction on the interior. This distinction allows of a broad treatment of the subject under two divisions:

I. Hypogene or Plutonic Action: The changes within the earth caused by internal heat, mechanical movement and chemical rearrangements.

II. Epigene or Surface Action: The changes produced on the superficial parts of the earth, chiefly by the circulation of air and water set in motion by the sun’s heat.

DIVISION I.—HYPOGENE OR PLUTONIC ACTION

In the discussion of this branch of the subject we must carry in our minds the conception of a globe still possessing a high internal temperature, radiating heat into space and consequently contracting in bulk. Portions of molten rocks from inside are from time to time poured out at the surface. Sudden shocks are generated by which destructive earthquakes are propagated through the diameter of the globe as well as to and along its surface. Wide geographical areas are pushed up or sink down. In the midst of these movements remarkable changes are produced upon the rocks of the crust; they are plicated, fractured, crushed, rendered crystalline and even fused.

(A) *Volcanoes and Volcanic Action.*

This subject is discussed in the article Volcano, and only a general view of its main features will be given here. Under the term volcanic action (vulcanism, vulcanicity) are embraced all the phenomena connected with the expulsion of heated materials from the interior of the earth to the surface. A volcano may be defined as a conical hill or mountain, built up wholly or mainly of materials which have been ejected from below, and which have accumulated around the central vent of eruption. As a rule its truncated summit presents a cup-shaped cavity, termed the crater, at the bottom of which is the opening of the main funnel or pipe whereby communication is maintained with the heated interior. From time to time, however, in large volcanoes rents are formed on the sides of the cone, whence steam and other hot vapours and also streams of molten lava are poured forth. On such rents smaller or parasitic cones are often formed, which imitate the operations of the parent cone and, after repeated eruptions, may rise to hills hundreds of feet in height. In course of centuries the result of the constant outpouring of volcanic materials may be to build up a large mountain like Etna, which towers above the sea to a height of 10,840 feet, and has some 200 minor cones along its flanks.

But all volcanic eruptions do not proceed from central orifices. In Iceland it has been observed that, from fissures opened in the ground and extending for long distances, molten material has issued in such abundance as to be spread over the surrounding country for many miles, while along the lines of fissure small cones or hillocks of fragmentary material have accumulated round more active parts of the rent. There is reason to believe that in the geological past this fissure-type of eruption has repeatedly been developed, as well as the more common form of central cones like Vesuvius or Etna.

In the operations of existing volcanoes only the superficial manifestations of volcanic action are observable. But when the rocks of the earth's crust are studied, they are found to enclose the relics of former volcanic eruptions. The roots of ancient volcanoes have thus been laid bare by geological revolutions; and some of the subterranean phases of volcanic action are thereby revealed which are wholly concealed in an active volcano. Hence to obtain as complete a conception as possible of the nature and history of volcanic action, regard must be had, not merely to modern volcanoes, but to the records of ancient eruptions which have been preserved within the crust.

The substances discharged from volcanic vents consist of—(1) Gases and vapours: which, dissolved in the molten magma of the interior, take the chief share in volcanic activity. They include in greatest abundance water-gas, which condenses into the clouds of steam so conspicuous in volcanic eruptions. Hydrochloric acid and sulphuretted hydrogen are likewise plentiful, together with many other substances which, sublimed by the high internal temperature, take a solid form on cooling at the surface. (2) Molten rock or lava: which ranges from the extremely acid type of the obsidians and rhyolites with 70% or more of silica, to the more basic and heavy varieties such as basalts and leucite-lavas with much iron, and sometimes no more than 45% of silica. The specific gravity of lavas varies between 2.37 and 3.22, and the texture ranges from nearly pure glass, like obsidian, to a coarse granitoid compound, as in some rhyolites. (3) Fragmentary materials, which are sometimes discharged in enormous quantity and dispersed over a wide extent of country, the finer particles being transported by upper air-currents for hundreds of miles. These materials arise either from the explosion of lava by the sudden expansion of the dissolved vapours and gases, as the molten rock rises to the surface, or from the breaking up and expulsion of portions of the walls of the vent, or of the lava, which happens to have solidified within these walls. They vary from the finest impalpable dust and ashes, through increasing stages of coarseness up to huge "bombs" torn from the upper surface of the molten rock in the vent, and large blocks of already solidified lava, or of non-volcanic rock detached from the sides of the pipe up which the eruptions take place.

Nothing is yet known as to the determining cause of any particular volcanic eruption. Some vents, like that of Stromboli, in the Mediterranean, are continually active, and have been so ever since man has observed them. Others again have been only intermittently in eruption, with intervals of centuries between their periods of activity. We are equally in the dark as to what has determined the sites on which volcanic action has manifested itself. There is reason, indeed, to believe that extensive fractures of the terrestrial crust have often provided passages up which the vapours, imprisoned in the internal magma, have been able to make their way, accompanied by other products. Where chains of volcanoes rise along definite lines, like those of Sumatra, Java, and many other tracts both in the Old and the New World, there appears to be little doubt that their linear distribution should be attributed to this cause. But where a volcano has appeared by itself, in a region previously exempt from volcanic action, the existence of a contributing fissure cannot be so confidently presumed. The study of certain ancient volcanoes, the roots of which have been exposed by long denudation, has shown an absence of any visible trace of their having availed themselves of fractures in the crust. The inference has been drawn that volcanic energy is capable of itself drilling an orifice through the crust, probably at some weaker part, and ejecting its products at the surface. The source of this energy is to be sought in the enormous expansive force of the vapours and gases dissolved in the magma. They are kept in solution by the enormous pressure within the earth; but as the lava approaches the surface and this pressure is relieved these dissolved vapours and gases rush out with explosive violence, blowing the upper part of the lava column into dust, and allowing portions of the liquid mass below to rise and escape, either from the crater or from some fissure which the vigour of explosion has opened on the side of the cone. So gigantic is the energy of these pent-up vapours, that, after a long period of volcanic quiescence, they sometimes burst forth with such violence as to blow off the whole of the upper part or even one side of a large cone. The history of Vesuvius, and the great eruptions of Krakatoa in 1883 and of Bandaizan in 1888 furnish memorable examples of great volcanic convulsions. It has been observed that such stupendous discharges of aeriform and fragmentary matter may be attended with the emission of little or no lava. On the other hand, some of the largest outflows of lava have been accompanied by comparatively little fragmentary material. Thus, the great lava-floods of Iceland in 1783 spread for 40 m. away from their parent fissure, which was marked only by a line of little cones of slag.

The temperature of lava as it issues from underground has been measured more or less satisfactorily, and affords an indication of that existing within the earth. At Vesuvius it has been ascertained to be more than 2000° Fahr. At first the molten rock glows with a white light, which rapidly reddens, and disappears under the rugged brown and black crust that forms on the surface. Underneath this badly conducting crust, the lava cools so slowly that columns of steam have been noticed rising from its surface more than 80 years after its eruption.

Considerable alteration in the topography of volcanic regions may be produced by successive eruptions. The fragmentary materials are sometimes discharged in such abundance as to cover the ground for many miles around with a deposit of loose ashes, cinders and slag. Such a deposit accumulating to a depth of many feet may completely bury 658
valleys and water-courses, and thus greatly affect the drainage. The coarsest materials accumulate nearest to the vent that emits them. The finer dust is not infrequently hurled forth with such an impetus as to be carried for thousands of feet into the tracks of upper air-currents, whereby it may be borne for hundreds of miles away from the vent so as ultimately to fall to the ground in countries far removed from any active volcano. Outflows of lava, from their greater solidity and durability, produce still more serious and lasting changes in the external features of the ground over which they flow. As they naturally seek the lowest levels, they find their way into the channels of streams. If they keep along the channels, they seal them up under a mass of compact stone which the running water, if not wholly diverted elsewhere, will take many long centuries to cut through. If, on the other hand, the lava crosses a stream, it forms a massive dam, above which the water is ponded back so as to form a lake.

As the result of prolonged activity a volcanic cone is gradually built up by successive outflows of lava and showers of dust and stones. These materials are arranged in beds, or sheets, inclined outwards from the central vent. On surrounding level ground the alternating beds are flat. In course of time, deep gullies are cut on the outer slopes of the cone by rain, and by the heavy showers that arise from the condensation of the copious discharges of steam during eruptions. Along the sides of these ravines instructive sections may be studied of the volcanic strata. The larger rivers of some volcanic regions have likewise eroded vast gorges in the more horizontal lavas and ashes of the flatter country, and have thus laid bare stupendous cliffs, along which the successive volcanic sheets can be seen piled above each other for many hundred feet. On a small scale, some of these features are well displayed among the rivers that drain the volcanic tracts of central France; on a great scale, they are presented in the course of the Snake river, and other streams that traverse the great volcanic country of western North America. Similar volcanic scenery has been produced in western Europe by the action of denudation in dissecting the flat Tertiary lavas of Scotland, the Faeroe Isles and Iceland.

Of special interest to the geologist are those volcanoes which have taken their rise on the sea-bottom; for the volcanic intercalations among the stratified formations of the earth’s crust are almost entirely of submarine origin. Many active volcanoes situated on islands have begun their eruptions below sea-level. Both Vesuvius and Etna sprang up on the floor of the Mediterranean sea, and have gradually built up their cones into conspicuous parts of the dry land. Examples of a similar history are to be found among the volcanic islands of the Pacific Ocean. In some of these cases a movement of elevation has carried the submarine lavas, tuffs and agglomerates above sea-level, and has furnished opportunities of comparing these materials with those of recent subaerial origin, and also with the ancient records of submarine eruptions which have been preserved among the stratified formations. From the evidence thus supplied, it can be shown that the materials ejected from modern submarine volcanic vents closely resemble those accumulated by subaerial volcanoes; that the dust, ashes and stones become intermingled or interstratified with coral-mud, or other non-volcanic deposit of the sea-bottom, that vesicular lavas may be intercalated among them as on land, and that between the successive sheets of volcanic origin, layers of limestone may be laid down which are composed chiefly, or wholly, of the remains of calcareous marine organisms.

Though active volcanoes are widely distributed over the globe, and are especially abundant around the vast basin of the Pacific Ocean, they afford an incomplete picture of the extent to which volcanic action has displayed itself on the surface of our planet. When the rocks of the land are attentively studied they disclose proofs of that action in many districts where there is now no outward sign of it. Not only so, but they reveal that volcanoes have been in eruption in some of these districts during many different periods of the past, back to the beginnings of geological history. The British Islands furnish a remarkable example of such a series of ancient eruptions. From the Cambrian period all through Palaeozoic times there rose at intervals in that country a succession of volcanic centres from some of which thousands of feet of lavas and tuffs were discharged. Again in older Tertiary times the same region witnessed a stupendous outpouring of basalt, the surviving relics of which are more than 3000 ft. thick, and cover many hundreds of square miles. Similar evidence is supplied in other countries both in the Old and the New world. Hence it is proved that, in the geological past, volcanic action has been vigorous at long intervals on the same sites during a vast series of ages, though no active vents are to be seen there now. The volcanoes now active form but a small proportion of the total number which has appeared on the surface of the earth.

With regard to the cause of volcanic action much has been speculated, but little can be confidently affirmed. That water in the form of occluded gas plays the chief part in forcing the lava column up a volcanic chimney, and in the violent explosions that accompany the rise of the molten material, is generally admitted. But opinions differ as to the source of this water. According to some investigators, it should be regarded as in large measure of meteoric origin, derived from the descent of rain into the earth, and its absorption by the molten magma in the interior. Others, contending that the supply so furnished, even if it could reach and be dissolved in the magma, would yet be insufficient to furnish the prodigious quantity of aqueous vapour discharged during an eruption, maintain that the water belongs to the magma itself. They point to the admitted fact that many substances, particularly metals in a state of fusion, can absorb large quantities of vapours and gases without chemical combination, and on cooling discharge them with eruptive phenomena somewhat like those of volcanoes. This question must be regarded as one of the still unsolved problems of geology.

(B) *Movements of the Earth’s Crust.*

Among the hypogene forces in geological dynamics an important place must be assigned to movements of the terrestrial crust. Though the expression “the solid earth” has become proverbial, it appears singularly inappropriate in the light of the results obtained in recent years by the use of delicate instruments of observation. With the facilities supplied by these instruments (see Seismometer), it has been ascertained that the ground beneath our feet is subject to continual slight tremors, and feeble pulsations of longer duration, some of which may be due to daily or seasonal variations of temperature, atmospheric pressure or other meteorological causes. The establishment of self-recording seismometers all over the world has led to the detection of many otherwise imperceptible shocks, over and above the appreciable earth-waves propagated from earthquake centres of disturbance. Moreover, it has been ascertained that some parts of the surface of the land are slowly rising, while others are falling with reference to the sea-level. From time to time the surface suffers calamitous devastation from earthquakes, when portions of the crust under great strain suddenly give way. Lastly, at intervals, probably separated from each other by vast periods of time, the terrestrial crust undergoes intense plication and fracture, and is consequently ridged up into mountain-chains. No event of this kind has been witnessed since man began to record his experiences. But from the structure of mountains, as laid open by prolonged denudation, it is possible to form a vivid conception of the nature and effects of these most stupendous of all geological revolutions.

In considering this department of geological inquiry it will be convenient to treat it under the following heads: (1) Slow depression and upheaval; (2) Earthquakes; (3) Mountain-making; (4) Metamorphism of rocks.

1. *Slow Depression and Upheaval.*—On the west side of Japan the land is believed to be sinking below the sea, for fields are replaced by beaches of sand or shingle, while the depth of the sea off shore has perceptibly increased. A subsidence of the south of Sweden has taken place in comparatively recent times, for streets and foundations of houses at successive levels are found below high-water mark. The west coast of Greenland over an extent of more than 600 m. is sinking, and old settlements are now submerged. Proofs of submergence of land are furnished by “submerged forests,” and beds of terrestrial peat now lying at various depths below the level of the sea, of which many examples have been collected along the shores of the British Isles, Holland and France. Interesting evidence that the west of Europe now stands at a lower level than it did at a late geological period is

supplied in the charts of the North Sea and Atlantic, which show that the valleys of the land are prolonged under the sea. These valleys have been eroded out of the rocks by the streams which flow in them, and the depth of their submerged portions below the sea level affords an indication of the extent of the subsidence.

The uprise of land has been detected in various parts of the world. One of the most celebrated instances is that of the shores of the Gulf of Bothnia, where, at Stockholm, the elevation, between the years 1774 and 1875, appears to have been 48 centimetres (18½ in.) in a century. But on the west side of Sweden, fronting the Skager Rak, the coast, between the years 1820 and 1870, rose 30 centimetres, which is at the rate of 60 centimetres, or nearly 2 ft. in a century. In the region of the Great Lakes in the interior of Canada and the United States it has been ascertained that the land is undergoing a slow tilt towards the south-west, of which the mean rate appears to be rather less than 6 in. in a century. If this rate of change should continue the waters of Lake Michigan, owing to the progress of the tilt, will, in some 500 or 600 years, submerge the city of Chicago, and eventually the drainage of the lakes will be diverted into the basin of the Mississippi. Proof of recent emergence of land is supplied by what are called “raised beaches” or “strand-lines,” that is, lines of former shores marked by sheets of littoral deposits, or platforms cut by shore-waves in rock and flanked by old sea-cliffs and lines of sea-worn caves. Admirable examples of these features are to be seen along the west coast of Europe from the south of England to the north of Norway. These lines of old shores become fainter in proportion to their antiquity. In Britain they occur at various heights, the platforms at 25, 50 and 100 ft. being well marked.

The cause of these slow upward and downward movements of the crust of the earth is still imperfectly understood. Upheaval might conceivably be produced by an ascent of the internal magma, and the consequent expansion of the overlying crust by heat; while depression might follow any subsidence of the magma, or its displacement to another 659
district. If, as is generally believed, the globe is still contracting, the shrinkage of the surface may cause both these movements. Subsidence will be in excess, but between subsiding tracts lateral thrust may suffice to push upward intervening more solid and stable ground; but no solution of the problem yet proposed is wholly satisfactory.

2. *Earthquakes.*—As this subject is discussed in a separate article it will be sufficient here to take note of its more important geological bearings. It was for many centuries taken for granted that earthquakes and volcanoes are due to a common cause. We have seen that in classical antiquity they were looked on as the results of the movements of wind imprisoned within the earth. Long after this notion was discarded, and a more scientific appreciation of volcanic action was reached, it was still thought that earthquakes should be regarded as manifestations of the same source of energy as that which displays itself in volcanic eruptions. It is true that earthquakes are frequent in districts of active volcanoes, and they may undoubtedly be often due there to the explosions of the magma, or to the rupture of rocks caused by its ascent towards the surface. But such shocks are comparatively local in their range and feeble in their effects. There is now a general agreement that between the great world-shaking earthquakes and volcanic phenomena, no immediate and intimate relationship can be traced, though they may be connected in ways which are not yet perceived. Some of the more recent great earthquakes on land have proved that the waves of shock are produced by the sudden rupture or collapse of rocks under great strain, either along lines of previous fracture or of new rents in the terrestrial crust; and that such ruptures may occur at a remote distance from any volcano. Thus the recent disastrous San Francisco earthquake has been recognized to have resulted from a slipping of ground along the line of an old fault, which has been traced for a long distance in California generally parallel to the coast. The position of this fault at the surface has long been clearly followed by its characteristic topography. After the earthquake these superficial features were found to have been removed by the same cause that had originated them. For some 300 m. on the track of this old fault-line a renewed slipping was seen to have taken place along one or both sides, and the ground at the surface was ruptured as well as displaced horizontally. Obviously, the jar occasioned by the sudden and simultaneous subsidence of a portion of the earth’s crust several hundred miles long, must be far more serious than could be produced by an earthquake radiating from a single local volcanic focus.

From their disastrous effects on buildings and human lives, an exaggerated importance has been imputed to earthquakes as agents of geological change. Experience shows that even after a severe shock which may have destroyed numerous towns and villages, together with thousands of their inhabitants, the face of the country has suffered scarcely any perceptible change, and that, in the course of a year or two, when the ruined houses and prostrate trees have been cleared away, little or no obvious trace of the catastrophe may remain. Among the more enduring records of a great earthquake may be enumerated (*a*) landslips, which lay bare hillsides, and sometimes pond back the drainage of valleys so as to give rise to lakes; (*b*) alterations of the topography, as in fissuring of the ground, or in the production of inequalities whereby the drainage is affected; new valleys and new lakes may thus be formed, while previously existing lakes may be emptied; (*c*) permanent changes of level, either in an upward or downward direction.

3. *Mountain-making.*—This subject may be referred to here for the striking evidence which it supplies of the importance of movements of the earth's crust among geological processes. The structure of a great mountain-chain such as the Alps proves that the crust of the earth has been intensely plicated, crumpled and fractured. Vast piles of sedimentary strata have been folded to such an extent as to occupy now only half of their original horizontal extent. This compression in the case of the Alps has been computed to amount to as much as 120,000 metres or 74 English miles, so that two points on the opposite sides of that chain have been brought by so much nearer to each other than they were originally before the movements. Besides such intense plication, extensive rupturing of the crust has taken place in the same range of mountains. Not only have the most ancient rocks been squeezed up into the central axis of the chain, but huge slices of them have been torn away from the main body, and thrust forward for many miles, so as now actually to form the summits of mountains, which are almost entirely composed of much younger formations. If these colossal disturbances occurred rapidly, they would give rise to cataclysms of inconceivable magnitude over the surface of the globe. No record has been discovered of such accompanying devastation. But whether sudden and violent, or prolonged and gradual, such stupendous upturnings of the crust did undoubtedly take place, as is clearly revealed in innumerable natural sections, which have been laid open by the denudation of the crests and sides of the mountains.

4. *Metamorphism of Rocks* (see Metamorphism).—During the movements to which the crust of the earth has been subject, not only have the rocks been folded and fractured, but they have likewise, in many regions, acquired new internal structures, and have thus undergone a process of "regional metamorphism." This rearrangement of their substance has been governed by conditions which are probably not yet all recognized, but among them we should doubtless include a high temperature, intense pressure, mechanical movement resulting in crushing, shearing and foliation, and the presence of water in their pores. It is among igneous rocks that the progressive stages of metamorphism can be most easily traced. Their definite original structure and mineral composition afford a starting-point from which the investigation may be begun and pursued. Where an igneous rock has been invaded by metamorphic changes, it may be observed to have been first broken down into separate lenticles, the cores of which may still retain, with little or no alteration, the original characteristic minerals and crystalline structure of the rock. Between these lenticles, the intervening portions have been crushed down into a powder or paste, which seems to have been squeezed round and past them, and shows a laminated arrangement that resembles the flow-structure in lavas. As the degree of metamorphism increases, the lenticles diminish in size, and the intervening crushed and foliated matrix increases in amount, until at last it may form the entire mass of the rock. While the original minerals are thus broken down, new varieties make their appearance. Of these, among the earliest to present themselves are usually the micas, that impart their characteristic silvery sheen to the surfaces of the folia along which they spread. Younger felspars, as well as mica, are developed, and there arise also sillimanite, garnet, andalusite and many others. The texture becomes more coarsely crystalline, and the segregation of the constituent minerals more definite along the lines of foliation. From the finest silky phyllites a graduation may be traced through successively coarser mica-schists, until we reach the almost granitic texture of the coarsest gneisses.

Regional metamorphism has arisen in the heart of mountain-chains, and in any other district where the deformation of the crust has been sufficiently intense. There is another type of alteration termed "contact-metamorphism," which is developed around masses of igneous rock, especially where these have been intruded in large bosses among stratified formations. It is particularly displayed around masses of granite, where sandstones are found altered into quartzite, shales and grits into schistose compounds, and where sometimes fossils are still recognizable among the metamorphic minerals.

DIVISION II.—EPIGENE OR SUPERFICIAL ACTION

It is on the surface of the globe, and by the operation of agents working there, that at present the chief amount of visible geological change is effected. In considering this branch of inquiry, we are not involved in a preliminary difficulty regarding the very nature of the agencies as is the case in the investigation of plutonic action. On the contrary, the surface agents are carrying on their work under our very eyes. We can watch it in all its stages, measure its progress, and mark in many ways how accurately it represents similar changes which, for long ages previously, must have been effected by the same means. But in the systematic treatment of this subject we encounter a difficulty of another kind. We discover that while the operations to be discussed are numerous and readily observable, they are so interwoven into one great network that any separation of them under different subdivisions is sure to be more or less artificial and to convey an erroneous impression. While, therefore, under the unavoidable necessity of making use of such a classification of subjects, we must always bear in mind that it is employed merely for convenience, and that in nature superficial geological action must be continually viewed as a whole, since the work of each agent has constant reference to that of the others, and is not properly intelligible unless that connexion be kept in view.

The movements of the air; the evaporation from land and sea; the fall of rain, hail and snow; the flow of rivers and glaciers; the tides, currents and waves of the ocean; the growth and decay of organized existence, alike on land and in the depths of the sea;—in short, the whole circle of movement, which is continually in progress upon the surface of our planet, are the subjects now to be examined. It is desirable to adopt some general term to embrace the whole of this range of inquiry. For this end the word epigene (Gr. ἐπί, upon) has been suggested as a convenient term, and antithetical to hypogene (Gr. ὑπό, under), or subterranean action.

A simple arrangement of this part of Geological Dynamics is in three sections:

A. *Air.*—The influence of the atmosphere in destroying and forming rocks.

B. *Water.*—The geological functions of the circulation of water through the air and between sea and land, and the action of the sea.

C. *Life.*—The part taken by plants and animals in preserving, destroying or reproducing geological formations. 660

The words destructive, reproductive and conservative, employed in describing the operations of the epigene agents, do not necessarily imply that anything useful to man is destroyed, reproduced or preserved. On the contrary, the destructive action of the atmosphere may turn barren rock into rich soil, while its reproductive effects sometimes turn rich land into barren desert. Again, the conservative influence of vegetation has sometimes for centuries retained as barren morass what might otherwise have become rich meadow or luxuriant woodland. The terms, therefore, are used in a strictly geological sense, to denote the removal and re-deposition of material, and its agency in preserving what lies beneath it.

(A) *The Air.*

As a geological agent, the air brings about changes partly by its component gases and partly by its movements. Its destructive action is both chemical and mechanical. The chemical changes are probably mainly, if not entirely, due to the moisture of the air, and particularly to the gases, vapours and organic matter which the moisture contains. Dry air seems to have little or no appreciable influence in promoting these reactions. As the changes in question are similar to those much more abundantly brought about by rain they are described in the following section under the division on rain.

Among the more recognizable mechanical changes effected in the atmosphere, one of considerable importance is to be seen in the result of great and rapid changes of temperature. Heat expands rocks, while cold contracts them. In countries with a great annual range of temperature, considerable difficulty is sometimes experienced in selecting building materials liable to be little affected by the alternate expansion and contraction, which prevents the joints of masonry from remaining close and tight. In dry tropical climates, where the days are intensely hot and the nights extremely cold, the rapid nocturnal contraction produces a strain so great as to rival frost in its influence upon the surface of exposed rocks, disintegrating them into sand, or causing them to crack or peel off in skins or irregular pieces. Dr Livingstone found in Africa (12° S. lat., 34° E. long.) that surfaces of rock which during the day were heated up to 137° Fahr., cooled so rapidly by radiation at night that, unable to sustain the strain of contraction, they split and threw off sharp angular fragments from a few ounces to 100 or 200 ℔ in weight. In temperate regions this action, though much less pronounced, still makes itself felt. In these climates, however, and still more in high latitudes, somewhat similar results are brought about by frost.

By its motion in wind the air drives loose sand over rocks, and in course of time abrades and smoothes them. “Desert polish” is the name given to the characteristic lustrous surface thus imparted. Holes are said to be drilled in window glass at Cape Cod by the same agency. Cavities are now and then hollowed out of rocks by the gyration in them of little fragments of stone or grains of sand kept in motion by the wind. Hurricanes form important geological agents upon land in uprooting trees, and thus sometimes impeding the drainage of a country and giving rise to the formation of peat mosses.

The reproductive action of the air arises partly from the effect of the chemical and mechanical disintegration involved in the process of “weathering,” and partly from the transporting power of wind and of aerial currents. The layer of soil, which covers so much of the surface of the land, is the result of the decay of the underlying rocks, mingled with mineral matter blown over the ground by wind, or washed thither by rain, and with the mouldering remains of plants and animals. The extent to which fine dust may be transported over the surface of the land can hardly be realized in countries clothed with a covering of vegetation, though even there, in dry weather during spring, clouds of dust may often be seen blown away by wind from bare ploughed fields. Intercepted by the leaves of plants and washed down to their roots by rain, this dust goes to increase the soil below. In arid climates, where dust clouds are dense and frequent, enormous quantities of fine mineral particles are thus borne along and accumulated. The remarkable deposit of “Loess,” which is sometimes more than 1500 ft. thick and covers extensive areas in China and other countries, is regarded as due to the drifting of dust by wind. Again the dunes of sand so abundant along the inner side of sandy sea-beaches in many different parts of the world are attributable to the same action.

(B) *Water.*

In treating of the epigene action of water in geological processes it will be convenient to deal first with its operations in traversing the land, and then with those which it performs in the sea. The circulation of water from land to sea and again from sea to land constitutes the fundamental cause of most of the daily changes by which the surface of the land is affected.

1. *Rain.*—Rain effects two kinds of changes upon the surface of the land. It acts *chemically* upon soils and stones, and sinking under ground continues a great series of similar reactions there. It acts *mechanically*, by washing away loose materials, and thus powerfully affecting the contours of the land. Its chemical action depends mainly upon the nature and proportion of the substances which, in descending to the earth, it abstracts from the atmosphere. Rain always absorbs a little air, which, in addition to its nitrogen and oxygen, contains carbonic acid, and in minute proportions, sodium chloride, sulphuric acid and other ingredients, especially inorganic dust, organic particles and living germs. Probably the most generally efficient of these constituents are oxygen, carbonic acid and organic matter. Armed with these reagents, rain effects a chemical decomposition of the rocks on which it falls, and through which it sinks underground. The principal changes thus produced are as follows: (*a*) Oxidation.—Owing to the prominence of oxygen in rain-water, and its readiness to unite with any substance which can contain more of it, a thin oxidized pellicle is formed on the surface of many rocks on which rain falls, and this oxidized layer if not at once washed off, sinks deeper until a crust is formed over the stone. A familiar illustration of this action is afforded by the rust, or oxide, which forms on iron when exposed to moisture, though this iron may be kept long bright if allowed to remain screened from moist air and rain. (*b*) Deoxidation.—Organic matter having an affinity for more oxygen decomposes peroxides by depriving them of some part of their share of that element and reducing them to protoxides. These changes are especially noticeable among the iron oxides so abundantly diffused among rocks. Hence rain-water, in sinking through soil and obtaining such organic matter, becomes thereby a reducing agent. (*c*) Solution.—This may take place either by the simple action of the water, as in the solution of rock-salt, or by the influence of the carbonic acid present in the rain. (*d*) Formation of Carbonates.—A familiar example of the action of carbonic acid in rain is to be seen in the corrosion of exposed marble slabs. The carbonic acid dissolves some of the lime, which, as a bicarbonate, is held in solution in the carbonated water, but is deposited again when the water loses its carbonic acid or evaporates. It is not merely carbonates, however, which are liable to this kind of destruction. Even silicates of lime, potash and soda, combinations existing abundantly as constituents of rocks, are attacked; their silica is liberated, and their alkalis or alkaline earths, becoming carbonates, are removed in solution. (*e*) Hydration.—Some minerals, containing little or no water, and therefore called anhydrous, when exposed to the action of the atmosphere, absorb water, or become hydrous, and are then usually more prone to further change. Hence the rocks of which they form part become disintegrated.

Besides the reactions here enumerated, a considerable amount of decay may be observed as the result of the presence of sulphuric and nitric acid in the air, especially in that of large towns and manufacturing districts, where much coal is consumed. Metallic surfaces, as well as various kinds of stone, are there corroded, while the mortar of walls may often be observed to be slowly swelling out and dropping off, owing to the conversion of the lime into sulphate. Great injury is likewise done from a similar cause to marble monuments in exposed graveyards.

The general result of the disintegrating action of the air and of rain, including also that of plants and animals, to be noticed in the sequel, is denoted by the term “weathering.” The amount of decay depends partly on conditions of climate, especially the range of temperature, the abundance of moisture, height above the sea and exposure to prevalent winds. Many rocks liable to be saturated with rain and rapidly dried under a warm sun are apt to disintegrate at the surface with comparative rapidity. The nature and progress of the weathering are mainly governed by the composition and texture of the rocks exposed to it. Rocks composed of particles liable to little chemical change from the influence of moisture are best fitted to resist weathering, provided they possess sufficient cohesion to withstand the mechanical processes of disintegration. Siliceous sandstones are excellent examples of this permanence. Consisting wholly or mainly of the durable mineral quartz, they are sometimes able so to withstand decay that buildings made of them still retain, after the lapse of centuries, the chisel-marks of the builders. Some rocks, which yield with comparative rapidity to the chemical attacks of moisture, may show little or no mark of disintegration on their surface. This is particularly the case with certain calcareous rocks. Limestone when pure is wholly soluble in acidulated water. Rain falling on such a rock removes some of it in solution, and will continue to do so until the whole is dissolved away. But where a limestone is full of impurities, a weathered crust of more or less insoluble particles remains after the solution of the calcareous part of the stone. Hence the relative purity of limestones may be roughly determined by examining their weathered surfaces, where, if they contain much sand, the grains will be seen projecting from the calcareous matrix, and where, should the rock be very ferruginous, the yellow hydrous peroxide, or ochre, will be found as a powdery crust. In limestones containing abundant encrinites, shells, or other organic remains, the weathered surface commonly presents the fossils standing out in relief. The crystalline arrangement of the lime in the organic structures enables them to resist disintegration better than the general mechanically aggregated matrix of the rock. An experienced fossil collector will always search well such weathered surfaces, for he often finds there, delicately picked out by the weather, minute and frail fossils which are wholly invisible on a freshly 661
broken surface of the stone. Many rocks weather with a thick crust, or even decay inwards for many feet or yards. Basalt, for example, often shows a yellowish-brown ferruginous layer on its surface, formed by the conversion of its felspar into kaolin, and the removal of its calcium silicate as carbonate, by the hydration of its olivine and augite and their conversion into serpentine, or some other hydrous magnesian silicate, and by the conversion of its magnetite into limonite. Granite sometimes shows in a most remarkable way the distance to which weathering can reach. It may occasionally be dug into for a depth of 20 or 30 ft., the quartz crystals and veins retaining their original positions, while the felspar is completely kaolinized. It is to the endlessly varied effects of weathering that the abundant fantastic shapes assumed by crags and other rocky masses are due. Most varieties of rock have their own characteristic modes of weathering, whereby they may be recognized even from a distance. To some of these features reference will be made in Part VIII.

The mechanical action of rain, which is intimately bound up with its chemical action, consists in washing off the fine superficial particles of rocks which have been corroded and loosened by the process of weathering, and in thus laying open fresh portions to the same influences of decay. The detritus so removed is partly carried down into the soil which is thereby enriched, partly held in suspension in the little runnels into which the rain-drops gather as they begin to flow over the land, partly pushed downwards along the surface of sloping ground. A good deal of it finds its way into the nearest brooks and rivers, which are consequently made muddy by heavy rain.

It is natural that a casual consideration of the subject should lead to an impression that, though the general result of the fall of rain upon a land-surface must lead to some amount of disintegration and lowering of that surface, the process must be so slow and slight as hardly to be considered of much importance among geological operations. But further attention will show such an impression to be singularly erroneous. It loses sight of the fact that a change which may be hardly appreciable within a human lifetime, or even within the comparatively brief span of geological time embraced in the compass of human history, may nevertheless become gigantic in its results in the course of immensely protracted periods. An instructive lesson in the erosive action of rain may be found in the pitted and channelled surface of ground lying under the drip of the eaves of a cottage. The fragments of stone and pebbles of gravel that form part of the soil can there be seen sticking out of the ground, because being hard they resist the impetus of the falling drops, protecting for a time the earth beneath them, while that which surrounded and covered them is washed away. From this familiar illustration the observer may advance through every stage in the disappearance of material which once covered the surface, until he comes to examples where once continuous and thick sheets of solid rock have been reduced to a few fragments or have been entirely removed. Since the whole land surface over which rain falls is exposed to this waste, the superficial covering of decayed rock or soil, as Hutton insisted, is constantly, though imperceptibly, travelling outward and downward to the sea. In this process of transport rain is an important carrying agent, while at the same time it serves to connect the work of the other disintegrating forces, and to make it conducive to the general degradation of the land. Though this decay is general and constant, it is obviously not uniform. In some places where, from the nature of the rock, from the flatness of the ground, or from other causes, rain works under great difficulties, the rate of waste may be extremely slow. In other places it may be rapid enough to be appreciable from year to year. A survey of this department of geological activity shows how unequal wasting by rain, combined with the operations of brooks and rivers, has produced the details of the present relief of the land, those tracts where the destruction has been greatest forming hollows and valleys, others, where it has been less, rising into ridges and hills (Part VIII.).

Rain-action is not merely destructive, but is accompanied with reproductive effects, chief of which is the formation of soil. In favourable situations it has gathered together accumulations of loam and earth from neighbouring higher ground, such as the “brick-earth,” “head,” and “rain-wash” of the south of England—earthy deposits, sometimes full of angular stones, derived from the subaerial waste of the rocks of the neighbourhood.

2. *Underground Water.*—Of the rain which falls upon the land one portion flows off into brooks and rivers by which the water is conducted back to the ocean; the larger part, however, sinks into the ground and disappears. It is this latter part which has now to be considered. Over and above the proportion of the rainfall which is absorbed by living vegetation and by the soil, there is a continual filtering down of the water from the surface into the rocks that lie below, where it partly lodges in pores and interstices, and partly finds its way into subterranean joints and fissures, in which it performs an underground circulation, and ultimately issues once more at the surface in the form of springs (*q.v.*). In the course of this circulation the water performs an important geological task. Not only carrying down with it the substances which the rain has abstracted from the air, but obtaining more acids and organic matter from the soil, it is enabled to effect chemical changes in the rocks underneath, and especially to dissolve limestone and other calcareous formations. So considerable is the extent of this solution in some places that the springs which come to the surface, and begin there to evaporate and lose some of their carbonic acid, contain more dissolved lime than they can hold. They consequently deposit it in the form of calcareous tuff or sinter (*q.v.*). Other subterranean waters issue with a large proportion of iron-salts in solution which form deposits of ochre. The various mineral springs so largely made use of for the mitigation or cure of diseases owe their properties to the various salts which they have dissolved out of rocks underground. As the result of prolonged subterranean solution in limestone districts, passages and caves (*q.v.*), sometimes of great width and length, are formed. When these lie near the surface their roofs sometimes fall in and engulf brooks and rivers, which then flow for some way underground until the tunnels conduct them back again to daylight on some lower ground.

Besides its chemical activity water exerts among subterranean rocks a mechanical influence which leads to important changes in the topography of the surface. In removing the mineral matter, either in solution or as fine sediment, it sometimes loosens the support of overlying masses of rock which may ultimately give way on sloping ground, and rush down the declivities in the form of landslips. These destructive effects are specially frequent on the sides of valleys in mountainous countries and on lines of sea-cliff.

3. *Brooks and Rivers.*—As geological agents the running waters on the face of the land play an important part in epigene changes. Like rain and springs they have both a chemical and a mechanical action. The latter receives most attention, as it undoubtedly is the more important; but the former ought not to be omitted in any survey of the general waste of the earth’s surface. The water of rivers must possess the powers of a chemical solvent like rain and springs, though its actual work in this respect can be less easily measured, seeing that river water is directly derived from rain and springs, and necessarily contains in solution mineral substances supplied to it by them and not by its own operation. Nevertheless, it is sometimes easy to prove that streams dissolve chemically the rocks of their channels. Thus, in limestone districts the base of the cliffs of river ravines may be found eaten away into tunnels, arches, and overhanging projections, presenting in their smooth surfaces a great contrast to the angular jointed faces of the same rock, where now exposed to the influence only of the weather on the higher parts of the cliff.

The mechanical action of rivers consists (*a*) in transporting mud, sand, gravel and blocks of stone from higher to lower levels; (*b*) in using these loose materials to widen and deepen their channels by erosion; (*c*) in depositing their load of detritus wherever possible and thus to make new geological formations.

(*a*) *Transporting Power.*—River-water is distinguished from that of springs by being less transparent, because it contains more or less mineral matter in suspension, derived mainly from what is washed down by rain, or carried in by brooks, but partly also from the abrasion of the water-channels by the erosive action of the rivers themselves. The progress of this burden of detritus may be instructively followed from the mountain-tributaries of a river down to the mouth of the main stream. In the high grounds the water-courses may be observed to be choked with large fragments of rock disengaged from the cliffs and crags on either side. Traced downwards the blocks are seen to become gradually smaller and more rounded. They are ground against each other, and upon the rocky sides and bottom of the channel, getting more and more reduced as they descend, and at the same time abrading the rocks over or against which they are driven. Hence a great deal of débris is produced, and is swept along by the onward and downward movement of the water. The finer portions, such as mud and fine sand, are carried in suspension, and impart the characteristic turbidity to river-water; the coarser sand and gravel are driven along the river-bottom. The proportion of suspended mineral matter has been ascertained with more or less precision for a number of rivers. As an illustrative example of a river draining a vast area with different climates, forms of surface and geological structure the Mississippi may be cited. The average proportion of sediment in its water was ascertained by Humphreys and Abbot to be $^1/_{1500}$ by weight or $^1/_{2900}$ by volume. These engineers found that, in addition to this suspended material, coarse detritus is constantly being pushed forward along the bed of the river into the Gulf of Mexico, to an amount which they estimated at about 750,000,000 cubic ft. of sand, earth and gravel; they concluded that the Mississippi carries into the gulf every year an amount of mechanically transported sediment sufficient to make a prism one square mile in area and 268 ft. in height.

(*b*) *Excavating Power.*—It is by means of the sand, gravel and stones which they drive against the sides and bottoms of their channels that streams have hollowed out the beds in which they flow. Not only is the coarse detritus reduced in size by the friction of the stones against each other, but, at the same time, these materials abrade the rocks against which they are driven by the current. Where, owing to the shape of the bottom of the channel, the stones are caught in eddies, and are kept whirling round there, they become more and more worn down themselves, and at the same time scour out basin-shaped cavities, or “pot-holes,” in the solid rock below. The uneven bed of a swiftly flowing stream may in this way be 662
honeycombed with such eroded basins which coalesce and thus appreciably lower the surface of the bed. The steeper the channel, other conditions being equal, the more rapid will be the erosion. Geological structure also affects the character and rate of the excavation. Where the rocks are so arranged as to favour the formation and persistence of a waterfall, a long chasm may be hollowed out like that of the Niagara below the falls, where a hard thick bed of nearly flat limestone lies on softer and more easily eroded shales. The latter are scooped out from underneath the limestone, which from time to time breaks off in large masses and the waterfall gradually retreats up stream, while the ravine is proportionately lengthened. To the excavating power of rivers the origin of the valley systems of the dry land must be mainly assigned (see Part VIII.).

(*c*) *Reproductive Power.*—So long as a stream flows over a steep declivity its velocity suffices to keep the sediment in suspension, but when from any cause, such as a diminution of slope, the velocity is checked, the transporting power is lessened and the sediment begins to fall to the bottom and to remain there. Hence various river-formed or “alluvial” deposits are laid down. These sometimes cover considerable spaces at the foot of mountains. The floors of valleys are strewn with detritus, and their level may thereby be sensibly raised. In floods the ground inundated on either side of a stream intercepts some part of the detritus, which is then spread over the flood-plain and gradually heightens it. At the same time the stream continues to erode the channel, and ultimately is unable to reach the old flood-plain. It consequently forms a new plain at a lower level, and thus, by degrees, it comes to be flanked on either side by a series of successive terraces or platforms, each of which marks one of its former levels. Where a river enters a large body of water its current is checked. Some of its sediment is consequently dropped, and by slow accumulation forms a delta (*q.v.*). On land, every lake in mountain districts furnishes instances of this kind of alluvium. But the most important deltas are those formed in the sea at the mouths of the larger rivers of the globe. Off many coast-lines the detritus washed from the land gathers into bars, which enclose long strips of water more or less completely separated from the sea outside and known as lagoons. A chain of such lagoon-barriers stretches for hundreds of miles round the Gulf of Mexico and the eastern shores of the United States.

4. *Lakes.*—These sheets of water, considered as a whole, do not belong to the normal system of drainage on the land whereby valleys are excavated. On the contrary they are exceptional to it; for the constant tendency of running water is to fill them up, or to drain them by wearing down the barriers that contain them at their outflow. Some of them are referable to movements of the terrestrial crust whereby depressions arise on the surface of

the land, as has been noted after earthquakes. Others have arisen from solution such as that of rock-salt or of limestone, the removal of which by underground water causes a subsidence of the ground above. A third type of lake-basin occurs in regions that are now or have once been subject to the erosive action of glaciers (see under next subdivision, *Terrestrial Ice*). Many small lakes or tarns have been caused by the deposit of débris across a valley as by landslips or moraines. Considered from a geological point of view, lakes perform an important function in regulating the drainage of the ground below their outfall and diminishing the destructive effects of floods, in filtering the water received from their affluent streams, and in providing undisturbed areas of deposit in which thick and extensive lacustrine formations may be accumulated. In the inland basins of some dry climates the lakes are salt, owing to excess of evaporation, and their bottoms become the sites of chemical deposits, particularly of chlorides of sodium and magnesium, and calcium sulphate and carbonate.

5. *Terrestrial Ice.*—Each of the forms assumed by frozen water has its own characteristic action in geological processes. Frost has a powerful influence in breaking up damp soils and surfaces of stone in the pores or cracks of which moisture has lodged. The water in freezing expands, and in so doing pushes asunder the component particles of soil or stone, or widens the space between the walls of joints or crevices. When the ice melts the loosened grains remain apart ready to be washed away by rain or blown off by wind, while by the widening of joints large blocks of rock are detached from the faces of cliffs. Where rivers or lakes are frozen over the ice exerts a marked pressure on their banks; and when it breaks up large sheets of it are driven ashore, pushing up quantities of gravel and stones above the level of the water. The piling up of the disrupted ice against obstructions in rivers ponds back the water, and often leads to destructive floods when the ice barriers break. Where the ice has formed round boulders in shallow water, or at the bottom ("anchor-ice"), it may lift these up when the frost gives way, and may transport them for some distance. Ice formed in the atmosphere, and descending to the ground in the form of hail, often causes great destruction to vegetation and not infrequently to animal life. Where the frozen moisture reaches the earth as snow, it serves to protect rock, soil and vegetation from the effects of frost; but on sloping ground it is apt to give rise to destructive avalanches or landslips, while indirectly, by its rapid melting, it may cause serious floods in rivers.

But the most striking geological work performed by terrestrial ice is that achieved by glaciers (*q.v.*) and ice-sheets. These vast masses of moving ice, when they descend from mountains where the steeper rocks are clear of snow, receive on their surface the débris detached by frost from the declivities above, and bear these materials to lower levels or to the sea. Enormous quantities of rock-rubbish are thus transported in the Alps and other high mountain ranges. When the ice retreats the boulders carried by it are dropped where it melts, and left there as memorials of the former extension of the glaciers. Evidence of this nature proves the much wider extent of the Alpine ice at a comparatively recent geological date. It can also be shown that detritus from Scandinavia has been ice-borne to the south-east of England and far into the heart of Europe.

The ice, by means of grains of sand and pieces of stone which it drags along, scores, scratches and polishes the surfaces of rock underneath it, and, in this way, produces the abundant fine sediment that gives the characteristic milky appearance to the rivers that issue from the lower ends of glaciers. By such long-continued attrition the rocks are worn down, portions of them of softer nature, or where the ice acts with especial vigour, are hollowed out into cavities which, on the disappearance of the ice, may be filled with water and become tarns or lakes. Rocks over which land-ice has passed are marked by a peculiar smooth, flowing outline, which forms a contrast to the more rugged surface produced by ordinary weathering. They are covered with groovings, which range from the finest striae left by sharp grains of sand to deep ruts ground out by blocks of stone. The trend of these markings shows the direction in which the ice flowed. By their evidence the position and movement of former glaciers in countries from which the ice has entirely vanished may be clearly determined (see Glacial Period).

6. *The Sea.*—The physical features of the sea are discussed in separate articles (see Ocean and Oceanography). The sea must be regarded as the great regulator of temperature and climate over the globe, and as thus exerting a profound influence on the distribution of plant and animal life. Its distinctly geological work is partly erosive and partly reproductive. As an eroding agent it must to some extent effect chemical decompositions in the rocks and sediments over which it spreads; but these changes have not yet been satisfactorily studied. Undoubtedly, its chief destructive power is of a mechanical kind, and arises from the action of its waves in beating upon shore-cliffs. By the alternate compression and expansion of the air in crevices of the rocks on which heavy breakers fall, and by the hydraulic pressure which these masses of sea-water exert on the walls of the fissures into which they rush, large masses of rock are loosened and detached, and caves and tunnels are drilled along the

base of sea-cliffs. Probably still more efficacious are the blows of the loose shingle, which, caught up and hurled forward by the waves, falls with great force upon the shore rocks, battering them as with a kind of artillery until they are worn away. The smooth surfaces of the rocks within reach of the waves contrasted with their angular forms above that limit bear witness to the amount of waste, while the rounded forms of the boulders and shingle show that they too are being continually reduced in size. Thus the sea, by its action on the coasts, produces much sediment, which is swept away by its waves and currents and strewn over its floor. Besides this material, it is constantly receiving the fine silt and sand carried down by rivers. As the floor of the ocean is thus the final receptacle for the waste of the land, it becomes the chief era on the surface of the globe for the accumulation of new stratified formations. And such has been one of its great functions since the beginning of geological time, as is proved by the rocks that form the visible part of the earth’s crust, and consist in great part of marine deposits. Chemical precipitates take place more especially in enclosed parts of the sea, where concentration of the water by evaporation can take place, and where layers of sodium chloride, calcium sulphate and carbonate, and other salts are laid down. But the chief marine accumulations are of detrital origin. Near the land and for a variable distance extending sometimes to 200 or 300 m. from shore the deposits consist chiefly of sediments derived from the waste of the land, the finer silts being transported farthest from their source. At greater depths and distances the ocean floor receives a slow deposit of exceedingly fine clay, which is believed to be derived from the decomposition of pumice and volcanic dust from insular or submarine volcanoes. Wide tracts of the bottom are covered with various forms of ooze derived from the accumulation of the remains of minute organisms.

(C) *Life.*

Among the agents by which geological changes are carried on upon the surface of the globe living organisms must be enumerated. Both plants and animals co-operate with the inorganic agents in promoting the degradation of the land. In some cases, on the other hand, they protect rocks from decay, while, by the accumulation of their remains, they give rise to extensive formations both upon the land and in the sea. Their operations may hence be described as alike destructive, conservative and reproductive. Under this heading also the influence of Man as a geological agent deserves notice.

(*a*) *Plants.*—Vegetation promotes the disintegration of rocks and soil in the following ways: (1) By keeping the surfaces of stone moist, and thus promoting both mechanical and chemical dissolution, as is especially shown by liverworts, mosses and other moisture-loving plants. (2) By producing through their decay carbonic and other acids, which, together with decaying organic matter taken up by passing moisture, become potent in effecting the chemical decomposition of rocks and in promoting the disintegration of soils. (3) By inserting their roots or branches between joints of rock, which are thereby loosened, so that large slices may be eventually wedged off. (4) By attracting rain, as thick woods, forests and peat-mosses do, and thus accelerating the general waste of a country by running water. (5) By promoting the decay of diseased and dead plants and animals, as when fungi overspread a damp rotting tree or the carcase of a dead animal. 663

That plants also exert a conservative influence on the surface of the land is shown in various ways. (1) The formation of a stratum of turf protects the soil and rocks underneath from being rapidly disintegrated and washed away by atmospheric action. (2) Many plants, even without forming a layer of turf, serve by their roots or branches to protect the loose sand or soil on which they grow from being removed by wind. The common sand-carex and other arenaceous plants bind the loose sand-dunes of our coasts, and give them a permanence, which would at once be destroyed were the sand laid bare again to storms. The growth of shrubs and brushwood along the course of a stream not only keeps the alluvial banks from being so easily undermined and removed as would otherwise be the case, but serves to arrest the sediment in floods, filtering the water and thereby adding to the height of the flood plain. (3) Some marine plants, like the calcareous nullipores, afford protection to shore rocks by covering them with a hard incrustation. The tangles and smaller Fuci which grow abundantly on the littoral zone break the force of the waves or diminish the effects of ground swell. (4) Forests and brushwood protect the soil, especially on slopes, from being washed away by rain or ploughed up by avalanches.

Plants contribute by the aggregation of their remains to the formation of stratified deposits. Some marine algae which secrete carbonate of lime not only encrust rocks but give rise to sheets of submarine limestone. An analogous part is played in fresh-water lakes by various lime-secreting plants, such as *Chara*. Long-continued growth of vegetation has, in some regions, produced thick accumulations of a dark loam, as in the black cotton soil (*regur*) of India, and the black earth (*tchernozom*) of Russia. Peat-mosses are formed in temperate and arctic climates by the growth of marsh-loving plants, sometimes to a thickness of 40 or 50 ft. In tropical regions the mangrove swamps on low moist shores form a dense jungle, sometimes 20 m. broad, which protects these shores from the sea until, by the arrest of sediment and the constant contribution of decayed vegetation, the spongy ground is at last turned into firm soil. Some plants (diatoms) can abstract silica and build it into their framework, so that their remains form a siliceous deposit or ooze which covers spaces of the deep sea-floor estimated at more than ten millions of square miles in extent.

(*b*) *Animals.*—These exert a destructive influence in the following ways: (1) By seriously affecting the composition and arrangement of the vegetable soil. Worms bring up the lower portions of the soil to the surface, and while thus promoting its fertility increase its liability to be washed away by rain. Burrowing animals, by throwing up the soil and subsoil, expose these to be dried and blown away by the wind. At the same time their subterranean passages serve to drain off the superficial water and to injure the stability of the surface of the ground above them. In Britain the mole and rabbit are familiar examples. (2) By interfering with or even diverting the flow of streams. Thus beaver-dams check the current of water-courses, intercept floating materials, and sometimes turn streams into new channels. The embankments of the Mississippi are sometimes weakened to such an extent by the burrowings of the cray-fish as to give way and allow the river to inundate the surrounding country. Similar results have happened in Europe from subterranean operations of rats. (3) Some mollusca bore into stone or wood and by the number of contiguous perforations greatly weaken the material. (4) Many animals exercise a ruinously destructive influence upon vegetation. Of the numerous plagues of this kind the locust, phylloxera and Colorado beetle may be cited.

The most important geological function performed by animals is the formation of new deposits out of their remains. It is chiefly by the lower grades of the animal kingdom that this work is accomplished, especially by molluscs, corals and foraminifera. Shell-banks are formed abundantly in such comparatively shallow and enclosed basins as that of the North Sea, and on a much more extensive scale on the floor of the West Indian seas. By the coral polyps thick masses of limestones have been built up in the warmer seas of the globe (see Coral Reefs). The floor of the Atlantic and other oceans is covered with a fine calcareous ooze derived mainly from the remains of foraminifera, while in other regions the bottom shows a siliceous ooze formed almost entirely of radiolaria. Vertebrate animals give rise to phosphatic deposits formed sometimes of their excrement, as in guano and coprolites, sometimes of an accumulation of their bones.

(*c*) *Man.*—No survey of the geological workings of plant and animal life upon the surface of the globe can be complete which does not take account of the influence of man—an influence of enormous and increasing consequence in physical geography, for man has introduced, as it were, an element of antagonism to nature. His interference shows itself in his relations to climate, where he has affected the meteorological conditions of different countries: (1) By removing forests, and laying bare to the sun and winds areas which were previously kept cool and damp under trees, or which, lying on the lee side, were protected from tempests. It is supposed that the wholesale destruction of the woodlands formerly existing in countries bordering the Mediterranean has been in part the cause of the present desiccation of these districts. (2) By drainage, whereby the discharged rainfall is rapidly removed, and the evaporation is lessened, with a consequent diminution of rainfall and some increase in the general temperature of a country. (3) By the other processes of agriculture, such as the transformation of moor and bog into cultivated land, and the clothing of bare hillsides with green crops or plantations of coniferous and hardwood trees.

Still more obvious are the results of human interference with the flow of water: (1) By increasing or diminishing the rainfall man directly affects the volume of rivers. (2) By his drainage operations he makes the rain to run off more rapidly than before, and thereby increases the magnitude of floods and of the destruction caused by them. (3) By wells, bores, mines, or other subterranean works he interferes with the underground waters, and consequently with the discharge of springs. (4) By embanking rivers he confines them to narrow channels, sometimes increasing their scour, and enabling them to carry their sediment further seaward, sometimes causing them to deposit it over the plains and raise their level. (5) By his engineering operations for water-supply he abstracts water from its natural basins and depletes the streams.

In many ways man alters the aspect of a country: (1) By changing forest into bare mountain, or clothing bare mountains with forest. (2) By promoting the growth or causing the removal of peat-mosses. (3) By heedlessly uncovering sand-dunes, and thereby setting in motion a process of destruction which may convert hundreds of acres of fertile land into waste sand, or by prudently planting the dunes with sand-loving vegetation and thus arresting their landward progress. (4) By so guiding the course of rivers as to make them aid him in reclaiming waste land, and bringing it under cultivation. (5) By piers and bulwarks, whereby the ravages of the sea are stayed, or by the thoughtless removal from the beach of stones which the waves had themselves thrown up, and which would have served for a time to protect the land. (6) By forming new deposits either designedly or incidentally. The roads, bridges, canals, railways, tunnels, villages and towns with which man has covered the surface of the land will in many cases form a permanent record of his presence. Under his hand the whole surface of civilized countries is very slowly covered with a stratum, either formed wholly by him or due in great measure to his operations and containing many relics of his presence. The soil of ancient towns has been increased to a depth of many feet by their successive destructions and renovations.

Perhaps the most subtle of human influences are to be seen in the distribution of plant and animal life upon the globe. Some of man's doings in this domain are indeed plain enough, such as the extirpation of wild animals, the diminution or destruction of some forms of vegetation, the introduction of plants and animals useful to himself, and especially the enormous predominance given by him to the cereals and to the spread of sheep and cattle. But no such extensive disturbance of the normal conditions of the distribution of life can take place without carrying with it many secondary effects, and setting in motion a wide cycle of change and of reaction in the animal and vegetable kingdoms. For example, the incessant warfare waged by man against birds and beasts of prey in districts given up to the chase leads sometimes to unforeseen results. The weak game is allowed to live, which would otherwise be killed off and give more room for the healthy remainder. Other animals which feed perhaps on the same materials as the game are by the same cause permitted to live unchecked, and thereby to act as a further hindrance to the spread of the protected species. But the indirect results of man's interference with the régime of plants and animals still require much prolonged observation.

Part V.—Geotectonic or Structural Geology

From a study of the nature and composition of minerals and rocks, and an investigation of the different agencies by which they are formed and modified, the geologist proceeds to inquire how these materials have been put together so as to build up the visible part of the earth's crust. He soon ascertains that they have not been thrown together wholly at random, but that they show a recognizable order of arrangement. Some of them, especially those of most recent growth, remain in their original condition and position, but, in proportion to their antiquity, they generally present increasing alteration, until it may no longer be possible to tell what was their pristine state. As by far the largest accessible portion of the terrestrial crust consists of stratified rocks, and as these furnish clear evidence of most of the modifications to which they have been subjected in the long course of geological history, it is convenient to take them into consideration first. They possess a number of structures 664 which belong to the original conditions in which they were accumulated. They present in addition other structures which have been superinduced upon them, and which they share with the unstratified or igneous rocks.

1. Original Structures

(*a*) *Stratified Rocks.*—This extensive and important series is above all distinguished by possessing a prevailing stratified arrangement. Their materials have been laid down in laminae, layers and strata, or beds, pointing generally to the intermittent deposition of the sediments of which they consist. As this stratification was, as a rule, originally nearly or quite horizontal, it serves as a base from which to measure any subsequent disturbance which the rocks have undergone. The occurrence of false-bedding, *i.e.* bands of inclined layers between the normal planes of stratification, does not form any real exception; but indicates the action of shifting currents whereby the sediment was transported and thrown down. Other important records of the original conditions of deposit are supplied by ripple-marks, sun-cracks, rain-prints and concretions.

From the nature of the material further light is cast on the geographical conditions in which the strata were accumulated. Thus, conglomerates indicate the proximity of old shore-lines, sandstones mark deposits in comparatively shallow water, clays and shales point to the tranquil accumulation of fine silt at a greater depth and further from land, while fossiliferous limestones bear witness to clearer water in which organisms flourished at some distance from deposits of sand and mud. Again, the alternation of different kinds of sediment suggests a variability in the conditions of deposition, such as a shifting of the sediment-bearing currents and of the areas of muddy and clear water. A thick group of conformable strata, that is, a series of deposits which show no discordance in their stratification, may usually be regarded as having been laid down on a sea-floor that was gently sinking. Here and there evidence is obtainable of the limits or of the progress of the subsidence by what is called "overlap." Of the absolute length of time represented by any strata or groups of strata no satisfactory estimates can yet be formed. Certain general conclusions may indeed be drawn, and comparisons may be made between different series of rocks. Sandstones full of false-bedding were probably accumulated more rapidly than finely-laminated shales or clays. It is not uncommon in certain Carboniferous formations to find coniferous and other trunks embedded in sandstone. Some of these trees seem to have been carried along and to have sunk, their heavier or root end touching the bottom and their upper end slanting upward in the direction of the current, exactly as in the case of the snags of the Mississippi. In other cases the trees have been submerged while still in their positions of growth. The continuous deposit of sand at last rose above the level of the trunks and buried them. It is clear then that the rate of deposit must have been sometimes sufficiently rapid to allow sand to accumulate to a depth of 30 ft. or more before the decay of the wood. Modern instances are known where, under certain circumstances, submerged trees may last for some centuries, but even the most durable must decay in what, after all, is a brief space of geological time. Since continuous layers of the same kind of deposit suggest a persistence of geological conditions, while numerous alternations of different kinds of sedimentary matter point to vicissitudes or alternations of conditions, it may be supposed that the time represented by a given thickness of similar strata was less than that shown by the same thickness of dissimilar strata, because the changes needed to bring new varieties of sediment into the area of deposit would usually require the lapse of some time for their completion. But this conclusion may often be erroneous. It will be best supported when, from the very nature of the rocks, wide variations in the character of the water-bottom can be established. Thus a group of shales followed by a fossiliferous limestone would almost always mark the lapse of a much longer period than an equal depth of sandy strata. A thick mass of limestone, made up of organic remains which lived and died upon the spot, and whose remains are crowded together generation above generation, must have demanded many years or centuries for its formation.

But in all speculations of this kind we must bear in mind that the length of time represented by a given depth of strata is not to be estimated merely from their thickness or lithological character. The interval between the deposit of two successive laminae of shale may have been as long as, or even longer than, that required for the formation of one of the laminae. In like manner the interval needed for the transition from one stratum or kind of strata to another may often have been more than equal to the time required for the formation of the strata on either side. But the relative chronological importance of the bars or lines in the geological record can seldom be satisfactorily discussed merely on lithological grounds. This must mainly be decided on the evidence of organic remains, as shown in Part VI., where the grouping of the stratified rocks into formations and systems is described.

(*b*) *Igneous Rocks.*—As part of the earth's crust these rocks present characters by which they are strongly differentiated from the stratified series. While the broad petrographical distinctions of their several varieties remain persistent, they present sufficient local variations of type to point to the existence of what have been called petrographic provinces, in each of which the eruptive masses are connected by a general family relationship, differing more or less from that of a neighbouring province. In each region presenting a long chronological series of eruptive rocks a petrographical sequence can be traced, which is observed to be not absolutely the same everywhere, though its general features may be persistent. The earliest manifestations of eruptive material in any district appear to have been most frequently of an intermediate type between acid and basic, passing thence into a thoroughly acid series and concluding with an effusion of basic material.

Considered as part of the architecture of the crust of the earth, igneous rocks are conveniently divisible into two great series: (1) those bodies of material which have been injected into the crust and have solidified there, and (2) those which have reached the surface and have been ejected there, either in a molten state as lava or in a fragmental form as dust, ashes and scoriae. The first of these divisions represents the plutonic, intrusive or subsequent phase of eruptivity; the second marks the volcanic, interstratified or contemporaneous phase.

1. The plutonic or intrusive rocks, which have been forced into the crust and have consolidated there, present a wide range of texture from the most coarse-grained granites to the most perfect natural glass. Seeing that they have usually cooled with extreme slowness underground, they are as a general rule more largely crystalline than the volcanic series. The form assumed by each individual body of intrusive material has depended upon the shape of the space into which it has been injected, and where it has cooled and become solid. This shape has been determined by the local structure of the earth's crust on the one hand and by the energy of the eruptive force on the other. It offers a convenient basis for the

classification of the intrusive rocks, which, as part of the framework of the crust, may thus be grouped according to the shape of the cavity which received them, as bosses, sills, dikes and necks.

Bosses, or stocks, are the largest and most shapeless extravasations of erupted material. They include the great bodies of granite which, in most countries of the world, have risen for many miles through the stratified formations and have altered the rocks around them by contact-metamorphism. Sills, or intrusive sheets, are bed-like masses which have been thrust between the planes of sedimentary or even of igneous rocks. The term laccolite has been applied to sills which are connected with bosses. Intrusive sheets are distinguishable from true contemporaneously intercalated lavas by not keeping always to the same platform, but breaking across and altering the contiguous strata, and by the closeness of their texture where they come in contact with the contiguous rocks, which, being cold, chilled the molten material and caused it to consolidate on its outer margins more rapidly than in its interior. Dikes or veins are vertical walls or ramifying branches of intrusive material which has consolidated in fissures or irregular clefts of the crust. Necks are volcanic chimneys which have been filled up with erupted material, and have now been exposed at the surface after prolonged denudation has removed not only the superficial volcanic masses originally associated with them, but also more or less of the upper part of the vents. Plutonic rocks do not present evidence of their precise geological age. All that can be certainly affirmed from them is that they must be younger than the rocks into which they have been intruded. From their internal structure, however, and from the evidence of the rocks associated with them, some more or less definite conjectures may be made as to the limits of time within which they were probably injected.

2. The interstratified or volcanic series is of special importance in geology, inasmuch as it contains the records of volcanic action during the past history of the globe. It was pointed out in Part I. that while towards the end of the 18th and in the beginning of the 19th century much attention was paid by Hutton and his followers to the proofs of intrusion afforded by what they called the “unerupted lavas” within the earth’s crust, these observers lost sight of the possibility that some of these rocks might have been erupted at the surface, and might thus be chronicles of volcanic action in former geological periods. It is not always possible to satisfactorily discriminate between the two types of contemporaneously intercalated and subsequently injected material. But rocks of the former type have not broken into or involved the overlying strata, and they are usually marked by the characteristic structures of superficial lavas and by their association with volcanic tuffs. By means of the evidence which they supply, it has been ascertained that volcanic action has been manifested in the globe since the earliest geological periods. In the British Isles, for example, the volcanic record is remarkably full for the long series of ages from Cambrian to Permian time, and again for the older Tertiary period. 665

2. Subsequently induced Structures

After their accumulation, whether as stratified or eruptive masses, all kinds of rocks have been subject to various changes, and have acquired in consequence a variety of superinduced structures. It has been pointed out in the part of this article dealing with dynamical geology that one of the most important forms of energy in the evolution of geological processes is to be found in the movements that take place within the crust of the earth. Some of these movements are so slight as to be only recognizable by means of delicate instruments; but from this inferior limit they range up to gigantic convulsions by which mountain-chains are upheaved. The crust must be regarded as in a perpetual state of strain, and its component materials are therefore subject to all the effects which flow from that condition. It is the one great object of the geotectonic division of geology to study the structures which have been developed in consequence of earth-movements, and to discover from this investigation the nature of the processes whereby the rocks of the crust have been brought into the condition and the positions in which we now find them. The details of this subject will be found in separate articles descriptive of each of the technical terms applied to the several kinds of superinduced structures. All that need be offered here is a general outline connecting the several portions of the subject together.

One of the most universal of these later structures is to be seen in the divisional planes, usually vertical or highly inclined, by which rocks are split into quadrangular or irregularly shaped blocks. To these planes the name of joints has been given. They are of prime importance from an industrial point of view, seeing that the art of quarrying consists mainly in detecting and making proper use of them. Their abundance in all kinds of rocks, from those of recent date up to those of the highest antiquity, affords a remarkable testimony to the strains which the terrestrial crust has suffered. They have arisen sometimes from tension, such as that caused by contraction from the drying and consolidation of an aqueous sediment or from the cooling of a molten mass; sometimes from torsion during movements of the crust.

Although the stratified rocks were originally deposited in a more or less nearly horizontal position on the floor of the sea, where now visible on the dry land they are seldom found to have retained their flatness. On the contrary, they are seen to have been generally tilted up at various angles, sometimes even placed on end (crop, dip, strike). When a sufficiently large area of ground is examined, the inclination into which the strata have been thrown may be observed not to continue far in the same direction, but to turn over to the opposite or another quarter. It can then be seen that in reality the rocks have been thrown into undulations. From the lowest and flattest arches where the departure from horizontality may be only trifling, every step may be followed up to intense curvature, where the strata have been compressed and plicated as if they had been piles of soft carpets (anticline, syncline, monocline, geo-anticline, geo-syncline, isoclinal, plication, curvature, quaquaversal). It has further happened abundantly all over the surface of the globe that relief from internal strain in the crust has been obtained by fracture, and the consequent subsidence or elevation of one or both sides of the fissure. The differential movement between the two sides may be scarcely perceptible in the feeblest dislocation, but in the extreme cases it may amount to many thousand feet (fault, fissure, dislocation, hade, slickensides). The great faults in a country are among its most important structural features, and as they not infrequently continue to be lines of weakness in the crust along which sudden slipping may from time to time take place, they become the lines of origin of earthquakes. The San Francisco earthquake of 1906, already cited, affords a memorable illustration of this connexion.

It is in a great mountain-chain that the extraordinary complication of plicated and faulted structures in the crust of the earth can be most impressively beheld. The combination of overturned folds with rupture has been already referred to as a characteristic feature in the Alps (Part IV.). The gigantic folds have in many places been pushed over each other so as to lie almost flat, while the upper limb has not infrequently been driven for many miles beyond the lower by a rupture along the axis. In this way successive slices of a thick series of formations have been carried northwards on the northern slope of the Alps, and have been piled so abnormally above each other that some of their oldest members recur several times on different thrust-planes, the whole being underlain by Tertiary strata (see Alps). Further proof of the colossal compression to which the rocks have been subjected is afforded by their intense crumpling and corrugation, and by the abundantly faulted and crushed condition to which they have been reduced. Similar evidence as to stresses in the terrestrial crust and the important changes which they produce among the rocks may also be obtained on a smaller scale in many non-mountainous countries.

Another marked result of the compression of the terrestrial crust has been induced in some rocks by the production of the fissile structure which is typically shown in roofing-slate (cleavage). Closely connected with this internal rearrangement has been the development of microscopic microlites or crystals (rutile, mica, &c.) in argillaceous slates which were undoubtedly originally fine marine mud and silt. From this incipient form of metamorphism successive stages may be traced through the various kinds of argillite and phyllite into mica-schist, and thence into more crystalline gneissoid varieties (foliation, slate, mica-schist, gneiss). The Alps afford excellent illustrations of these transformations.

The fissures produced in the crust are sometimes clean, sharply defined divisional planes, like cracks across a pane of glass. Much more usually, however, the rocks on either side have been broken up by the friction of movement, and the fault is marked by a variable breadth of this broken material. Sometimes the walls have separated and molten rock has risen from below and solidified between them as a dike. Occasionally the fissures have opened to the surface, and have been filled in from above with detritus, as in the sandstone-dikes of Colorado and California. In mineral districts the fissures have been filled with various spars and ores, forming what are known as mineral veins.

Where one series of rocks is covered by another without any break or discordance in the stratification they are said to be conformable. But where the older series has been tilted up or visibly denuded before being overlain by the younger, the latter is termed unconformable. This relation is one of the greatest value in structural geology, for it marks a gap in the geological record, which may represent a vast lapse of time not there recorded by strata.

Part VI.—Paleontological Geology

This division of the science deals with fossils, or the traces of plants and animals preserved in the rocks of the earth’s crust, and endeavours to gather from them information as to the history of the globe and its inhabitants. The term “fossil” (Lat. *fossilis*, from *fodere*, to dig up), meaning literally anything “dug up,” was formerly applied indiscriminately to any mineral substance taken out of the earth’s crust, whether organized or not. Since the time of Lamarck, however, the meaning of the word has been restricted, so as to include only the remains or traces of plants and animals preserved in any natural formation whether hard rock or superficial deposit. It includes not merely the petrified structures of organisms, but whatever was directly connected with or produced by these organisms. Thus the resin which was exuded from trees of long-perished forests is as much a fossil as any portion of the stem, leaves, flowers or fruit, and in some respects is even more valuable to the geologist than more determinable remains of its parent trees, because it has often preserved in admirable perfection the insects which flitted about in the woodlands. The burrows and trails of a worm preserved in sandstone and shale claim recognition as fossils, and indeed are commonly the only indications to be met with of the existence of annelid life among old geological formations. The droppings of fishes and reptiles, called coprolites, are excellent fossils, and tell their tale as to the presence and food of vertebrate life in ancient waters. The little agglutinated cases of the caddis-worm remain as fossils in formations from which, perchance, most other traces of life may have passed away. Nay, the very handiwork of man, when preserved in any natural manner, is entitled to rank among fossils; as where his flint-implements have been dropped into the pre-historic gravels of river-valleys or where his canoes have been buried in the silt of lake-bottoms.

A study of the land-surfaces and sea-floors of the present time shows that there are so many chances against the conservation of the remains of either terrestrial or marine animals and plants that if, as is probable, the same conditions existed in former geological periods, we should regard the occurrence of organic remains among the stratified formations of the earth’s crust as generally the result of various fortunate accidents.

Let us consider, in the first place, the chances for the preservation of remains of the present fauna and flora of a country. The surface of the land may be densely clothed with forest and abundantly peopled with animal life. But the trees die and moulder into soil. The 666
animals, too, disappear, generation after generation, and leave few or no perceptible traces of their existence. If we were not aware from authentic records that central and northern Europe were covered with vast forests at the beginning of our era, how could we know this fact? What has become of the herds of wild oxen, the bears, wolves and other denizens of primeval Europe? How could we prove from the examination of the surface soil of any country that those creatures had once abounded there? The conditions for the preservation of any relics of the plant and animal life of a terrestrial surface must obviously be always exceptional. They are supplied only where the organic remains can be protected from the air and superficial decay. Hence they may be observed in (1) the deposits on the floors of lakes; (2) in peat-mosses; (3) in deltas at river-mouths; and (4) under the stalagmite of caverns in limestone districts. But in these and other favourable places a mere infinitesimal fraction of the fauna or flora of a land-surface is likely to be entombed or preserved.

In the second place, although in the sea the conditions for the preservation of organic remains are in many respects more favourable than on land, they are apt to be frustrated by many adverse circumstances. While the level of the land remains stationary, there can be but little effective entombment of marine organisms in littoral deposits; for only a limited accumulation of sediment will be formed until subsidence of the sea-floor takes place. In the trifling beds of sand or gravel thrown up on a stationary shore, only the harder and more durable forms of life, such as gastropods and lamellibranchs, which can withstand the triturating effects of the beach waves, are likely to remain uneffaced.

Below tide-marks, along the margin of the land where sediment is gradually deposited, the conditions are more favourable for the preservation of marine organisms. In the sheets of sand and mud there laid down the harder parts of many forms of life may be entombed and protected from decay. But only a small proportion of the total marine fauna may be expected to appear in such deposits. At the best, merely littoral and shallow-water forms will occur, and, even under the most favourable conditions, they will represent but a fraction of the whole assemblage of life in these juxta-terrestrial parts of the ocean. As we recede from the land the rate of deposition of sediment on the sea-floor must become feebler, until, in the remote central abysses, it reaches a hardly appreciable minimum. Except, therefore, where some kind of ooze or other deposit is accumulating in these more pelagic regions, the conditions must be on the whole unfavourable for the preservation of any adequate representation of the deep-sea fauna. Hard durable objects, such as teeth and bones, may slowly accumulate, and be protected by a coating of peroxide of manganese, or of some of the silicates now forming here and there over the deep-sea bottom; or the rate of growth of the abysmal deposit may be so tardy that most of the remains of at least the larger animals will disappear, owing to decay, before they can be covered up and preserved. Any such deep-sea formation, if raised into land, would supply but a meagre picture of the whole life of the sea.

It would thus appear that the portion of the sea-floor best suited for receiving and preserving the most varied assemblage of marine organic remains is the area in front of the land, to which rivers and currents bring continual supplies of sediment. The most favourable conditions for the accumulation of a thick mass of marine fossiliferous strata will arise when the area of deposit is undergoing a gradual subsidence. If the rate of depression and that of deposit were equal, or nearly so, the movement might proceed for a vast period without producing any great apparent change in marine geography, and even without seriously affecting the distribution of life over the sea-floor within the area of subsidence. Hundreds or thousands of feet of sedimentary strata might in this way be heaped up round the continents, containing a fragmentary series of organic remains belonging to those forms of comparatively shallow-water life which had hard parts capable of preservation. There can be little doubt that such has, in fact, been the history of the main mass of stratified formations in the earth’s crust. By far the largest proportion of these piles of marine strata has unquestionably been laid down in water of no great depth within the area of deposit of terrestrial sediment. The enormous thickness to which they attain seems only explicable by prolonged and repeated movements of subsidence, interrupted, however, as we know, by other movements of a contrary kind.

Since the conditions for the preservation of organic remains exist more favourably under the sea than on land, marine organisms must be far more abundantly conserved than those of the land. This is true to-day, and has, as far as known, been true in all past geological time. Hence for the purposes of the geologist the fossil remains of marine forms of life far surpass all others in value. Among them there will necessarily be a gradation of importance, regulated chiefly by their relative abundance. Now, of all the marine tribes which live within the juxta-terrestrial belt of sedimentation, unquestionably the Mollusca stand in the place of pre-eminence as regards their aptitude for becoming fossils. They almost all possess a hard, durable shell, capable of resisting considerable abrasion and readily passing into a mineralized condition. They are extremely abundant both as to individuals and genera. They occur on the shore within tide mark, and range thence down into the abysses. Moreover, they appear to have possessed these qualifications from early geological times. In the marine Mollusca, therefore, we have a common ground of comparison between the stratified formations of different periods. They have been styled the alphabet of palaeontological inquiry.

There are two main purposes to which fossils may be put in geological research: (1) to throw light upon former conditions of physical geography, such as the presence of land, rivers, lakes and seas, in places where they do not now exist, changes of climate, and the former distribution of plants and animals; and (2) to furnish a guide in geological chronology whereby rocks may be classified according to relative date, and the facts of geological history may be arranged and interpreted as a connected record of the earth’s progress.

1. As examples of the first of these two directions of inquiry reference may be made to (*a*) former land-surfaces revealed by the occurrence of layers of soil with tree-stumps and roots still in the position of growth (see Purbeckian); (*b*) ancient lakes proved by beds of marl or limestone full of lacustrine shells; (*c*) old sea-bottoms marked by the occurrence of marine organisms; (*d*) variations in the quality of the water, such as freshness or saltness, indicated by changes in the size and shape of the fossils; (*e*) proximity to former land, suggested by the occurrence of abundant drift-wood in the strata; (*f*) former conditions of climate, different from the present, as evidenced by such organisms as tropical types of plants and animals intercalated among the strata of temperate or northern countries.

2. In applying fossils to the determination of geological chronology it is first necessary to ascertain the order of superposition of the rocks. Obviously, in a continuous series of undisturbed sedimentary deposits the lowest must necessarily be the oldest, and the plants or animals which they contain must have lived and died before any of the organisms that occur in the overlying strata. This order of superposition having been settled in a series of formations, it is found that the fossils at the bottom are not quite the same as those at the top of the series. Tracing the beds upward, we discover that species after species of the lowest platforms disappears, until perhaps not one of them is found. With the cessation of these older species others make their entrance. These, in turn, are found to die out, and to be replaced by newer forms. After patient examination of the rocks, it has been ascertained that every well-marked “formation,” or group of strata, is characterized by its own species or genera, or by a general assemblage, or *facies*, of organic forms. Such a generalization can only, of course, be determined by actual practical experience over an area of some size. When the typical fossils of a formation are known, they serve to identify that formation in its progress across a country. Thus, in tracts where the true order of superposition cannot be determined, owing to the want of sections or to the disturbed condition of the rocks, fossils

serve as a means of identification and furnish a guide to the succession of the rocks. They even demonstrate that in some mountainous ground the beds have been turned completely upside down, where it can be shown that the fossils in what are now the uppermost strata ought properly to lie underneath those in the beds below them.

It is by their characteristic fossils that the stratified rocks of the earth’s crust can be most satisfactorily subdivided into convenient groups of strata and classed in chronological order. Each “formation” is distinguished by its own peculiar assemblage of organic remains, by means of which it can be followed and recognized, even amid the crumplings and dislocations of a disturbed region. The same general succession of organic types can be observed over a large part of the world, though, of course, with important modifications in different countries. This similarity of succession has been termed *homotaxis*, a term which expresses the fact that the order in which the leading types of organized existence have appeared upon the earth has been similar even in widely separated regions. It is evident that, in this way, a reliable method of comparison is furnished, whereby the stratified formations of different parts of the earth’s crust can be brought into relation with each other. Had the geologist continued to remain, as in the days of Werner, hampered by the limitations imposed by a reliance on mere lithological characters, he would have made little or no progress in deciphering the record of the successive phases of the history of the globe chronicled in the crust. Just as, at the present time, sheets of gravel in one place are contemporaneous with sheets of mud at another, so in the past all kinds of sedimentation have been in progress simultaneously, and those of one period may not be distinguishable in themselves from those of another. Little or no reliance can be placed upon lithological resemblances or differences in comparing the sedimentary formations of different countries.

In making use of fossil evidence for the purpose of subdividing the stratified rocks of the earth’s crust, it is found to be applicable to the smaller details of stratigraphy as well as to the definition of large groups of strata. Thus a particular stratum may be marked by the occurrence in it of various fossils, one or more of which may be distinctive, either from occurring in no other bed above and below or from special abundance in that stratum. One
or more of these species is therefore used as a guide to the occurrence of the bed in 667
question, which is called by the name of the most abundant species. In this way what is called a “geological horizon,” or “zone,” is marked off, and its exact position in the series of formations is fixed.

Perhaps the most distinctive feature in the progress of palaeontological geology during the last half century has been the recognition and wide application of this method of zonal stratigraphy, which, in itself, was only a further development of William Smith’s famous idea, “Strata identified by Organized Fossils.” It was first carried out in detail by various palaeontologists in reference to the Jurassic formations, notably by F.A. von Quenstedt and C.A. Oppel in Germany and A.D. d’Orbigny in France. The publication of Oppel’s classic work *Die Juraformation Englands, Frankreichs und des südwestlichen Deutschlands* (1856-1858) marked an epoch in the development of stratigraphical geology. Combining what had been done by various observers with his own laborious researches in France, England, Württemberg and Bavaria, he drew up a classification of the Jurassic system, grouping its several formations into zones, each characterized by some distinctly predominant fossil after which it was named (see Lias). The same method of classification was afterwards extended to the Cretaceous series by A.D. d’Orbigny, E. Hébert and others, until the whole Mesozoic rocks from the Trias to the top of the Chalk has now been partitioned into zones, each named after some characteristic species or genus of fossils. More recently the principle has been extended to the Palaeozoic formations, though as yet less fully than to the younger parts of the geological record. It has been successfully applied by Professor C. Lapworth to the investigation of the Silurian series (see Silurian; Ordovician System). He found that the species of graptolites have each a comparatively narrow vertical range, and they may consequently be used for stratigraphical purposes. Applying the method, in the first instance, to the highly plicated Silurian rocks of the south of Scotland, he found that by means of graptolites he was able to work out the structure of the ground. Each great group of strata was seen to possess its own graptolitic zones, and by their means could be identified not only in the original complex Scottish area, but in England and Wales and in Ireland. It was eventually ascertained that the succession of zones in Great Britain could be recognized on the Continent, in North America and even in Australia. The brachiopods and trilobites have likewise been made use of for zonal purposes among the oldest sedimentary formations. The most ancient of the Palaeozoic systems has as its fitting base the *Olenellus* zone.

Within undefined and no doubt variable geographical limits palaeontological zones have been found to be remarkably persistent. They follow each other in the same general order, but not always with equal definiteness. The type fossil may appear in some districts on a higher or a lower platform than it does in others. Only to a limited degree is there any coincidence between lithological variations in the strata and the sequence of the zones. In the Jurassic formations, indeed, where frequent alternations of different sedimentary materials are to be met with, it is in some cases possible to trace a definite upward or downward limit for a zone by some abrupt change in the sedimentation, such as from limestone to shale. But such a precise demarcation is impossible where no distinct bands of different sediments are to be seen. The zones can then only be vaguely determined by finding their characteristic fossils, and noting where these begin to appear in the strata and where they cease. It would seem, therefore, that the sequence of palaeontological zones, or life-horizons, has not depended merely upon changes in the nature of the conditions under which the organisms lived. We should naturally expect that these changes would have had a marked influence; that, for instance, a difference should be perceptible between the character of the fossils in a limestone and that of those in a shale or a sandstone. The environment, when a limestone was in course of deposition, would generally be one of clear water, favourable for a more vigorous and more varied fauna than where a shale series was accumulating, when the water would be discoloured, and only such animals would continue to live in it, or on the bottom, as could maintain themselves in the midst of mud. But no such lithological reason, betokening geographical changes that would affect living creatures, can be adduced as a universally applicable explanation of the occurrence and limitation of palaeontological zones. One of these zones may be only a few inches, or feet or yards in vertical extent, and no obvious lithological or other cause can be seen why its specially characteristic fossils should not be found just as frequently in the similar strata above and below. There is often little or no evidence of any serious change in the conditions of sedimentation, still less of any widespread physical disturbance, such as the catastrophes by which the older geologists explained the extinction of successive types of life.

It has been suggested that, where the life-zones are well defined, sedimentation has been extremely slow, and that though these zones follow each other with no break in the sedimentation, they were really separated by prolonged intervals of time during which organic evolution could come effectively into play. But it is not easy to explain how, for example in the Lower Lias, there could have been a succession of prodigious intervals, when practically no sediment was laid down, and yet that the strata should show no sign of contemporaneous disturbance or denudation, but succeed each other as if they had been accumulated by one continuous process of deposit. It must be admitted that the problem of life-zones in stratigraphical geology has not yet been solved.

As Darwin first cogently showed, the history of life has been very imperfectly registered in the stratified parts of the earth’s crust. Apart from the fact that, even under the most favourable conditions, only a small proportion of the total flora and fauna of any period would be preserved in the fossil state, enormous gaps occur where no record has survived at all. It is as if whole chapters and books were missing from a historical work. Some of these lacunae are sufficiently obvious. Thus, in some cases, powerful dislocations have thrown considerable portions of the rocks out of sight. Sometimes extensive metamorphism has so affected them that their original characters, including their organic contents, have been destroyed. Oftenest of all, denudation has come into play, and vast masses of fossiliferous rock have been entirely worn away, as is demonstrated by the abundant unconformabilities in the structure of the earth’s crust.

While the mere fact that one series of rocks lies unconformably on another proves the lapse of a considerable interval between their respective dates, the relative length of this interval may sometimes be proved by means of fossil evidence, and by this alone. Let us suppose, for example, that a certain group of formations has been disturbed, upraised, denuded and covered unconformably by a second group. In lithological characters the two may closely resemble each other, and there may be nothing to show that the gap represented by their unconformability is of an important character. In many cases, indeed, it would be quite impossible to pronounce any well-grounded judgment as to the amount of interval, even measured by the vague relative standards of geological chronology. But if each group contains a well-preserved suite of organic remains, it may not only be possible, but easy, to say exactly how much of the geological record has been left out between the two sets of formations. By comparing the fossils with those obtained from regions where the geological record is more complete, it may be ascertained, perhaps, that the lower rocks belong to a certain platform or stage in geological history which for our present purpose we may call D, and that the upper rocks can in like manner be paralleled with stage H. It would be then apparent that at this locality the chronicles of three great geological periods E, F, and G were wanting, which are elsewhere found to be intercalated between D and H. The lapse of time represented by this unconformability would thus be equivalent to that required for the accumulation of the three missing formations in those regions where sedimentation was more continuous.

Fossil evidence may be made to prove the existence of gaps which are not otherwise apparent. As has been already remarked, changes in organic forms must, on the whole, have been extremely slow in the geological past. The whole species of a sea-floor could not pass entirely away, and be replaced by other forms, without the lapse of long periods of time. If then among the conformable stratified formations of former ages we encounter sudden and abrupt changes in the *facies* of the fossils, we may be certain that these must mark omissions in the record, which we may hope to fill in from a more perfect series elsewhere. The complete biological contrasts between the fossil contents of unconformable strata are sufficiently explicable. It is not so easy to give a satisfactory account of those which occur where the beds are strictly conformable, and where no evidence can be observed of any considerable change of physical conditions at the time of deposit. A group of strata having the same general lithological characters throughout may be marked by a great discrepance between the fossils above and below a certain line. A few species may pass from the one into the other, or perhaps every species may be different. In cases of this kind, when proved to be not merely local but persistent over wide areas, we must admit, notwithstanding the apparently undisturbed and continuous character of the original deposition of the strata, that the abrupt transition from the one *facies* of fossils to the other represents a long interval of time which has not been recorded by the deposit of strata. A.C. Ramsay, who called attention to these gaps, termed them “breaks in the succession of organic remains.” He showed that they occur abundantly among the Palaeozoic and Secondary rocks of England. It is obvious, of course, that such breaks, even though traceable over wide regions, were not general over the whole globe. There have never been any universal interruptions in the continuity of the chain of being, so far as geological evidence can show. But the physical changes which caused the breaks may have been general over a zoological district or minor region. They no doubt often caused the complete extinction of genera and species which had a small geographical range.

From all these facts it is clear that the geological record, as it now exists, is at the best but an imperfect chronicle of geological history. In no country is it complete. The lacunae of one region must be supplied from another. Yet in proportion to the geographical distance between the localities where the gaps occur and those whence the missing intervals are supplied, the element of uncertainty in our reading of the record is increased. The most desirable method of research is to exhaust the evidence for each area or province, and to compare the general order of its succession as a whole with that which can be established for other provinces.

Part VII.—Stratigraphical Geology

This branch of the science arranges the rocks of the earth’s crust in the order of their appearance, and interprets the sequence of events of which they form the records. Its province is to cull from the other departments of geology the facts which may be needed to show what has been the progress of our planet, and of each continent and country, from the earliest times of which the rocks have preserved any memorial. Thus from mineralogy and petrography it contains information regarding the origin and subsequent mutations of minerals and rocks. From dynamical geology it learns by what agencies the materials of the earth’s crust have been formed, altered, broken, upheaved and melted. From geotectonic geology it understands the various processes whereby these materials were put together so as to build up the complicated crust of the earth. From palaeontological geology it receives in well-determined fossil remains a clue by which to discriminate the different stratified formations, and to trace the grand onward march of organized existence upon this planet. Stratigraphical geology thus gathers up the sum of all that is made known by the other departments of the science, and makes it subservient to the interpretation of the geological history of the earth.

The leading principles of stratigraphy may be summed up as follows:

1. In every stratigraphical research the fundamental requisite is to establish the order of superposition of the strata. Until this is accomplished it is impossible to arrange the dates, and make out the sequence of geological history.

2. The stratified portion of the earth’s crust, or what has been called the “geological record,” can be subdivided into natural groups, or series of strata, characterized by distinctive organic remains and recognizable by these remains, in spite of great changes in lithological character from place to place. A bed, or a number of beds, linked together by containing one or more distinctive species or genera of fossils is termed a *zone* or *horizon*, and usually bears the name of one of its more characteristic fossils, as the *Planorbis*-zone of the Lower Lias, which is so called from the prevalence in it of the ammonite *Psiloceras planorbis*. Two or more such zones related to each other by the possession of a number of

the same characteristic species or genera have been designated *beds* or an *assise*. Two or more sets of beds or assises similarly related form a *group* or *stage*; a number of groups or stages make a *series*, *formation* or *section*, and a succession of formations may be united into a *system*.

3. Some living species of plants and animals can be traced downwards through the more recent geological formations; but the number which can be so followed grows smaller as the examination is pursued into more ancient deposits. With their disappearance other species or genera present themselves which are no longer living. These in turn may be traced backward into earlier formations, till they too cease and their places are taken by yet older forms. It is thus shown that the stratified rocks contain the records of a gradual progression of organic forms. A species which has once died out does not seem ever to have reappeared.

4. When the order of succession of organic remains among the stratified rocks has been determined, they become an invaluable guide in the investigation of the relative age of rocks and the structure of the land. Each zone and formation, being characterized by its own species or genera, may be recognized by their means, and the true succession of strata may thus be confidently established even in a country wherein the rocks have been shattered by dislocation, folded, inverted or metamorphosed.

5. Though local differences exist in regard to the precise zone in which a given species of organism may make its first appearance, the general order of succession of the organic forms found in the rocks is never inverted. The record is nowhere complete in any region, but the portions represented, even though extremely imperfect, always follow each other in their proper chronological order, unless where disturbance of the crust has intervened to destroy the original sequence.

6. The relative chronological value of the divisions of the geological record is not to be measured by mere depth of strata. While it may be reasonably assumed that, in general, a great thickness of stratified rock must mark the passage of a long period of time, it cannot safely be affirmed that a much less thickness elsewhere must represent a correspondingly diminished period. The need for this caution may sometimes be made evident by an unconformability between two sets of rocks, as has already been explained. The total depth of both groups together may be, say 1000 ft. Elsewhere we may find a single unbroken formation reaching a depth of 10,000 ft.; but it would be unwarrantable to assume that the latter represents ten times the length of time indicated by the former two. So far from this being the case, it might not be difficult to show that the minor thickness of rock really denotes by far the longer geological interval. If, for instance, it could be proved that the upper part of both the sections lies on one and the same geological platform, but that the lower unconformable series in the one locality belongs to a far lower and older system of rocks than the base of the thick conformable series in the other, then it would be clear that the gap marked by the unconformability really indicates a longer period than the massive succession of deposits.

7. Fossil evidence furnishes the chief means of comparing the relative value of formations and groups of rock. A “break in the succession of organic remains,” as already explained, marks an interval of time often unrepresented by strata at the place where the break is found. The relative importance of these breaks, and therefore, probably, the comparative intervals of time which they mark, may be estimated by the difference of the *facies* or general character of the fossils on each side. If, for example, in one case we find every species to be dissimilar above and below a certain horizon, while in another locality only half of the species on each side are peculiar, we naturally infer, if the total number of species seems large enough to warrant the inference, that the interval marked by the former break was much longer than that marked by the second. But we may go further and compare by means of fossil evidence the relation between breaks in the succession of organic remains and the depth of strata between them.

Three formations of fossiliferous strata, A, C, and H, may occur conformably above each other. By a comparison of the fossil contents of all parts of A, it may be ascertained that, while some species are peculiar to its lower, others to its higher portions, yet the majority extend throughout the formation. If now it is found that of the total number of species in the upper portion of A only one-third passes up into C, it may be inferred with some plausibility that the time represented by the break between A and C was really longer than that required for the accumulation of the whole of the formation A. It might even be possible to discover elsewhere a thick intermediate formation B filling up the gap between A and C. In like manner were it to be discovered that, while the whole of the formation C is characterized by a common suite of fossils, not one of the species and only one half of the genera pass up into H, the inference could hardly be resisted that the gap between the two formations marks the

> passage of a far longer interval than was needed for the deposition of the whole of C. And thus we reach the remarkable conclusion that, thick though the stratified formations of a country may be, in some cases they may not represent so long a total period of time as do the gaps in their succession,—in other words, that non-deposition was more frequent and prolonged than deposition, or that the intervals of time which have been recorded by strata have not been so long as those which have not been so recorded.

In all speculations of this nature, however, it is necessary to reason from as wide a basis of observation as possible, seeing that so much of the evidence is negative. Especially needful is it to bear in mind that the cessation of one or more species at a certain line among the rocks of a particular district may mean nothing more than that, onward from the time marked by that line, these species, owing to some change in the conditions of life, were compelled to migrate or became locally extinct or, from some alteration in the conditions of fossilization, were no longer imbedded and preserved as fossils. They may have continued to flourish abundantly in neighbouring districts for a long period afterward. Many examples of this obvious truth might be cited. Thus in a great succession of mingled marine, brackish-water and terrestrial strata, like that of the Carboniferous Limestone series of Scotland, corals, crinoids and brachiopods abound in the limestones and accompanying shales, but 669
disappear as the sandstones, ironstones, clays, coals and bituminous shales supervene. An observer meeting for the first time with an instance of this disappearance, and remembering what he had read about breaks in succession, might be tempted to speculate about the extinction of these organisms, and their replacement by other and later forms of life, such as the ferns, lycopods, estuarine or fresh-water shells, ganoid fishes and other fossils so abundant in the overlying strata. But further research would show him that high above the plant-bearing sandstones and coals other limestones and shales might be observed, once more charged with the same marine fossils as before, and still farther overlying groups of sandstones, coals and carbonaceous beds followed by yet higher marine limestones. He would thus learn that the same organisms, after being locally exterminated, returned again and again to the same area. After such a lesson he would probably pause before too confidently asserting that the highest bed in which we can detect certain fossils marks their final appearance in the history of life. Some breaks in the succession may thus be extremely local, one set of organisms having been driven to a different part of the same region, while another set occupied their place until the first was enabled to return.

8. The geological record is at the best but an imperfect chronicle of the geological history of the earth. It abounds in gaps, some of which have been caused by the destruction of strata owing to metamorphism, denudation or otherwise, others by original non-deposition, as above explained. Nevertheless from this record alone can the progress of the earth be traced. It contains the registers of the appearance and disappearance of tribes of plants and animals which have from time to time flourished on the earth. Only a small proportion of the total number of species which have lived in past time have been thus chronicled, yet by collecting the broken fragments of the record an outline at least of the history of life upon the earth can be deciphered.

It cannot be too frequently stated, nor too prominently kept in view, that, although gaps occur in the succession of organic remains as recorded in the rocks, they do not warrant the conclusion that any such blank intervals ever interrupted the progress of plant and animal life upon the globe. There is every reason to believe that the march of life has been unbroken, onward and upward. Geological history, therefore, if its records in the stratified formations were perfect, ought to show a blending and gradation of epoch with epoch. But the progress has been constantly interrupted, now by upheaval, now by volcanic outbursts, now by depression. These interruptions serve as natural divisions in the chronicle, and enable the geologist to arrange his history into periods. As the order of succession among stratified rocks was first made out in Europe, and as many of the gaps in that succession were found to be widespread over the European area, the divisions which experience established for that portion of the globe came to be regarded as typical, and the names adopted for them were applied to the rocks of other and far distant regions. This application has brought out the fact that some of the most marked breaks in the European series do not exist elsewhere, and, on the other hand, that some portions of that series are much more complete than the corresponding sections in other regions. Hence, while the general similarity of succession may remain, different subdivisions and nomenclature are required as we pass from continent to continent.

The nomenclature adopted for the subdivisions of the geological record bears witness to the rapid growth of geology. It is a patch-work in which no system nor language has been adhered to, but where the influences by which the progress of the science has been moulded may be distinctly traced. Some of the earliest names are lithological, and remind us of the fact that mineralogy and petrography preceded geology in the order of birth—Chalk, Oolite, Greensand, Millstone Grit. Others are topographical, and often recall the labours of the early geologists of England—London Clay, Oxford Clay, Purbeck, Portland, Kimmeridge beds. Others are taken from local English provincial names, and remind us of the debt we owe to William Smith, by whom so many of them were first used—Lias, Gault, Crag, Cornbrash. Others of later date recognize an order of superposition as already established among formations—Old Red Sandstone, New Red Sandstone. By common consent it is admitted that names taken from the region where a formation or group of rocks is typically developed are best adapted for general use. Cambrian, Silurian, Devonian, Permian, Jurassic are of this class, and have been adopted all over the globe.

But whatever be the name chosen to designate a particular group of strata, it soon comes to be used as a chronological or homotaxial term, apart altogether from the stratigraphical character of the strata to which it is applied. Thus we speak of the Chalk or Cretaceous system, and embrace under that term formations which may contain no chalk; and we may describe as Silurian a series of strata utterly unlike in lithological characters to the formations in the typical Silurian country. In using these terms we unconsciously allow the idea of relative date to arise prominently before us. Hence such a word as “chalk” or “cretaceous” does not suggest so much to us the group of strata so called as the interval of geological history which these strata represent. We speak of the Cretaceous, Jurassic, and Cambrian periods, and of the Cretaceous fauna, the Jurassic flora, the Cambrian trilobites, as if these adjectives denoted simply epochs of geological time.

The stratified formations of the earth’s crust, or geological record, are classified into five main divisions, which in their order of antiquity are as follows: (1) Archean or Pre-Cambrian, called also sometimes Azoic (lifeless) or Eozoic (dawn of life); (2) Palaeozoic (ancient life) or Primary; (3) Mesozoic (middle life) or Secondary; (4) Cainozoic (recent life) or Tertiary; (5) Quaternary or Post-Tertiary. These divisions are further ranged into systems, formations, groups or stages, assises and zones. Accounts of the various subdivisions named are given in separate articles under their own headings. In order, however, that the sequence of the formations and their parallelism in Europe and North America may be presented together a stratigraphical table is given on next page.

Part VIII.—Physiographical Geology

This department of geological inquiry investigates the origin and history of the present topographical features of the land. As these features must obviously be related to those of earlier time which are recorded in the rocks of the earth’s crust, they cannot be satisfactorily studied until at least the main outlines of the history of these rocks have been traced. Hence physiographical research comes appropriately after the other branches of the science have been considered.

From the stratigraphy of the terrestrial crust we learn that by far the largest part of the area of dry land is built up of marine formations; and therefore that the present land is not an aboriginal portion of the earth’s surface, but has been overspread by the sea in which its rocks were mainly accumulated. We further discover that this submergence of the land did not happen once only, but again and again in past ages and in all parts of the world. Yet although the terrestrial areas varied much from age to age in their extent and in their distribution, being at one time more continental, at another more insular, there is reason to believe that these successive diminutions and expansions have on the whole been effected within, or not far outside, the limits of the existing continents. There is no evidence that any portion of the present land ever lay under the deeper parts of the ocean. The abysmal deposits of the ocean-floor have no true representatives among the sedimentary formations anywhere visible on the land. Nor, on the other hand, can it be shown that any part of the existing ocean abysses ever rose above sea-level into dry land. Hence geologists have drawn the inference that the ocean basins have probably been always where they now are; and that although the continental areas have often been narrowed by submergence and by denudation, there has probably seldom or never been a complete disappearance of land. The fact that the sedimentary formations of each successive geological period consist to so large an extent of mechanically formed terrigenous detritus, affords good evidence of the coexistence of tracts of land as well as of extensive denudation.

The Geological Record or Order of Succession of the Stratified Formations of the Earth’s

		Europe.	North America.
Quaternary or Post-Tertiary.	Recent, Post-glacial or Human.	Historic, up to the present time. Prehistoric, comprising deposits of the Iron, Bronze, and later Stone Ages. Neolithic—alluvium, peat, lake-dwellings, loess, &c. Palaeolithic—river-gravels, cave-deposits, &c.	Similar to the European development, but with scantier traces of the presence of man.
	Pleistocene or Glacial.	Older Loess and valley-gravels; cave-deposits. Strand-lines or raised beaches; youngest moraines. Upper Boulder-clays; eskers; marine sands and clays. Interglacial deposits. Lower boulder-clay or Till, with striated rock-surfaces below.	As in Europe, it is hardly possible to assign a definite chronological place to each of the various deposits of this period, terrestrial and marine. They generally resemble the European series. The characteristic marine, fluviatile and lacustrine terraces, which overlie the older drifts, have been classed as the Champlain Group.
Cainozoic or Tertiary.	Pliocene.	Newer:—English Forest-Bed Group; Red and Norwich Crag; Amstelian and Scaldesian groups of Belgium and Holland; Sicilian and Astian of France and Italy. Older:—English Coralline Crag; Diestian of Belgium; Plaisancian of southern France and Italy.	On the Atlantic border represented by the marine Floridian series; in the interior by a subaerial and lacustrine series; and on the Pacific border by the thick marine series of San Francisco.
	Miocene.	Wanting in Britain; well developed in France, S. E. Europe and Italy; divisible into the following groups in descending order: (1) Pontian; (2) Sarmatian; (3) Tortonian; (4) Helvetian; (5) Langhian (Burdigalian).	Represented in the Eastern States by a marine series (Yorktown or Chesapeake, Chipola and Chattahoochee groups), and in the interior by the lacustrine Loup Fork (Nebraska), Deep River, and John Day groups.
	Oligocene.	In Britain the “fluvio-marine series” of the Isle of Wight; also the volcanic plateaux of Antrim and Inner Hebrides and those of the Faeroe Isles and Iceland. In continental Europe the following subdivisions have been established in descending order: (1) Aquitanian, (2) Stampian (Rupelian), (3) Tongrain (Sannoisian).	On the Atlantic border no equivalents have been satisfactorily recognised, but on the Pacific side there are marine deposits in N. W. Oregon, which may represent this division. In the interior the equivalent is believed to be the fresh-water White River series, including (1) *Protoceras* beds, (2) *Oreodon* beds, and (3) *Titanothervum* beds.
	Eocene.	Barton sands and clays; Ludian series of France. Bracklesham Beds; Lutetian (Calcaire grossier and Caillasses) of Paris basin. London clay, Woolwich and Reading Beds; Thanet sands; Ypresian or Londinian of N. France and Belgium; Sparnacian and Thanetian groups	Woodstock and Aquia Creek groups of Potomac River; Vicksburg, Jackson, Claiborne, Buhrstone, and Lignitic groups of Mississippi. In the interior a thick series of fresh-water formations, comprising, in descending order, the Uinta, Bridger, Wind River, Wasatch, Torrejon,

			and Puerco groups. On the Pacific side the marine Tejon series of Oregon and California.
Mesozoic or Secondary.	Cretaceous. Upper.	Danian—wanting in Britain; uppermost limestone of Denmark. Senonian—Upper Chalk with Flints of England; Aturian and Emscherian stages on the European continent. Turonian—Middle Chalk with few flints, and comprising the Angoumian and Ligerian stages. Cenomanian—Lower Chalk and Chalk Marl. Albian—Upper Greensand and Gault.	On the Atlantic border both marine strata and others containing a terrestrial flora represent the Cretaceous series of formations. In the interior there is also a commingling of marine with lacustrine deposits. At the top lies the Laramie or Lignitic series with an abundant terrestrial flora, passing down into the lacustrine and brackish-water Montana series. Of older date, the Colorado series contains an abundant marine fauna, yet includes also some Niobrara marls and limestones are likewise of marine origin, but the lower members of the series (Benton and Dakota) show another great representation of fresh-water sedimentation with lignites and coals. In California a vast succession of marine deposits (Shasta-Chico) represents the Cretaceous system; and in western British N. America coal-seams also occur.
	Cretaceous. Lower.	Aptian—Lower Greensand; Marls and limestones of Provence, &c. Urgonian (Barremian)—Atherfield clay; massive Hippurite limestones of southern France. Neocomian—Weald clay and Hastings sand; Hauterivian and Valanginian sub-stages of Switzerland and France.	
	Jurassic.	Purbeckian—Purbeck beds; Münder Mergel; largely present in Westphalia. Portlandian—Portland group of England, represented in S. France by the thick Tithonian limestones. Kimmeridgian— Kimmeridge Clay of England; Virgulian and Pterocerian groups of N. France; represented by thick limestones in the Mediterranean basin. Corallian—Coral Rag, Coralline Oolite; Sequanian stages of the Continent, comprising the sub-stages of Astartian and Rauracian. Oxfordian—Oxford Clay; Axgovian and Neuvizyan stages. Callovian—Kellaways Rock, Divesian sub-stage of N. France. Bathonian—series of English strata from Cornbrash down to Fuller’s Earth. Bajocian—Inferior Oolite of England. Lassic—divisible into (1) Upper Lias or Toarcian, (2) Middle Lias, Marlstone or Charmouthian, (3) Lower Lias of Sinemurian and Hettangian.	Representatives of the Middle and lower Jurassic formations have been found in California and Oregon, and farther north among the Arctic islands. Strata containing Lower Jurassic marine fossils appear in Wyoming and Dakota; and above them come the *Atlantosaurus* and *Baptanodon* beds, which have yielded so large a variety of deinosaurs and other vertebrates, and especially the remains of a number of genera of small mammals.
		In Germany and western Europe this division represents the deposits of	In New York, Connecticut, New Brunswick, and

<table>
<tr><td></td><td>Triassic.</td><td colspan="2">inland seas or lagoons, and is divisible into the following stages in descending order: (1) Rhaetic, (2) Keuper, (3) Muschelkalk, (4) Bunter. In the eastern Alps and the Mediterranean basin the contemporaneous sedimentary formations are those of open clear sea, in which a thickness of many thousand feet of strata was accumulated.</td><td>Nova Scotia a series of red sandstone (Newark series) contains land-plants and labyrinthodonts like the lagoon type of central and western Europe. On the Pacific slope, however, marine equivalents occur, representing the pelagic type of south-eastern Europe.</td></tr>
<tr><td rowspan="9">Palaeozoic or Primary.</td><td>Permian.</td><td colspan="2">Thuringian—Zechstein, Magnesian Limestone; named from its development in Thuringia; well represented also in Saxony, Bavaria and Bohemia.
Saxonian—Rothliegendes Group; Red Sandstones, &c.
Autunian—where the strata present the lagoon facies, well displayed at Autun in France; where the marine type is predominant, as in Russia, the group has been termed Artinskian.</td><td>To this division of the geological record the Upper Barren Measures of the coal-fields of Pennsylvania, Prince Edward Island, Nova Scotia and New Brunswick have been assigned.
Farther south in Kansas, Texas, and Nebraska the representatives of the division have an abundant marine fauna.</td></tr>
<tr><td>Carboniferous.</td><td colspan="2">Stephanian or Uralian—represented in Russia by marine formations, and in central and western Europe by numerous small basins containing a peculiar flora and in some places a great variety of insects.
Westphalian or Moscovian—Coal-measures, Millstone Grit.
Culm or Dinantian—Carboniferous Limestone and Calciferous Sandstone series.</td><td>Upper productive Coal-measures.
Lower Barren measures.
Lower productive Coal-measures.
Pottsville conglomerate.
Mauch Chunk shales; limestones of Chester, St Louis, &c.
Pocono series; Kinderhook limestone.</td></tr>
<tr><td rowspan="4">Devonian and Old Red Sandstone.</td><td>Devonian type.</td><td>Old Red Sandstone type.</td><td></td></tr>
<tr><td>Upper
Famennian.
Frasnian.</td><td>Yellow and red sandstone with *Holoptychius*, *Bothriolepis*, &c.</td><td>Catskill red sandstone; Old Red Sandstone type: the strata below show the Devonian type.
Chemung Group.
Genesee Group.</td></tr>
<tr><td>Middle
Givetian.
Eifelian.</td><td>Caithness Flagstones with *Osteolepus*, *Dipterus*, *Homosteus*, &c.</td><td>Hamilton Group.
Marcellus Group.</td></tr>
<tr><td>Lower
Coblentizian.
Gedinnian.</td><td>Red and purple sandstones and conglomerates with *Cephalaspis*, *Pteraspis*,</td><td>Corniferous Limestone.
Onondaga Limestone.
Upper Helderberg Group.
Oriskany Sandstone.</td></tr>
<tr><td rowspan="2">Silurian.</td><td colspan="2">Upper
Ludlow Group.
Wenlock Group.
Llandovery Group.</td><td>Lower Helderberg Group.
Water-Lime.
Niagara Shale and Limestone.
Clinton Group.
Medina Group.</td></tr>
<tr><td colspan="2">Lower (Ordovician)
Ludlow Group.
Wenlock Group.
Llandovery Group.</td><td>Cincinnati Group.
Utica Group.
Trenton Group.
Chazy Group.
Calciferous Group.</td></tr>
<tr><td></td><td colspan="2">Upper or *Olenus* series—Tremadoc slates and *Lingula* Flags.
Middle or *Pardoxides* series—Menevian</td><td>Upper or Potsdam series with *Olenus* and *Dicelocephalus* fauna.</td></tr>
</table>

	Cambrian.	Group. Lower or *Olenellus* series—Llanberis and Harlech Group, and *Olenellus*-zone.	Middle or Acadian series with *Paradoxides* fauna. Lower or Georgian series with *Olenellus* fauna.
Archean, Pre-Cambrian Eozoic.		In Scotland, underneath the Cambrian Olenellus group, lies unconformably a mass of red sandstone and conglomerate (Torridonian) 8000 or 10,000 ft. thick, which rests with a strong gneisses and schists (Lewisian). A thick series of slates and phyllites lies below the oldest Palaeozoic rocks in central Europe, with coarse gneisses below.	In Canada and the Lake Superior region of the United States a vast succession of rocks of Pre-Cambrian age has been grouped into the following subdivisions in descending order: (1) Keweenwan, lying unconformably on (2) Animikie, separated by a strong unconformability from (3) Upper Huronian, (4) Lower Huronian with an unconformable base, (5) Goutchiching, (6) Laurentian. In the eastern part of Canada, Newfoundland, &c., and also in Montana, sedimentary formations of great thickness below the lowest Cambrian zone have been found to contain some obscure organisms.

From these general considerations we proceed to inquire how the existing topographical features of the land arose. Obviously the co-operation of the two great geological agencies of hypogene and epigene energy, which have been at work from the beginning of our globe’s decipherable history, must have been the cause to which these features are to be assigned; and the task of the geologist is to ascertain, if possible, the part that has been taken by each. There is a natural tendency to see in a stupendous piece of scenery, such as a deep ravine, a range of hills, a line of precipice or a chain of mountains, evidence only of subterranean convulsion; and before the subject was taken up as a matter of strict scientific induction, an appeal to former cataclysms was considered a sufficient solution of the problems presented by such features of landscape. The rise of the modern Huttonian school, however, led to a more careful examination of these problems. The important share taken by erosion in the determination of the present features of landscape was then recognized, while a fuller appreciation of the relative parts played by the hypogene and epigene causes has gradually been reached.

1. The study of the progress of denudation at the present time has led to the conclusion that even if the rate of waste were not more rapid than it is to-day, it would yet suffice in a comparatively brief geological period to reduce the dry land to below the sea-level. But not only would the area of the land be diminished by denudation, it could hardly fail to be more or less involved in those widespread movements of subsidence, during which the thick sedimentary formations of the crust appear to have been accumulated. It is thus manifest that there must have been from time to time during the history of our globe upward movements of the crust, whereby the balance between land and sea was redressed. Proofs of such movements have been abundantly preserved among the stratified formations. We there learn that the uplifts have usually followed each other at long intervals between which subsidence prevailed, and thus that there has been a prolonged oscillation of the crust over the great continental areas of the earth’s surface.

An examination of that surface leads to the recognition of two great types of upheaval. In the one, the sea-floor, with all its thick accumulations of sediment, has been carried upwards, sometimes for several thousand feet, so equably that the strata retain their original flatness with hardly any sensible disturbance for hundreds of square miles. In the other type the solid crust has been plicated, corrugated and dislocated, especially along particular lines, and has attained its most stupendous disruption in lofty chains of mountains. Between these two phases of uplift many intermediate stages have been developed, according to the direction and intensity of the subterranean force and the varying nature and disposition of the rocks Of the crust.

(*a*) Where the uplift has extended over wide spaces, without appreciable deformation of the crust, the flat strata have given rise to low plains, or if the amount of uprise has been great enough, to high plains, plateaux or tablelands. The plains of Russia, for example, lie for the most part on such tracts of equably uplifted strata. The great plains of the western interior of the United States form a great plateau or tableland, 5000 or 6000 ft. above the sea, and many thousands of square miles in extent, on which the Rocky Mountains have been ridged up.

(*b*) It is in a great mountain-chain that the complicated structures developed during disturbances of the earth’s crust can best be studied (see Parts IV. and V. of this article), and where the influence of these structures on the topography of the surface is most effectively displayed. Such a chain may be the result of one colossal disturbance; but those of high geological antiquity usually furnish proofs of successive uplifts with more or less intervening denudation. Formed along lines of continental displacement in the crust, they have again and again given relief from the strain of compression by fresh crumpling, fracture and uprise. The chief guide in tracing these successive stages of growth is supplied by unconformability. If, for example, a mountain-range consists of upraised Silurian rocks, upon the upturned and denuded edges of which the Carboniferous Limestone lies transgressively, it is clear that its original upheaval must have taken place in the period of geological time represented by the interval between the Silurian and the Carboniferous Limestone formations. If, as the range is followed along its course, the Carboniferous Limestone is found to be also highly inclined and covered unconformably by the Upper Coal-measures, a second uplift of that portion of the ground can be proved to have taken place between the time of the Limestone and that of the Upper Coal-measures. By this simple and obvious kind of evidence the relative ages of different mountain-chains may be compared. In most great chains, however, the rocks have been so intensely crumpled, and even inverted, that much labour may be required before their true relations can be determined.

The Alps furnish an instructive example of the long series of revolutions through which a great mountain-system may have passed before reaching its present development. The first beginnings of the chain may have been upraised before the oldest Palaeozoic formations were laid down. There are at least traces of land and shore-lines in the Carboniferous period. Subsequent submergences and uplifts appear to have occurred during the Mesozoic periods. There is evidence that thereafter the whole region sank deep under the sea, in which the older Tertiary sediments were accumulated, and which seems to have spread right across the heart of the Old World. But after the deposition of the Eocene formations came the gigantic disruptions whereby all the rocks of the Alpine region were folded over each other, crushed, corrugated, fractured and displaced, some of their older portions, including the fundamental gneisses and schists, being squeezed up, torn off, and pushed horizontally for many miles over the younger rocks. But this upheaval, though the most momentous, was not the last which the chain has undergone, for at a later epoch in Tertiary time renewed disturbance gave rise to a further series of ruptures and plications. The chain thus successively upheaved has been continuously exposed to denudation and has consequently lost much of its original height. That it has been left in a state of instability is indicated by the frequent earthquakes of the Alpine region, which doubtless arise from the sudden snapping of rocks under intense strain.

A distinct type of mountain due to direct hypogene action is to be seen in a volcano. It has been already pointed out (Part IV. sect. 1) that at the vents which maintain a communication between the molten magma of the earth’s interior and the surface, eruptions take place whereby quantities of lava and fragmentary materials are heaped round each orifice of discharge. A typical volcanic mountain takes the form of a perfect cone, but as it grows in size and its main vent is choked, while the sides of the cone are unable to withstand the force of the explosions or the pressure of the ascending column of lava, eruptions take place laterally, and numerous parasitic cones arise on the flanks of the parent mountain. Where lava flows out from long fissures, it may pile up vast sheets of rock, and bury the surrounding country under several thousand feet of solid stone, covering many hundreds of square miles. In this way volcanic tablelands have been formed which, attacked by the denuding forces, are gradually trenched by valleys and ravines, until the original level surface of the lava-field may be almost or wholly lost. As striking examples of this physiographical type reference may be made to the plateau of Abyssinia, the Ghats of India, the plateaux of Antrim, the Inner Hebrides and Iceland, and the great lava-plains of the western territories of the United States.

2. But while the subterranean movements have upraised portions of the surface of the

lithosphere above the level of the ocean, and have thus been instrumental in producing the existing tracts of land, the detailed topographical features of a landscape are not solely, nor in general even chiefly, attributable to these movements. From the time that any portion of the sea-floor appears above sea-level, it undergoes erosion by the various epigene agents. Each climate and geological region has its own development of these agents, which include air, aridity, rapid and frequent alternations of wetness and dryness or of heat and cold, rain, springs, frosts, rivers, glaciers, the sea, plant and animal life. In a dry climate subject to great extremes of temperature the character and rate of decay will differ from those of a moist or an arctic climate. But it must be remembered that, however much they may vary in activity and in the results which they effect, the epigene forces work without intermission, while the hypogene forces bring about the upheaval of land only after long intervals. Hence, trifling as the results during a human life may appear, if we realize the multiplying influence of time we are led to perceive that the apparently feeble superficial agents can, in the course of ages, achieve stupendous transformations in the aspect of the land. If this efficacy may be deduced from what can be seen to be in progress now, it may not less convincingly be shown, from the nature of the sedimentary rocks of the earth’s crust, to have been in progress from the early beginnings of geological history. Side by side with the various upheavals and subsidences, there has been a continuous removal of materials from the land, and an equally persistent deposit of these materials under water, with the consequent growth of new rocks. Denudation has been aptly compared to a process of sculpturing wherein, while each of the implements employed by nature, like a special kind of graving tool, produces its own characteristic impress on the land, they all combine harmoniously towards the achievement of their one common task. Hence the present contours of the land depend partly on the original configuration of the ground, and the influence it may have had in guiding the operations of the erosive agents, partly on the vigour with which these agents perform their work, and partly on the varying structure and powers of resistance possessed by the rocks on which the erosion is carried on.

Where a new tract of land has been raised out of the sea by such an energetic movement as broke up the crust and produced the complicated structure and tumultuous external forms of a great mountain chain, the influence of the hypogene forces on the topography attains its highest development. But even the youngest existing chain has suffered so greatly from denudation that the aspect which it presented at the time of its uplift can only be dimly perceived. No more striking illustration of this feature can be found than that supplied by the Alps, nor one where the geotectonic structures have been so fully studied in detail. On the outer flanks of these mountains the longitudinal ridges and valleys of the Jura correspond with lines of anticline and syncline. Yet though the dominant topographical elements of the region have obviously been produced by the plication of the stratified formations, each ridge has suffered so large an amount of erosion that the younger rocks have been removed from its crest where the older members of the series are now exposed to view, while on every slope proofs may be seen of extensive denudation. If from these long wave-like undulations of the ground, where the relations between the disposition of the rocks below and the forms of the surface are so clearly traceable, the observer proceeds inwards to the main chain, he finds that the plications and displacements of the various formations assume an increasingly complicated character; and that although proofs of great denudation continue to abound, it becomes increasingly difficult to form any satisfactory conjecture as to the shape of the ground when the upheaval ended or any reliable estimate of the amount of material which has since then been removed. Along the central heights the mountains lift themselves towards the sky like the storm-swept crests of vast earth-billows. The whole aspect of the ground suggests intense commotion, and the impression thus given is often much intensified by the twisted and crumpled strata, visible from a long distance, on the crags and crests. On this broken-up surface the various agents of denudation have been ceaselessly engaged since it emerged from the sea. They have excavated valleys, sometimes along depressions provided for them by the subterranean disturbances, sometimes down the slopes of the disrupted blocks of ground. So powerful has been this erosion that valleys cut out along lines of anticline, which were natural ridges, have sometimes become more important than those in lines of syncline, which were structurally depressions. The same subaerial forces have eroded lake-basins, dug out corries or cirques, notched the ridges, splintered the crests and furrowed the slopes, leaving no part of the original surface of the uplifted chain unmodified.

It has often been noted with surprise that features of underground structure which, it might have been confidently anticipated, should have exercised a marked influence on the topography of the surface have not been able to resist the levelling action of the denuding agents, and do not now affect the surface at all. This result is conspicuously seen in coal-fields where the strata are abundantly traversed by faults. These dislocations, having sometimes a displacement of several hundred feet, might have been expected to break up the surface into a network of cliffs and plains; yet in general they do not modify the level character of the ground above. One of the most remarkable faults in Europe is the great thrust which bounds the southern edge of the Belgian coal-field and brings the Devonian rocks above the Coal-measures. It can be traced across Belgium into the Boulonnais, and may not improbably run beneath the Secondary and Tertiary rocks of the south of England. It is crossed by the valleys of the Meuse and other northerly-flowing streams. Yet so indistinctly is it marked in the Meuse valley that no one would suspect its existence from any peculiarity in the general form of the ground, and even an experienced geologist, until he had learned the structure of the district, would scarcely detect any fault at all.

Where faults have influenced the superficial topography, it is usually by giving rise to a hollow along which the subaerial agents and especially running water can act effectively. Such a hollow may be eventually widened and deepened into a valley. On bare crags and crests, lines of fault are apt to be marked by notches or clefts, and they thus help to produce the pinnacles and serrated outlines of these exposed uplands.

It was cogently enforced by Hutton and Playfair, and independently by Lamarck, that no co-operation of underground agency is needed to produce such topography as may be seen in a great part of the world, but that if a tract of sea-floor were upraised into a wide plain, the fall of rain and the circulation of water over its surface would in the end carve out such a system of hills and valleys as may be seen on the dry land now. No such plain would be a dead-level. It would have inequalities on its surface which would serve as channels to guide the drainage from the first showers of rain. And these channels would be slowly widened and deepened until they would become ravines and valleys, while the ground between them would be left projecting as ridges and hills. Nor would the erosion of such a system of water-courses require a long series of geological periods for its accomplishment. From measurements and estimates of the amount of erosion now taking place in the basin of the Mississippi river it has been computed that valleys 800 ft. deep might be carved out in less than a million years. In the vast tablelands of Colorado and other western regions of the United States an impressive picture is presented of the results of mere subaerial erosion on undisturbed and nearly level strata. Systems of stream-courses and valleys, river gorges unexampled elsewhere in the world for depth and length, vast winding lines of escarpment, like ranges of sea-cliffs, terraced slopes rising from plateau to plateau, huge buttresses and solitary stacks standing like islands out of the plains, great mountain-masses towering into picturesque peaks and pinnacles cleft by innumerable gullies, yet everywhere marked by the parallel bars of the horizontal strata out of which they have been carved—these are the orderly symmetrical characteristics of a country where the scenery is due entirely to the action of subaerial agents on the one hand and the varying resistance of perfectly regular 673
stratified rocks on the other.

The details of the sculpture of the land have mainly depended on the nature of the materials on which nature’s erosive tools have been employed. The joints by which all rocks are traversed have been especially serviceable as dominant lines down which the rain has filtered, up which the springs have risen and into which the frost wedges have been driven. On the high bare scarps of a lofty mountain the inner structure of the mass is laid open, and there the system of joints even more than faults is seen to have determined the lines of crest, the vertical walls of cliff and precipice, the forms of buttress and recess, the position of cleft and chasm, the outline of spire and pinnacle. On the lower slopes, even under the tapestry of verdure which nature delights to hang where she can over her naked rocks, we may detect the same pervading influence of the joints upon the forms assumed by ravines and crags. Each kind of stone, too, gives rise to its own characteristic form of scenery. Massive crystalline rocks, such as granite, break up along their joints and often decay into sand or earth along their exposed surfaces, giving rise to rugged crags with long talus slopes at their base. The stratified rocks besides splitting at their joints are especially distinguished by parallel ledges, cornices and recesses, produced by the irregular decay of their component strata, so that they often assume curiously architectural types of scenery. But besides this family feature they display many minor varieties of aspect according to their lithological composition. A range of sandstone hills, for example, presents a marked contrast to one of limestone, and a line of chalk downs to the escarpments formed by alternating bands of harder and softer clays and shales.

It may suffice here merely to allude to a few of the more important parts of the topography of the land in their relation to physiographical geology. A true mountain-chain, viewed from the geological side, is a mass of high ground which owes its prominence to a ridging-up of the earth’s crust, and the intense plication and rupture of the rocks of which it is composed. But ranges of hills almost mountainous in their bulk may be formed by the gradual erosion of valleys out of a mass of original high ground, such as a high plateau or tableland. Eminences which have been isolated by denudation from the main mass of the formations of which they originally formed part are known as “outliers” or “hills of circumdenudation.”

Tablelands, as already pointed out, may be produced either by the upheaval of tracts of horizontal strata from the sea-floor into land; or by the uprise of plains of denudation, where rocks of various composition, structure and age have been levelled down to near or below the level of the sea by the co-operation of the various erosive agents. Most of the great tablelands of the globe are platforms of little-disturbed strata which have been upraised bodily to a considerable elevation. No sooner, however, are they placed in that position than they are attacked by running water, and begin to be hollowed out into systems of valleys. As the valleys sink, the platforms between them grow into narrower and more definite ridges, until eventually the level tableland is converted into a complicated network of hills and valleys, wherein, nevertheless, the key to the whole arrangement is furnished by a knowledge of the disposition and effects of the flow of water. The examples of this process brought to light in Colorado, Wyoming, Nevada and the other western regions by Newberry, King, Hayden, Powell and other explorers, are among the most striking monuments of geological operations in the world.

Examples of ancient and much decayed tablelands formed by the denudation of much disturbed rocks are furnished by the Highlands of Scotland and of Norway. Each of these tracts of high ground consists of some of the oldest and most dislocated formations of Europe, which at a remote period were worn down into a plain, and in that condition may have lain long submerged under the sea and may possibly have been overspread there with younger formations. Having at a much later time been raised several thousand feet above sea-level the ancient platforms of Britain and Scandinavia have been since exposed to denudation, whereby each of them has been so deeply channeled into glens and fjords that it presents to-day a surface of rugged hills, either isolated or connected along the flanks, while only fragments of the general surface of the tableland can here and there be recognized amidst the general destruction.

Valleys have in general been hollowed out by the greater erosive action of running water along the channels of drainage. Their direction has been probably determined in the great majority of cases by irregularities of the surface along which the drainage flowed on the first emergence of the land. Sometimes these irregularities have been produced by folds of the terrestrial crust, sometimes by faults, sometimes by the irregularities on the surface of an uplifted platform of deposition or of denudation. Two dominant trends may be observed among them. Some are longitudinal and run along the line of flexures in the upraised tract of land, others are transverse where the drainage has flowed down the slopes of the ridges into the longitudinal valleys or into the sea. The forms of valleys have been governed partly by the structure and composition of the rocks, and partly by the relative potency of the different denuding agents. Where the influence of rain and frost has been slight, and the streams, supplied from distant sources, have had sufficient declivity, deep, narrow, precipitous ravines or gorges have been excavated. The canyons of the arid region of the Colorado are a magnificent example of this result. Where, on the other hand, ordinary atmospheric action has been more rapid, the sides of the river channels have been attacked, and open sloping glens and valleys have been hollowed out. A gorge or defile is usually due to the action of a waterfall, which, beginning with some abrupt declivity or precipice in the course of the river when it first commenced to flow, or caused by some hard rock crossing the channel, has eaten its way backward.

Lakes have been already referred to, and their modes of origin have been mentioned. As they are continually being filled up with the detritus washed into them from the surrounding regions they cannot be of any great geological antiquity, unless where by some unknown process their basins are from time to time widened and deepened.

In the general subaerial denudation of a country, innumerable minor features are worked out as the structure of the rocks controls the operations of the eroding agents. Thus, among comparatively undisturbed strata, a hard bed resting upon others of a softer kind is apt to form along its outcrop a line of cliff or escarpment. Though a long range of such cliffs resembles a coast that has been worn by the sea, it may be entirely due to mere atmospheric waste. Again, the more resisting portions of a rock may be seen projecting as crags or knolls. An igneous mass will stand out as a bold hill from amidst the more decomposable strata through which it has risen. These features, often so marked on the lower grounds, attain their most conspicuous development among the higher and barer parts of the mountains, where subaerial disintegration is most rapid. The torrents tear out deep gullies from the sides of the declivities. Corries or cirques are scooped out on the one hand and naked precipices are left on the other. The harder bands of rock project as massive ribs down the slopes, shoot up into prominent *aiguilles*, or help to give to the summits the notched saw-like outlines they so often present.

The materials worn from the surface of the higher are spread out over the lower grounds. The streams as they descend begin to drop their freight of sediment when, by the lessening of their declivity, their carrying power is diminished. The great plains of the earth's surface are due to this deposit of gravel, sand and loam. They are thus monuments at once of the destructive and reproductive processes which have been in progress unceasingly since the first land rose above the sea and the first shower of rain fell. Every pebble and particle of their soil, once part of the distant mountains, has travelled slowly and fitfully to lower levels. Again and again have these materials been shifted, ever moving downward and sea-ward. For centuries, perhaps, they have taken their share in the fertility of the plains and have 674
ministered to the nurture of flower and tree, of the bird of the air, the beast of the field and of man himself. But their destiny is still the great ocean. In that bourne alone can they find undisturbed repose, and there, slowly accumulating in massive beds, they will remain until, in the course of ages, renewed upheaval shall raise them into future land, there once more to pass through the same cycle of change.

(A. Ge.)

LITERATURE.—*Historical*: The standard work is Karl A. von Zittel's *Geschichte der Geologie und Paläontologie* (1899), of which there is an abbreviated, but still valuable, English translation; D'Archiac, *Histoire des progrès de la géologie*, deals especially with the period 1834-1850; Keferstein, *Geschichte und Literatur der Geognosie*, gives a summary up to 1840; while Sir A. Geikie's *Founders of Geology* (1897; 2nd ed., 1906) deals more particularly with the period 1750-1820. General treatises: Sir Charles Lyell's *Principles of Geology* is a classic. Of modern English works, Sir A. Geikie's *Text Book of Geology* (4th ed., 1903) occupies the first place; the work of T.C. Chamberlin and R.D. Salisbury, *Geology*; *Earth History* (3 vols., 1905-1906), is especially valuable for American geology. A. de Lapparent's *Traité de géologie* (5th ed., 1906), is the standard French work. H. Credner's *Elemente der Geologie* has gone through several editions in Germany. Dynamical and physiographical geology are elaborately treated by E. Suess, *Das Antlitz der Erde*, translated into English, with the title *The Face of the Earth*. The practical study of the science is treated of by F. von Richthofen, *Führer für Forschungsreisende* (1886); G.A. Cole, *Aids in Practical Geology* (5th ed., 1906); A. Geikie, *Outlines of Field Geology* (5th ed., 1900). The practical applications of Geology are discussed by J.V. Elsden, *Applied Geology* (1898-1899). The relations of Geology to scenery are dealt with by Sir A. Geikie, *Scenery of Scotland* (3rd ed., 1901); J.E. Marr, *The Scientific Study of Scenery* (1900); Lord Avebury, *The Scenery of Switzerland* (1896); *The Scenery of England* (1902); and J. Geikie, *Earth Sculpture* (1898). A detailed bibliography is given in Sir A. Geikie's *Text Book of Geology*. See also the separate articles on geological subjects for special references to authorities.

1 In De Luc's *Lettres physiques et morales sur les montagnes* (1778), the word "cosmology" is used for our science, the author stating that "geology" is more appropriate, but it "was not a word in use." In a completed edition, published in 1779, the same statement is made, but "geology" occurs in the text; in the same year De Saussure used the word without any explanation, as if it were well known.

2 The subject of the age of the earth has also been discussed by Professor J. Joly and Professor W.J. Sollas. The former geologist, approaching the question from a novel point of view, has estimated the total quantity of sodium in the water of the ocean and the quantity of that element received annually by the ocean from the denudation of the land. Dividing the one sum by the other, he arrives at the result that the probable age of the earth is between 90 and 100 millions of years (*Trans. Roy. Dublin Soc.* ser. ii. vol. vii., 1899, p. 23: *Geol. Mag.*, 1900, p. 220). Professor Sollas believes that this limit exceeds what is required for the evolution of geological history, that the lower limit assigned by Lord Kelvin falls short of what the facts demand, and that geological time will probably be found to have been comprised within some indeterminate period between these limits. (Address to Section C, *Brit. Assoc. Report*, 1900; *Age of the Earth*, London, 1905.)

GEOMETRICAL CONTINUITY. In a report of the Institute prefixed to Jean Victor Poncelet's *Traité des propriétés projectives des figures* (Paris, 1822), it is said that he employed “ce qu’il appelle le principe de continuité.” The law or principle thus named by him had, he tells us, been tacitly assumed as axiomatic by “les plus savans géomètres.” It had in fact been enunciated as “lex continuationis,” and “la loi de la continuité,” by Gottfried Wilhelm Leibnitz (Oxf. N.E.D.), and previously under another name by Johann Kepler in cap. iv. 4 of his *Ad Vitellionem paralipomena quibus astronomiae pars optica traditur* (Francofurti, 1604). Of sections of the cone, he says, there are five species from the “recta linea” or line-pair to the circle. From the line-pair we pass through an infinity of hyperbolas to the parabola, and thence through an infinity of ellipses to the circle. Related to the sections are certain remarkable points which have no name. Kepler calls them foci. The circle has one focus at the centre, an ellipse or hyperbola two foci equidistant from the centre. The parabola has one focus within it, and another, the “caecus focus,” which may be imagined to be *at infinity* on the axis *within or without the curve*. The line from it to any point of the section is parallel to the axis. To carry out the analogy we must speak paradoxically, and say that the line-pair likewise has foci, which in this case coalesce as in the circle and fall upon the lines themselves; for our geometrical terms should be subject to analogy. Kepler dearly loves analogies, his most trusty teachers, acquainted with all the secrets of nature, “*omnium naturae arcanorum conscios*.” And they are to be especially regarded in geometry as, by the use of “however absurd expressions,” classing extreme limiting forms with an infinity of intermediate cases, and placing the whole essence of a thing clearly before the eyes.

Here, then, we find formulated by Kepler the doctrine of the concurrence of parallels at a single point at infinity and the principle of continuity (under the name analogy) in relation to the infinitely great. Such conceptions so strikingly propounded in a famous work could not escape the notice of contemporary mathematicians. Henry Briggs, in a letter to Kepler from Merton College, Oxford, dated “10 Cal. Martiis 1625,” suggests improvements in the *Ad Vitellionem paralipomena*, and gives the following construction: Draw a line CBADC, and let an ellipse, a parabola, and a hyperbola have B and A for focus and vertex. Let CC be the other foci of the ellipse and the hyperbola. Make AD equal to AB, and with centres CC and radius in each case equal to CD describe circles. Then any point of the ellipse is equidistant from the focus B and one circle, and any point of the hyperbola from the focus B and the other circle. Any point P of the parabola, in which the second focus is missing or infinitely distant, is equidistant from the focus B and the line through D which we call the directrix, this taking the place of either circle when its centre C is at infinity, and every line CP being then parallel to the axis. Thus Briggs, and we know not how many “savans géomètres” who have left no record, had already taken up the new doctrine in geometry in its author’s lifetime. Six years after Kepler’s death in 1630 Girard Desargues, “the Monge of his age,” brought out the first of his remarkable works founded on the same principles, a short tract entitled *Méthode universelle de mettre en perspective les objets donnés réellement ou en devis* (Paris, 1636); but “Le privilége étoit de 1630.” (Poudra, *Œuvres de Des.*, i. 55). Kepler as a modern geometer is best known by his *New Stereometry of Wine Casks* (Lincii, 1615), in which he replaces the circuitous Archimedean method of exhaustion by a direct “royal road” of infinitesimals, treating a vanishing arc as a straight line and regarding a curve as made up of a succession of short chords. Some 2000 years previously one Antipho, probably the well-known opponent of Socrates, has regarded a circle in like manner as the limiting form of a many-sided inscribed rectilinear figure. Antipho’s notion was rejected by the men of his day as unsound, and when reproduced by Kepler it was again stoutly opposed as incapable of any sort of geometrical demonstration—not altogether without reason, for it rested on an assumed law of continuity rather than on palpable proof.

To complete the theory of continuity, the one thing needful was the idea of imaginary points implied in the algebraical geometry of René Descartes, in which equations between variables representing co-ordinates were found often to have imaginary roots. Newton, in his two sections on “Inventio orbium” (*Principia* i. 4, 5), shows in his brief way that he is familiar with the principles of modern geometry. In two propositions he uses an auxiliary line which is supposed to cut the conic in X and Y, but, as he remarks at the end of the second (prop. 24), it may not cut it at all. For the sake of brevity he passes on at once with the observation that the required constructions are evident from the case in which the line cuts the trajectory. In the scholium appended to prop. 27, after saying that an asymptote is a tangent at infinity, he gives an unexplained general construction for the axes of a conic, which seems to imply that it has asymptotes. In all such cases, having equations to his loci in the background, he may have thought of elements of the figure as passing into the imaginary state in such manner as not to vitiate conclusions arrived at on the hypothesis of their reality.

Roger Joseph Boscovich, a careful student of Newton’s works, has a full and thorough

discussion of geometrical continuity in the third and last volume of his *Elementa universae matheseos* (ed. prim. Venet, 1757), which contains *Sectionum conicarum elementa nova quadam methodo concinnata et dissertationem de transformatione locorum geometricorum, ubi de continuitatis lege, et de quibusdam infiniti mysteriis*. His first principle is that all varieties of a defined locus have the same properties, so that what is demonstrable of one should be demonstrable in like manner of all, although some artifice may be required to bring out the underlying analogy between them. The opposite extremities of an infinite straight line, he says, are to be regarded as joined, as if the line were a circle having its centre at the infinity on either side of it. This leads up to the idea of a *veluti plus quam infinita extensio*, a line-circle containing, as we say, the line infinity. Change from the real to the imaginary state is contingent upon the passage of some element of a figure through zero or infinity and never takes place *per saltum*. Lines being some positive and some negative, there must be negative rectangles and negative squares, such as those of the exterior diameters of a hyperbola. Boscovich's first principle was that of Kepler, by whose *quantumvis absurdis locutionibus* the boldest applications of it are covered, as when we say 675
with Poncelet that all concentric circles in a plane touch one another in two imaginary fixed points at infinity. In G.K. Ch. von Staudt's *Geometrie der Lage and Beiträge zur G. der L.* (Nürnberg, 1847, 1856-1860) the geometry of position, including the extension of the field of pure geometry to the infinite and the imaginary, is presented as an independent science, "welche des Messens nicht bedarf." (See Geometry: *Projective*.)

Ocular illusions due to distance, such as Roger Bacon notices in the *Opus majus* (i. 126, ii. 108, 497; Oxford, 1897), lead up to or illustrate the mathematical uses of the infinite and its reciprocal the infinitesimal. Specious objections can, of course, be made to the anomalies of the law of continuity, but they are inherent in the higher geometry, which has taught us so much of the "secrets of nature." Kepler's excursus on the "analogy" between the conic sections hereinbefore referred to is given at length in an article on "The Geometry of Kepler and Newton" in vol. xviii. of the *Transactions of the Cambridge Philosophical Society* (1900). It had been generally overlooked, until attention was called to it by the present writer in a note read in 1880 (*Proc. C.P.S.* iv. 14-17), and shortly afterwards in *The Ancient and Modern Geometry of Conics, with Historical Notes and Prolegomena* (Cambridge 1881).

(C. T.*)

GEOMETRY, the general term for the branch of mathematics which has for its province the study of the properties of space. From experience, or possibly intuitively, we characterize existent space by certain fundamental qualities, termed axioms, which are insusceptible of proof; and these axioms, in conjunction with the mathematical entities of the point, straight line, curve, surface and solid, appropriately defined, are the premises from which the geometer draws conclusions. The geometrical axioms are merely conventions; on the one hand, the system may be based upon inductions from experience, in which case the deduced geometry may be regarded as a branch of physical science; or, on the other hand, the system may be formed by purely logical methods, in which case the geometry is a phase of pure mathematics. Obviously the geometry with which we are most familiar is that of existent space—the three-dimensional space of experience; this geometry may be termed Euclidean, after its most famous expositor. But other geometries exist, for it is possible to frame systems of axioms which definitely characterize some other kind of space, and from these axioms to deduce a series of non-contradictory propositions; such geometries are called non-Euclidean.

It is convenient to discuss the subject-matter of geometry under the following headings:

I. *Euclidean Geometry*: a discussion of the axioms of existent space and of the geometrical entities, followed by a synoptical account of Euclid's Elements.

II. *Projective Geometry*: primarily Euclidean, but differing from I. in employing the notion of geometrical continuity (*q.v.*)—points and lines at infinity.

III. *Descriptive Geometry*: the methods for representing upon planes figures placed in space of three dimensions.

IV. *Analytical Geometry*: the representation of geometrical figures and their relations by algebraic equations.

V. *Line Geometry*: an analytical treatment of the line regarded as the space element.

VI. *Non-Euclidean Geometry*: a discussion of geometries other than that of the space of experience.

VII. *Axioms of Geometry*: a critical analysis of the foundations of geometry.

Special subjects are treated under their own headings: *e.g.* Projection, Perspective; Curve, Surface; Circle, Conic Section; Triangle, Polygon, Polyhedron; there are also articles on special curves and figures, *e.g.* Ellipse, Parabola, Hyperbola; Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron; Cardioid, Catenary, Cissoid, Conchoid, Cycloid, Epicycloid, Limaçon, Oval, Quadratrix, Spiral, &c.

History.—The origin of geometry (Gr. γῆ, earth, μέτρον, a measure) is, according to Herodotus, to be found in the etymology of the word. Its birthplace was Egypt, and it arose from the need of surveying the lands inundated by the Nile floods. In its infancy it therefore consisted of a few rules, very rough and approximate, for computing the areas of triangles and quadrilaterals; and, with the Egyptians, it proceeded no further, the geometrical entities—the point, line, surface and solid—being only discussed in so far as they were involved in practical affairs. The point was realized as a mark or position, a straight line as a stretched string or the tracing of a pole, a surface as an area; but these units were not abstracted; and for the Egyptians geometry was only an art—an auxiliary to surveying.[1] The first step towards its elevation to the rank of a science was made by Thales (*q.v.*) of Miletus, who transplanted the elementary Egyptian mensuration to Greece. Thales clearly abstracted the notions of points and lines, founding the geometry of the latter unit, and discovering *per saltum* many propositions concerning areas, the circle, &c. The empirical rules of the Egyptians were corrected and developed by the Ionic School which he founded, especially by Anaximander and Anaxagoras, and in the 6th century B.C. passed into the care of the Pythagoreans. From this time geometry exercised a powerful influence on Greek thought. Pythagoras (*q.v.*), seeking the key of the universe in arithmetic and geometry, investigated logically the principles underlying the known propositions; and this resulted in the formulation of definitions, axioms and postulates which, in addition to founding a *science* of geometry, permitted a crystallization, fractional, it is true, of the amorphous collection of material at hand. Pythagorean geometry was essentially a geometry of areas and solids; its goal was the regular solids—the tetrahedron, cube, octahedron, dodecahedron and icosahedron—which symbolized the five elements of Greek cosmology. The geometry of the circle, previously studied in Egypt and much more seriously by Thales, was somewhat neglected, although this curve was regarded as the most perfect of all plane figures and the sphere the most perfect of all solids. The circle, however, was taken up by the Sophists, who made most of their discoveries in attempts to solve the classical problems of squaring the circle, doubling the cube and trisecting an angle. These problems, besides stimulating pure geometry, *i.e.* the geometry of constructions made by the ruler and compasses, exercised considerable influence in other directions. The first problem led to the discovery of the method of *exhaustion* for determining areas. Antiphon inscribed a square in a circle, and on each side an isosceles triangle having its vertex on the circle; on the sides of the octagon so obtained, isosceles triangles were again constructed, the process leading to inscribed polygons of 8, 16 and 32 sides; and the areas of these polygons, which are easily determined, are successive approximations to the area of the circle. Bryson of Heraclea took an important step when he circumscribed, in addition to inscribing, polygons to a circle, but he committed an error in treating the circle as the mean of the two polygons. The method of Antiphon, in assuming that by continued division a polygon can be constructed coincident with the circle, demanded that magnitudes are not infinitely divisible. Much controversy ranged about this point; Aristotle supported the doctrine of infinite divisibility; Zeno attempted to show its absurdity. The mechanical tracing of loci, a principle initiated by Archytas of Tarentum to solve the last two problems, was a frequent subject for study, and several mechanical curves were thus discovered at subsequent dates (cissoid, conchoid, quadratrix). Mention may be made of Hippocrates, who, besides developing the known methods, made a study of similar figures, and, as a consequence, of proportion. This step is important as bringing into line discontinuous number and continuous magnitude.

A fresh stimulus was given by the succeeding Platonists, who, accepting in part the Pythagorean cosmology, made the study of geometry preliminary to that of philosophy. The many discoveries made by this school were facilitated in no small measure by the clarification of the axioms and definitions, the logical sequence of propositions which was adopted, and, more especially, by the formulation of the analytic method, *i.e.* of assuming the truth of a proposition and then reasoning to a known truth. The main strength of the Platonist geometers lies in stereometry or the geometry of solids. The Pythagoreans had dealt with the sphere and regular solids, but the pyramid, prism, cone and cylinder were but little known until the Platonists took them in hand. Eudoxus established their mensuration, proving the pyramid and cone to have one-third the content of a prism and cylinder on the same base and of the same height, and was probably the discoverer of a proof that the volumes of spheres are as the cubes of their radii. The discussion of sections of the cone and cylinder led to the discovery of the three curves named the parabola, ellipse and hyperbola (see CONIC SECTION); it is difficult to over-estimate the importance of this discovery; its investigation marks the crowning achievement of Greek geometry, and led in later years to the fundamental theorems and methods of modern geometry.

The presentation of the subject-matter of geometry as a connected and logical series of propositions, prefaced by Ὅροι or foundations, had been attempted by many; but it is to Euclid that we owe a complete exposition. Little indeed in the *Elements* is probably original except the arrangement; but in this Euclid surpassed such predecessors as Hippocrates, Leon, pupil of Neocleides, and Theudius of Magnesia, devising an apt logical model, although when scrutinized in the light of modern mathematical conceptions the proofs are riddled with fallacies. According to the commentator Proclus, the *Elements* were written with a twofold object, first, to introduce the novice to geometry, and secondly, to lead him to the regular solids; conic sections found no place therein. What Euclid did for the line and circle, Apollonius did for the conic sections, but there we have a discoverer as well as editor. These two works, which contain the greatest contributions to ancient geometry, are treated in detail in Section I. *Euclidean Geometry* and the articles EUCLID; CONIC SECTION; APPOLONIUS. Between Euclid and Apollonius there flourished the illustrious Archimedes, whose geometrical discoveries are mainly concerned with the mensuration of the circle and conic sections, and of the sphere, cone and cylinder, and whose greatest contribution to geometrical method is the elevation of the method of exhaustion to the dignity of an instrument of research. Apollonius was followed by Nicomedes, the inventor of the conchoid; Diocles, the inventor of the cissoid; Zenodorus, the founder of the study of isoperimetrical figures; Hipparchus, the founder of trigonometry; and Heron the elder, who wrote after the manner of the Egyptians, and primarily directed attention to problems of practical surveying.

Of the many isolated discoveries made by the later Alexandrian mathematicians, those of Menelaus are of importance. He showed how to treat spherical triangles, establishing their properties and determining their congruence; his theorem on the products of the segments in which the sides of a triangle are cut by a line was the foundation on which Carnot erected his theory of transversals. These propositions, and also those of Hipparchus, were utilized and developed by Ptolemy (*q.v.*), the expositor of trigonometry and discoverer of many isolated propositions. Mention may be made of the commentator Pappus, whose *Mathematical Collections* is valuable for its wealth of historical matter; of Theon, an editor of Euclid's *Elements* and commentator of Ptolemy's *Almagest*; of Proclus, a commentator of Euclid; and of Eutocius, a commentator of Apollonius and Archimedes.

The Romans, essentially practical and having no inclination to study science *qua* science, only had a geometry which sufficed for surveying; and even here there were abundant inaccuracies, the empirical rules employed being akin to those of the Egyptians and Heron. The Hindus, likewise, gave more attention to computation, and their geometry was either of Greek origin or in the form presented in trigonometry, more particularly connected with arithmetic. It had no logical foundations; each proposition stood alone; and the results were empirical. The Arabs more closely followed the Greeks, a plan adopted as a sequel to the translation of the works of Euclid, Apollonius, Archimedes and many others into Arabic. Their chief contribution to geometry is exhibited in their solution of algebraic equations by intersecting conics, a step already taken by the Greeks in isolated cases, but only elevated into a *method* by Omar al Hayyami, who flourished in the 11th century. During the middle ages little was added to Greek and Arabic geometry. Leonardo of Pisa wrote a *Practica geometriae* (1220), wherein Euclidean methods are employed; but it was not until the 14th century that geometry, generally Euclid's *Elements*, became an essential item in university curricula. There was, however, no sign of original development, other branches of mathematics, mainly algebra and trigonometry, exercising a greater fascination until the 16th century, when the subject again came into favour.

The extraordinary mathematical talent which came into being in the 16th and 17th centuries reacted on geometry and gave rise to all those characters which distinguish modern from ancient geometry. The first innovation of moment was the formulation of the principle of geometrical continuity by Kepler. The notion of infinity which it involved permitted generalizations and systematizations hitherto unthought of (see GEOMETRICAL

Continuity); and the method of indefinite division applied to rectification, and quadrature and cubature problems avoided the cumbrous method of exhaustion and provided more accurate results. Further progress was made by Bonaventura Cavalieri, who, in his *Geometria indivisibilibus continuorum* (1620), devised a method intermediate between that of exhaustion and the infinitesimal calculus of Leibnitz and Newton. The logical basis of his system was corrected by Roberval and Pascal; and their discoveries, taken in conjunction with those of Leibnitz, Newton, and many others in the fluxional calculus, culminated in the branch of our subject known as differential geometry (see Infinitesimal Calculus; Curve; Surface).

A second important advance followed the recognition that conics could be regarded as projections of a circle, a conception which led at the hands of Desargues and Pascal to modern *projective geometry* and *perspective*. A third, and perhaps the most important, advance attended the application of algebra to geometry by Descartes, who thereby founded *analytical geometry*. The new fields thus opened up were diligently explored, but the calculus exercised the greatest attraction and relatively little progress was made in geometry until the beginning of the 19th century, when a new era opened.

Gaspard Monge was the first important contributor, stimulating analytical and differential geometry and founding *descriptive geometry* in a series of papers and especially in his lectures at the École polytechnique. Projective geometry, founded by Desargues, Pascal, Monge and L.N.M. Carnot, was crystallized by J.V. Poncelet, the creator of the modern methods. In his *Traité des propriétés des figures* (1822) the line and circular points at infinity, imaginaries, polar reciprocation, homology, cross-ratio and projection are systematically employed. In Germany, A.F. Möbius, J. Plücker and J. Steiner were making far-reaching contributions. Möbius, in his *Barycentrische Calcul* (1827), introduced homogeneous co-ordinates, and also the powerful notion of geometrical transformation, including the special cases of collineation and duality; Plücker, in his *Analytisch-geometrische Entwickelungen* (1828-1831), and his *System der analytischen Geometrie* (1835), introduced the abridged notation, line and plane co-ordinates, and the conception of generalized space elements; while Steiner, besides enriching geometry in numerous directions, was the first to systematically generate figures by projective pencils. We may also notice M. Chasles, whose *Aperçu historique* (1837) is a classic. Synthetic geometry, characterized by its fruitfulness and beauty, attracted most attention, and it so happened that its originally weak logical foundations became replaced by a more substantial set of axioms. These were found in the anharmonic ratio, a device leading to the liberation of synthetic geometry from metrical relations, and in involution, which yielded rigorous definitions of imaginaries. These innovations were made by K.J.C. von Staudt. Analytical geometry was stimulated by the algebra of invariants, a subject much developed by A. Cayley, G. Salmon, S.H. Aronhold, L.O. Hesse, and more particularly by R.F.A. Clebsch.

The introduction of the line as a space element, initiated by H. Grassmann (1844) and 677
Cayley (1859), yielded at the hands of Plücker a new geometry, termed *line geometry*, a subject developed more notably by F. Klein, Clebsch, C.T. Reye and F.O.R. Sturm (see Section V., *Line Geometry*).

Non-euclidean geometries, having primarily their origin in the discussion of Euclidean parallels, and treated by Wallis, Saccheri and Lambert, have been especially developed during the 19th century. Four lines of investigation may be distinguished:—the naïve-synthetic, associated with Lobatschewski, Bolyai, Gauss; the metric differential, studied by Riemann, Helmholtz, Beltrami; the projective, developed by Cayley, Klein, Clifford; and the critical-synthetic, promoted chiefly by the Italian mathematicians Peano, Veronese, Burali-Forte, Levi Cività, and the Germans Pasch and Hilbert.

(C. E.*)

I. Euclidean Geometry

This branch of the science of geometry is so named since its methods and arrangement are those laid down in Euclid’s *Elements*.

§ 1. *Axioms.*—The object of geometry is to investigate the properties of space. The first step must consist in establishing those fundamental properties from which all others follow by processes of deductive reasoning. They are laid down in the Axioms, and these ought to form such a system that nothing need be added to them in order fully to characterize space, and that nothing may be omitted without making the system incomplete. They must, in fact, completely “define” space.

§ 2. *Definitions.*—The axioms of Euclidean Geometry are obtained from inspection of existent space and of solids in existent space,—hence from experience. The same source gives us the notions of the geometrical entities to which the axioms relate, viz. solids, surfaces, lines or curves, and points. A solid is directly given by experience; we have only to abstract all material from it in order to gain the notion of a geometrical solid. This has shape, size, position, and may be moved. Its boundary or boundaries are called surfaces. They separate one part of space from another, and are said to have no thickness. Their boundaries are curves or lines, and these have length only. Their boundaries, again, are points, which have no magnitude but only position. We thus come in three steps from solids to points which have no magnitude; in each step we lose one extension. Hence we say a solid has three dimensions, a surface two, a line one, and a point none. Space itself, of which a solid forms only a part, is also said to be of three dimensions. The same thing is intended to be expressed by saying that a solid has length, breadth and thickness, a surface length and breadth, a line length only, and a point no extension whatsoever.

Euclid gives the essence of these statements as definitions:—

Def. 1, I. *A point is that which has no parts, or which has no magnitude.*

Def. 2, I. *A line is length without breadth.*

Def. 5, I. *A superficies is that which has only length and breadth.*

Def. 1, XI. *A solid is that which has length, breadth and thickness.*

It is to be noted that the synthetic method is adopted by Euclid; the analytical derivation of the successive ideas of "surface," "line," and "point" from the experimental realization of a "solid" does not find a place in his system, although possessing more advantages.

If we allow motion in geometry, we may generate these entities by moving a point, a line, or a surface, thus:—

The path of a moving point is a line.
The path of a moving line is, in general, a surface.
The path of a moving surface is, in general, a solid.

And we may then assume that the lines, surfaces and solids, as defined before, can all be generated in this manner. From this generation of the entities it follows again that the boundaries—the first and last position of the moving element—of a line are points, and so on; and thus we come back to the considerations with which we started.

Euclid points this out in his definitions,—Def. 3, I., Def. 6, I., and Def. 2, XI. He does not, however, show the connexion which these definitions have with those mentioned before. When points and lines have been defined, a statement like Def. 3, I., "The extremities of a line are points," is a proposition which either has to be proved, and then it is a theorem, or which has to be taken for granted, in which case it is an axiom. And so with Def. 6, I., and Def. 2, XI.

§ 3. Euclid's definitions mentioned above are attempts to describe, in a few words, notions which we have obtained by inspection of and abstraction from solids. A few more notions have to be added to these, principally those of the simplest line—the straight line, and of the simplest surface—the flat surface or plane. These notions we possess, but to define them accurately is difficult. Euclid's Definition 4, I., "A straight line is that which lies evenly between its extreme points," must be meaningless to any one who has not the notion of straightness in his mind. Neither does it state a property of the straight line which can be used in any further investigation. Such a property is given in Axiom 10, I. It is really this axiom, together with Postulates 2 and 3, which characterizes the straight line.

Whilst for the straight line the verbal definition and axiom are kept apart, Euclid mixes them up in the case of the plane. Here the Definition 7, I., includes an axiom. It defines a plane as a surface which has the property that every straight line which joins any two points in it lies altogether in the surface. But if we take a straight line and a point in such a surface, and draw all straight lines which join the latter to all points in the first line, the surface will be fully determined. This construction is therefore sufficient as a definition. That every other straight line which joins any two points in this surface lies altogether in it is a further property, and to assume it gives another axiom.

Thus a number of Euclid's axioms are hidden among his first definitions. A still greater confusion exists in the present editions of Euclid between the postulates and axioms so called, but this is due to later editors and not to Euclid himself. The latter had the last three axioms put together with the postulates (αἰτήματα), so that these were meant to include all assumptions relating to space. The remaining assumptions, which relate to magnitudes in general, viz. the first eight "axioms" in modern editions, were called "common notions" (κοιναὶ ἔννοιαι). Of the latter a few may be said to be definitions. Thus the eighth might be taken as a definition of "equal," and the seventh of "halves." If we wish to collect the axioms used in Euclid's *Elements*, we have therefore to take the three postulates, the last three axioms as generally given, a few axioms hidden in the definitions, and an axiom used by Euclid in the proof of Prop. 4, I, and on a few other occasions, viz. that figures may be moved in space without change of shape or size.

§ 4. *Postulates.*—The assumptions actually made by Euclid may be stated as follows:—

(1) Straight lines exist which have the property that any one of them may be produced both ways without limit, that through any two points in space such a line may be drawn, and that any two of them coincide throughout their indefinite extensions as soon as two points in the one coincide with two points in the other. (This gives the contents of Def. 4, part of Def. 35, the first two Postulates, and Axiom 10.)

(2) Plane surfaces or planes exist having the property laid down in Def. 7, that every straight line joining any two points in such a surface lies altogether in it.

(3) Right angles, as defined in Def. 10, are possible, and all right angles are equal; that is to say, wherever in space we take a plane, and wherever in that plane we construct a right angle, all angles thus constructed will be equal, so that any one of them may be made to coincide with any other. (Axiom 11.)

(4) The 12th Axiom of Euclid. This we shall not state now, but only introduce it when we cannot proceed any further without it.

(5) Figures maybe freely moved in space without change of shape or size. This is assumed by Euclid, but not stated as an axiom.

(6) In any plane a circle may be described, having any point in that plane as centre, and its distance from any other point in that plane as radius. (Postulate 3.)

The definitions which have not been mentioned are all "nominal definitions," that is to say, they fix a name for a thing described. Many of them overdetermine a figure.

§ 5. Euclid's *Elements* (see Euclid) are contained in thirteen books. Of these the first four and the sixth are devoted to "plane geometry," as the investigation of figures in a plane is generally called. The 5th book contains the theory of proportion which is used in Book VI. 678
The 7th, 8th and 9th books are purely arithmetical, whilst the 10th contains a most ingenious treatment of geometrical irrational quantities. These four books will be excluded from our survey. The remaining three books relate to figures in space, or, as it is generally called, to "solid geometry." The 7th, 8th, 9th, 10th, 13th and part of the 11th and 12th books are now generally omitted from the school editions of the *Elements*. In the first four and in the 6th book it is to be understood that all figures are drawn in a plane.

Book I. of Euclid's "Elements."

§ 6. According to the third postulate it is possible to draw in any plane a circle which has its centre at any given point, and its radius equal to the distance of this point from any other point given in the plane. This makes it possible (Prop. 1) to construct on a given line AB an equilateral triangle, by drawing first a circle with A as centre and AB as radius, and then a circle with B as centre and BA as radius. The point where these circles intersect—that they intersect Euclid quietly assumes—is the vertex of the required triangle. Euclid does not suppose, however, that a circle may be drawn which has its radius equal to the distance between any two points unless one of the points be the centre. This implies also that we are not supposed to be able to make any straight line equal to any other straight line, or to carry a distance about in space. Euclid therefore next solves the problem: It is required along a given straight line from a point in it to set off a distance equal to the length of another straight line given anywhere in the plane. This is done in two steps. It is shown in Prop. 2 how a straight line may be drawn from a given point equal in length to another given straight line not drawn from that point. And then the problem itself is solved in Prop. 3, by drawing first through the given point some straight line of the required length, and then about the same point as centre a circle having this length as radius. This circle will cut off from the given straight line a length equal to the required one. Nowadays, instead of going through this long process, we take a pair of compasses and set off the given length by its aid. This assumes that we may move a length about without changing it. But Euclid has not assumed it, and this proceeding would be fully justified by his desire not to take for granted more than was necessary, if he were not obliged at his very next step actually to make this assumption, though without stating it.

§ 7. We now come (in Prop. 4) to the first theorem. It is the fundamental theorem of Euclid's whole system, there being only a very few propositions (like Props. 13, 14, 15, I.), except those in the 5th book and the first half of the 11th, which do not depend upon it. It is stated very accurately, though somewhat clumsily, as follows:—

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite.

That is to say, the triangles are "identically" equal, and one may be considered as a copy of the other. The proof is very simple. The first triangle is taken up and placed on the second, so that the parts of the triangles which are known to be equal fall upon each other. It is then easily seen that also the remaining parts of one coincide with those of the other, and that they are therefore equal. This process of applying one figure to another Euclid scarcely uses again, though many proofs would be simplified by doing so. The process introduces motion into geometry, and includes, as already stated, the axiom that figures may be moved without change of shape or size.

If the last proposition be applied to an isosceles triangle, which has two sides equal, we obtain the theorem (Prop. 5), *if two sides of a triangle are equal, then the angles opposite these sides are equal*.

Euclid's proof is somewhat complicated, and a stumbling-block to many schoolboys. The proof becomes much simpler if we consider the isosceles triangle ABC (AB = AC) twice over, once as a triangle BAC, and once as a triangle CAB; and now remember that AB, AC in the first are equal respectively to AC, AB in the second, and the angles included by these sides are equal. Hence the triangles are equal, and the angles in the one are equal to those in the other, viz. those which are opposite equal sides, *i.e.* angle ABC in the first equals angle ACB in the second, as they are opposite the equal sides AC and AB in the two triangles.

There follows the converse theorem (Prop. 6). *If two angles in a triangle are equal, then the sides opposite them are equal,—i.e.* the triangle is isosceles. The proof given consists in what is called a *reductio ad absurdum*, a kind of proof often used by Euclid, and principally in proving the converse of a previous theorem. It assumes that the theorem to be proved is wrong, and then shows that this assumption leads to an absurdity, *i.e.* to a conclusion which is in contradiction to a proposition proved before—that therefore the assumption made cannot be true, and hence that the theorem is true. It is often stated that Euclid invented this kind of proof, but the method is most likely much older.

§ 8. It is next proved that *two triangles which have the three sides of the one equal respectively to those of the other are identically equal, hence that the angles of the one are equal respectively to those of the other, those being equal which are opposite equal sides*. This is Prop. 8, Prop. 7 containing only a first step towards its proof.

These theorems allow now of the solution of a number of problems, viz.:—

To bisect a given angle (Prop. 9).

To bisect a given finite straight line (Prop. 10).

To draw a straight line perpendicularly to a given straight line through a given point in it (Prop. 11), *and also through a given point not in it* (Prop. 12).

The solutions all depend upon properties of isosceles triangles.

§ 9. The next three theorems relate to angles only, and might have been proved before Prop. 4, or even at the very beginning. The first (Prop. 13) says, *The angles which one straight line makes with another straight line on one side of it either are two right angles or are together equal to two right angles*. This theorem would have been unnecessary if Euclid had admitted the notion of an angle such that its two limits are in the same straight line, and had besides defined the sum of two angles.

Its converse (Prop. 14) is of great use, inasmuch as it enables us in many cases to prove that two straight lines drawn from the same point are one the continuation of the other. So also is

Prop. 15. *If two straight lines cut one another, the vertical or opposite angles shall be equal.*

§ 10. Euclid returns now to properties of triangles. Of great importance for the next steps (though afterwards superseded by a more complete theorem) is

Prop. 16. *If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.*

Prop. 17. *Any two angles of a triangle are together less than two right angles, is an immediate consequence of it.* By the aid of these two, the following fundamental properties of triangles are easily proved:—

Prop. 18. *The greater side of every triangle has the greater angle opposite to it*;

Its converse, Prop. 19. *The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it*;

Prop. 20. *Any two sides of a triangle are together greater than the third side*;

And also Prop. 21. *If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

§ 11. Having solved two problems (Props. 22, 23), he returns to two triangles which have two sides of the one equal respectively to two sides of the other. It is known (Prop. 4) that if the included angles are equal then the third sides are equal; and conversely (Prop. 8), if the third sides are equal, then the angles included by the first sides are equal. From this it follows that if the included angles are not equal, the third sides are not equal; and conversely, that if the third sides are not equal, the included angles are not equal. Euclid now completes this knowledge by proving, that "*if the included angles are not equal, then the third side in that triangle is the greater which contains the greater angle*"; and conversely, that "*if the third sides are unequal, that triangle contains the greater angle which contains the greater side*." These are Prop. 24 and Prop. 25.

§ 12. The next theorem (Prop. 26) says that *if two triangles have one side and two angles of the one equal respectively to one side and two angles of the other, viz. in both triangles either the angles adjacent to the equal side, or one angle adjacent and one angle opposite it, then the two triangles are identically equal*.

This theorem belongs to a group with Prop. 4 and Prop. 8. Its first case might have been given immediately after Prop. 4, but the second case requires Prop. 16 for its proof.

§ 13. We come now to the investigation of parallel straight lines, *i.e.* of straight lines which lie in the same plane, and cannot be made to meet however far they be produced either way. The investigation which starts from Prop. 16, will become clearer if a few names be explained which are not all used by Euclid. If two straight lines be cut by a third, the latter is now generally called a "transversal" of the figure. It forms at the two points where it cuts the given lines four angles with each. Those of the angles which lie between the given lines are called interior angles, and of these, again, any two which lie on opposite sides of the transversal but one at each of the two points are called "alternate angles."

We may now state Prop. 16 thus:—*If two straight lines which meet are cut by a transversal, their alternate angles are unequal*. For the lines will form a triangle, and one of the alternate angles will be an exterior angle to the triangle, the other interior and opposite to it.

From this follows at once the theorem contained in Prop. 27. *If two straight lines which are cut by a transversal make alternate angles equal, the lines cannot meet, however far they be produced, hence they are parallel.* This proves the existence of parallel lines.

Prop. 28 states the same fact in different forms. *If a straight line, falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right* 679 *angles, the two straight lines shall be parallel to one another*.

Hence we know that, "if two straight lines which are cut by a transversal meet, their alternate angles are not equal"; and hence that, "if alternate angles are equal, then the lines are parallel."

The question now arises, Are the propositions converse to these true or not? That is to say, "If alternate angles are unequal, do the lines meet?" And "if the lines are parallel, are alternate angles necessarily equal?"

The answer to either of these two questions implies the answer to the other. But it has been found impossible to prove that the negation or the affirmation of either is true.

The difficulty which thus arises is overcome by Euclid assuming that the first question has to be answered in the affirmative. This gives his last axiom (12), which we quote in his own words.

Axiom 12.—*If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.*

The answer to the second of the above questions follows from this, and gives the theorem Prop. 29:—*If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side, and also the two interior angles on the same side together equal to two right angles*.

§ 14. With this a new part of elementary geometry begins. The earlier propositions are independent of this axiom, and would be true even if a wrong assumption had been made in it. They all relate to figures in a plane. But a plane is only one among an infinite number of conceivable surfaces. We may draw figures on any one of them and study their properties. We may, for instance, take a sphere instead of the plane, and obtain “spherical” in the place of “plane” geometry. If on one of these surfaces lines and figures could be drawn, answering to all the definitions of our plane figures, and if the axioms with the exception of the last all hold, then all propositions up to the 28th will be true for these figures. This is the case in spherical geometry if we substitute “shortest line” or “great circle” for “straight line,” “small circle” for “circle,” and if, besides, we limit all figures to a part of the sphere which is less than a hemisphere, so that two points on it cannot be opposite ends of a diameter, and therefore determine always one and only one great circle.

For spherical triangles, therefore, all the important propositions 4, 8, 26; 5 and 6; and 18, 19 and 20 will hold good.

This remark will be sufficient to show the impossibility of proving Euclid’s last axiom, which would mean proving that this axiom is a consequence of the others, and hence that the theory of parallels would hold on a spherical surface, where the other axioms do hold, whilst parallels do not even exist.

It follows that the axiom in question states an inherent difference between the plane and other surfaces, and that the plane is only fully characterized when this axiom is added to the other assumptions.

§ 15. The introduction of the new axiom and of parallel lines leads to a new class of propositions.

After proving (Prop. 30) that “*two lines which are each parallel to a third are parallel to each other*,” we obtain the new properties of triangles contained in Prop. 32. Of these the second part is the most important, viz. the theorem, *The three interior angles of every triangle are together equal to two right angles*.

As easy deductions not given by Euclid but added by Simson follow the propositions about the angles in polygons, they are given in English editions as corollaries to Prop. 32.

These theorems do not hold for spherical figures. The sum of the interior angles of a spherical triangle is always greater than two right angles, and increases with the area.

§ 16. The theory of parallels as such may be said to be finished with Props. 33 and 34, which state properties of the parallelogram, *i.e.* of a quadrilateral formed by two pairs of parallels. They are—

Prop. 33. *The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel*; and

Prop. 34. *The opposite sides and angles of a parallelogram are equal to one another, and the diameter (diagonal) bisects the parallelogram, that is, divides it into two equal parts.*

§ 17. The rest of the first book relates to areas of figures.

The theory is made to depend upon the theorems—

Prop. 35. *Parallelograms on the same base and between the same parallels are equal to one another*; and

Prop. 36. *Parallelograms on equal bases and between the same parallels are equal to one another*.

As each parallelogram is bisected by a diagonal, the last theorems hold also if the word parallelogram be replaced by “triangle,” as is done in Props. 37 and 38.

It is to be remarked that Euclid proves these propositions only in the case when the parallelograms or triangles have their bases in the same straight line.

The theorems converse to the last form the contents of the next three propositions, viz.: Props, 40 and 41.—*Equal triangles, on the same or on equal bases, in the same straight line, and on the same side of it, are between the same parallels*.

That the two cases here stated are given by Euclid in two separate propositions proved separately is characteristic of his method.

§ 18. To compare areas of other figures, Euclid shows first, in Prop. 42, how *to draw a parallelogram which is equal in area to a given triangle, and has one of its angles equal to a given angle*. If the given angle is right, then the problem is solved *to draw a "rectangle" equal in area to a given triangle*.

Next this parallelogram is transformed into another parallelogram, *which has one of its sides equal to a given straight line*, whilst its angles remain unaltered. This may be done by aid of the theorem in

Prop. 43. *The complements of the parallelograms which are about the diameter of any parallelogram are equal to one another.*

Thus the problem (Prop. 44) is solved to *construct a parallelogram on a given line, which is equal in area to a given triangle, and which has one angle equal to a given angle* (generally a right angle).

As every polygon can be divided into a number of triangles, we can now construct a parallelogram having a given angle, say a right angle, and being equal in area to a given polygon. For each of the triangles into which the polygon has been divided, a parallelogram may be constructed, having one side equal to a given straight line and one angle equal to a given angle. If these parallelograms be placed side by side, they may be added together to form a single parallelogram, having still one side of the given length. This is done in Prop. 45.

Herewith a means is found to compare areas of different polygons. We need only construct two rectangles equal in area to the given polygons, and having each one side of given length. By comparing the unequal sides we are enabled to judge whether the areas are equal, or which is the greater. Euclid does not state this consequence, but the problem is taken up again at the end of the second book, where it is shown how to construct a square equal in area to a given polygon.

Prop. 46 is: *To describe a square on a given straight line*.

§ 19. The first book concludes with one of the most important theorems in the whole of geometry, and one which has been celebrated since the earliest times. It is stated, but on doubtful authority, that Pythagoras discovered it, and it has been called by his name. If we call that side in a right-angled triangle which is opposite the right angle the hypotenuse, we may state it as follows:—

Theorem of Pythagoras (Prop. 47).—*In every right-angled triangle the square on the hypotenuse is equal to the sum of the squares of the other sides.*

And conversely—

Prop. 48. *If the square described on one of the sides of a triangle be equal to the squares described on the other sides, then the angle contained by these two sides is a right angle.*

On this theorem (Prop. 47) almost all geometrical measurement depends, which cannot be directly obtained.

Book II.

§ 20. The propositions in the second book are very different in character from those in the first; they all relate to areas of rectangles and squares. Their true significance is best seen by stating them in an algebraic form. This is often done by expressing the lengths of lines by aid of numbers, which tell how many times a chosen unit is contained in the lines. If there is a unit to be found which is contained an exact number of times in each side of a rectangle, it is easily seen, and generally shown in the teaching of arithmetic, that the rectangle contains a number of unit squares equal to the product of the numbers which measure the sides, a unit square being the square on the unit line. If, however, no such unit can be found, this process requires that connexion between lines and numbers which is only established by aid of ratios of lines, and which is therefore at this stage altogether inadmissible. But there exists another way of connecting these propositions with algebra, based on modern notions which seem destined greatly to change and to simplify mathematics. We shall introduce here as much of it as is required for our present purpose.

At the beginning of the second book we find a definition according to which “a rectangle is said to be ‘contained’ by the two sides which contain one of its right angles”; in the text this phraseology is extended by speaking of rectangles contained by any two straight lines, meaning the rectangle which has two adjacent sides equal to the two straight lines.

We shall denote a finite straight line by a single small letter, a, b, c, ... x, and the area of the rectangle contained by two lines a and b by ab, and this we shall call the product of the two lines a and b. It will be understood that this definition has nothing to do with the definition of a product of numbers.

We define as follows:—

The *sum* of two straight lines a and b means a straight line c which may be divided in two parts equal respectively to a and b. This sum is denoted by $a + b$.

The *difference* of two lines a and b (in symbols, a-b) means a line c which when added to b gives a; that is,

$$a - b = c \text{ if } b + c = a.$$

The *product* of two lines a and b (in symbols, ab) means the area of the rectangle contained by the lines a and b. For aa, which means the square on the line a, we write a^2. 680

§ 21. The first ten of the fourteen propositions of the second book may then be written in the form of formulae as follows:—

Prop.	1.	$a (b + c + d + \dots) = ab + ac + ad + \dots$
”	2.	$ab + ac = a^2$ if $b + c = a$.
”	3.	$a (a + b) = a^2 + ab$.
”	4.	$(a + b)^2 = a^2 + 2ab + b^2$.
”	5.	$(a + b)(a - b) + b^2 = a^2$.
”	6.	$(a + b)(a - b) + b^2 = a^2$.
”	7.	$a^2 + (a - b)^2 = 2a (a - b) + b^2$.
”	8.	$4(a + b)a + b^2 = (2a + b)^2$.
”	9.	$(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$.
”	10.	$(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$.

It will be seen that 5 and 6, and also 9 and 10, are identical. In Euclid’s statement they do not look the same, the figures being arranged differently.

If the letters a, b, c, ... denoted numbers, it follows from algebra that each of these formulae is true. But this does not prove them in our case, where the letters denote lines, and their products areas without any reference to numbers. To prove them we have to discover the laws which rule the operations introduced, viz. addition and multiplication of segments. This we shall do now; and we shall find that these laws are the same with those which hold in algebraical addition and multiplication.

§ 22. In a sum of numbers we may change the order in which the numbers are added, and we may also add the numbers together in groups and then add these groups. But this also holds for the sum of segments and for the sum of rectangles, as a little consideration shows. That the sum of rectangles has always a meaning follows from the Props. 43-45 in the first book. These laws about addition are reducible to the two—

$$a + b = b + a \tag*{(1),}$$

$$a + (b + c) = a + b + c \tag*{(2);}$$

or, when expressed for rectangles,

$$ab + ed = ed + ab \tag*{(3),}$$

$$ab + (cd + ef) = ab + cd + ef \tag*{(4).}$$

The brackets mean that the terms in the bracket have been added together before they are added to another term. The more general cases for more terms may be deduced from the above.

For the product of two numbers we have the law that it remains unaltered if the factors be interchanged. This also holds for our geometrical product. For if ab denotes the area of the rectangle which has a as base and b as altitude, then ba will denote the area of the rectangle which has b as base and a as altitude. But in a rectangle we may take either of the two lines which contain it as base, and then the other will be the altitude. This gives

$$ab = ba \tag{5}$$

In order further to multiply a sum by a number, we have in algebra the rule:—Multiply each term of the sum, and add the products thus obtained. That this holds for our geometrical products is shown by Euclid in his first proposition of the second book, where he proves that the area of a rectangle whose base is the sum of a number of segments is equal to the sum of rectangles which have these segments separately as bases. In symbols this gives, in the simplest case,

$$a(b + c) = ab + ac$$

and

$$(b + c)a = ba + ca \tag{6}$$

To these laws, which have been investigated by Sir William Hamilton and by Hermann Grassmann, the former has given special names. He calls the laws expressed in

(1) and (3) the commutative law for addition;
(5) the commutative law for multiplication;
(2) and (4) the associative laws for addition;
(6) the distributive law.

§ 23. Having proved that these six laws hold, we can at once prove every one of the above propositions in their algebraical form.

The first is proved geometrically, it being one of the fundamental laws. The next two propositions are only special cases of the first. Of the others we shall prove one, viz. the fourth:—

$$(a + b)^2 = (a + b)(a + b) = (a + b)a + (a + b)b \qquad \text{by (6).}$$

But

$$(a + b)a = aa + ba \qquad \text{by (6),}$$

$$= aa + ab \qquad \text{by (5);}$$

and

$$(a + b)b = ab + bb \qquad \text{by (6).}$$

Therefore

$$\begin{aligned}(a + b)^2 &= aa + ab + (ab + bb)\\ &= aa + (ab + ab) + bb\\ &= aa + 2ab + bb\end{aligned} \qquad \text{by (4).}$$

This gives the theorem in question.

In the same manner every one of the first ten propositions is proved.

It will be seen that the operations performed are exactly the same as if the letters denoted numbers.

Props. 5 and 6 may also be written thus—

$$(a + b)(a - b) = a^2 - b^2.$$

Prop. 7, which is an easy consequence of Prop. 4, may be transformed. If we denote by c the line a + b, so that

$$c = a + b,\ a = c - b,$$

we get

$$\begin{aligned}c^2 + (c - b)^2 &= 2c(c - b) + b^2\\ &= 2c^2 - 2bc + b^2.\end{aligned}$$

Subtracting c^2 from both sides, and writing a for c, we get

$$(a - b)^2 = a^2 - 2ab + b^2.$$

In Euclid’s *Elements* this form of the theorem does not appear, all propositions being so stated that the notion of subtraction does not enter into them.

§ 24. The remaining two theorems (Props. 12 and 13) connect the square on one side of a triangle with the sum of the squares on the other sides, in case that the angle between the latter is acute or obtuse. They are important theorems in trigonometry, where it is possible to include them in a single theorem.

§ 25. There are in the second book two problems, Props. 11 and 14.

If written in the above symbolic language, the former requires to find a line x such that $a(a - x) = x^2$. Prop. 11 contains, therefore, the solution of a quadratic equation, which we may write $x^2 + ax = a^2$. The solution is required later on in the construction of a regular decagon.

More important is the problem in the last proposition (Prop. 14). It requires the construction of a square equal in area to a given rectangle, hence a solution of the equation

$$x^2 = ab.$$

In Book I., 42-45, it has been shown how a rectangle may be constructed equal in area to a given figure bounded by straight lines. By aid of the new proposition we may therefore now determine a line such that the square on that line is equal in area to any given rectilinear figure, or we can *square* any such figure.

As of two squares that is the greater which has the greater side, it follows that now the comparison of two areas has been reduced to the comparison of two lines.

The problem of reducing other areas to squares is frequently met with among Greek mathematicians. We need only mention the problem of squaring the circle (see Circle).

In the present day the comparison of areas is performed in a simpler way by reducing all areas to rectangles having a common base. Their altitudes give then a measure of their areas.

The construction of a rectangle having the base u, and being equal in area to a given rectangle, depends upon Prop. 43, I. This therefore gives a solution of the equation

$$ab = ux,$$

where x denotes the unknown altitude.

Book III.

§ 26. The third book of the *Elements* relates exclusively to properties of the circle. A circle and its circumference have been defined in Book I., Def. 15. We restate it here in slightly different words:—

Definition.—The circumference of a circle is a plane curve such that all points in it have the same distance from a fixed point in the plane. This point is called the “centre” of the circle.

Of the new definitions, of which eleven are given at the beginning of the third book, a few only require special mention. The first, which says that circles with equal radii are equal, is in part a theorem, but easily proved by applying the one circle to the other. Or it may be considered proved by aid of Prop. 24, equal circles not being used till after this theorem.

In the second definition is explained what is meant by a line which “touches” a circle. Such a line is now generally called a tangent to the circle. The introduction of this name allows us to state many of Euclid’s propositions in a much shorter form.

For the same reason we shall call a straight line joining two points on the circumference of a circle a “chord.”

Definitions 4 and 5 may be replaced with a slight generalization by the following:—

Definition.—By the distance of a point from a line is meant the length of the perpendicular drawn from the point to the line.

§ 27. From the definition of a circle it follows that every circle has a centre. Prop. 1 requires to find it when the circle is given, *i.e.* when its circumference is drawn.

To solve this problem a chord is drawn (that is, any two points in the circumference are joined), and through the point where this is bisected a perpendicular to it is erected. Euclid then proves, first, that no point off this perpendicular can be the centre, hence that the centre must lie in this line; and, secondly, that of the points on the perpendicular one only can be the centre, viz. the one which bisects the parts of the perpendicular bounded by the circle. In the second part Euclid silently assumes that the perpendicular there used does cut the circumference in two, and only in two points. The proof therefore is incomplete. The proof of the first part, however, is exact. By drawing two non-parallel chords, and the perpendiculars which bisect them, the centre will be found as the point where these perpendiculars intersect.

§ 28. In Prop. 2 it is proved that a chord of a circle lies altogether within the circle.

What we have called the first part of Euclid’s solution of Prop. 1 may be stated as a theorem:— 681

Every straight line which bisects a chord, and is at right angles to it, passes through the centre of the circle.

The converse to this gives Prop. 3, which may be stated thus:—

If a straight line through the centre of a circle bisect a chord, then it is perpendicular to the chord, and if it be perpendicular to the chord it bisects it.

An easy consequence of this is the following theorem, which is essentially the same as Prop. 4:—

Two chords of a circle, of which neither passes through the centre, cannot bisect each other.

These last three theorems are fundamental for the theory of the circle. It is to be remarked that Euclid never proves that a straight line cannot have more than two points in common with a circumference.

§ 29. The next two propositions (5 and 6) might be replaced by a single and a simpler theorem, viz:—

Two circles which have a common centre, and whose circumferences have one point in common, coincide.

Or, more in agreement with Euclid’s form:—

Two different circles, whose circumferences have a point in common, cannot have the same centre.

That Euclid treats of two cases is characteristic of Greek mathematics.

The next two propositions (7 and 8) again belong together. They may be combined thus:—

If from a point in a plane of a circle, which is not the centre, straight lines be drawn to the different points of the circumference, then of all these lines one is the shortest, and one the longest, and these lie both in that straight line which joins the given point to the centre. Of all the remaining lines each is equal to one and only one other, and these equal lines lie on opposite sides of the shortest or longest, and make equal angles with them.

Euclid distinguishes the two cases where the given point lies within or without the circle, omitting the case where it lies in the circumference.

From the last proposition it follows that if from a point more than two equal straight lines can be drawn to the circumference, this point must be the centre. This is Prop. 9.

As a consequence of this we get

If the circumferences of the two circles have three points in common they coincide.

For in this case the two circles have a common centre, because from the centre of the one three equal lines can be drawn to points on the circumference of the other. But two circles which have a common centre, and whose circumferences have a point in common, coincide. (Compare above statement of Props. 5 and 6.)

This theorem may also be stated thus:—

Through three points only one circumference may be drawn; or, Three points determine a circle.

Euclid does not give the theorem in this form. He proves, however, *that the two circles cannot cut another in more than two points* (Prop. 10), and *that two circles cannot touch one another in more points than one* (Prop. 13).

§ 30. Propositions 11 and 12 assert that *if two circles touch, then the point of contact lies* *on the line joining their centres*. This gives two propositions, because the circles may touch either internally or externally.

§ 31. Propositions 14 and 15 relate to the length of chords. The first says *that equal chords are equidistant from the centre, and that chords which are equidistant from the centre are equal*;

Whilst Prop. 15 compares unequal chords, viz. *Of all chords the diameter is the greatest, and of other chords that is the greater which is nearer to the centre*; and conversely, *the greater chord is nearer to the centre*.

§ 32. In Prop. 16 the tangent to a circle is for the first time introduced. The proposition is meant to show that the straight line at the end point of the diameter and at right angles to it is a tangent. The proposition itself does not state this. It runs thus:—

Prop. 16. *The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle.*

Corollary.—The straight line at right angles to a diameter drawn through the end point of it touches the circle.

The statement of the proposition and its whole treatment show the difficulties which the tangents presented to Euclid.

Prop. 17 solves the problem *through a given point, either in the circumference or without it, to draw a tangent to a given circle*.

Closely connected with Prop. 16 are Props. 18 and 19, which state (Prop. 18), *that the line joining the centre of a circle to the point of contact of a tangent is perpendicular to the tangent*; and conversely (Prop. 19), *that the straight line through the point of contact of, and perpendicular to, a tangent to a circle passes through the centre of the circle*.

§ 33. The rest of the book relates to angles connected with a circle, viz. angles which have the vertex either at the centre or on the circumference, and which are called respectively angles at the centre and angles at the circumference. Between these two kinds of angles exists the important relation expressed as follows:—

Prop. 20. *The angle at the centre of a circle is double of the angle at the circumference on the same base, that is, on the same arc.*

This is of great importance for its consequences, of which the two following are the principal:—

Prop. 21. *The angles in the same segment of a circle are equal to one another*;

Prop. 22. *The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.*

Further consequences are:—

Prop. 23. *On the same straight line, and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another*;

Prop. 24. *Similar segments of circles on equal straight lines are equal to one another.*

The problem Prop. 25. *A segment of a circle being given to describe the circle of which it is a segment*, may be solved much more easily by aid of the construction described in relation to Prop. 1, III., in § 27.

§ 34. There follow four theorems connecting the angles at the centre, the arcs into which they divide the circumference, and the chords subtending these arcs. They are expressed for angles, arcs and chords in equal circles, but they hold also for angles, arcs and chords in the same circle.

The theorems are:—

Prop. 26. *In equal circles equal angles stand on equal arcs, whether they be at the centres or circumferences*;

Prop. 27. (converse to Prop. 26). *In equal circles the angles which stand on equal arcs are equal to one another, whether they be at the centres or the circumferences*;

Prop. 28. *In equal circles equal straight lines* (equal chords) *cut off equal arcs, the greater equal to the greater, and the less equal to the less*;

Prop. 29 (converse to Prop. 28). *In equal circles equal arcs are subtended by equal straight lines.*

§ 35. Other important consequences of Props. 20-22 are:—

Prop. 31. *In a circle the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle*;

Prop. 32. *If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.*

§ 36. Propositions 30, 33, 34, contain problems which are solved by aid of the propositions preceding them:—

Prop. 30. *To bisect a given arc, that is, to divide it into two equal parts*;

Prop. 33. *On a given straight line to describe a segment of a circle containing an angle equal to a given rectilineal angle*;

Prop. 34. *From a given circle to cut off a segment containing an angle equal to a given rectilineal angle*.

§ 37. If we draw chords through a point A within a circle, they will each be divided by A into two segments. Between these segments the law holds that the rectangle contained by them has the same area on whatever chord through A the segments are taken. The value of this rectangle changes, of course, with the position of A.

A similar theorem holds if the point A be taken without the circle. On every straight line through A, which cuts the circle in two points B and C, we have two segments AB and AC, and the rectangles contained by them are again equal to one another, and equal to the square on a tangent drawn from A to the circle.

The first of these theorems gives Prop. 35, and the second Prop. 36, with its corollary, whilst Prop. 37, the last of Book III., gives the converse to Prop. 36. The first two theorems may be combined in one:—

If through a point A in the plane of a circle a straight line be drawn cutting the circle in B and C, then the rectangle AB.AC has a constant value so long as the point A be fixed; and if from A a tangent AD can be drawn to the circle, touching at D, then the above rectangle equals the square on AD.

Prop. 37 may be stated thus:—

If from a point A without a circle a line be drawn cutting the circle in B and C, and another line to a point D on the circle, and AB.AC = AD^2*, then the line AD touches the circle at D.*

It is not difficult to prove also the converse to the general proposition as above stated. This proposition and its converse may be expressed as follows:—

If four points ABCD be taken on the circumference of a circle, and if the lines AB, CD, produced if necessary, meet at E, then

$$EA \cdot EB = EC \cdot ED;$$

and conversely, if this relation holds then the four points lie on a circle, that is, the circle drawn through three of them passes through the fourth.

That a circle may always be drawn through three points, provided that they do not lie in a straight line, is proved only later on in Book IV.

682

Book IV.

§ 38. The fourth book contains only problems, all relating to the construction of triangles and polygons inscribed in and circumscribed about circles, and of circles inscribed in or circumscribed about triangles and polygons. They are nearly all given for their own sake, and not for future use in the construction of figures, as are most of those in the former books. In seven definitions at the beginning of the book it is explained what is understood by figures inscribed in or described about other figures, with special reference to the case where one figure is a circle. Instead, however, of saying that one figure is described about another, it is now generally said that the one figure is circumscribed about the other. We may then state the definitions 3 or 4 thus:—

Definition.—A polygon is said to be inscribed in a circle, and the circle is said to be circumscribed about the polygon, if the vertices of the polygon lie in the circumference of the circle.

And definitions 5 and 6 thus:—

Definition.—A polygon is said to be circumscribed about a circle, and a circle is said to be inscribed in a polygon, if the sides of the polygon are tangents to the circle.

§ 39. The first problem is merely constructive. It requires to draw in a given circle a chord equal to a given straight line, which is not greater than the diameter of the circle. The problem is not a determinate one, inasmuch as the chord may be drawn from any point in the circumference. This may be said of almost all problems in this book, especially of the next two. They are:—

Prop. 2. *In a given circle to inscribe a triangle equiangular to a given triangle;*

Prop. 3. *About a given circle to circumscribe a triangle equiangular to a given triangle.*

§ 40. Of somewhat greater interest are the next problems, where the triangles are given and the circles to be found.

Prop. 4. *To inscribe a circle in a given triangle.*

The result is that the problem has always a solution, viz. the centre of the circle is the point where the bisectors of two of the interior angles of the triangle meet. The solution shows, though Euclid does not state this, that the problem has but one solution; and also,

The three bisectors of the interior angles of any triangle meet in a point, and this is the centre of the circle inscribed in the triangle.

The solutions of most of the other problems contain also theorems. Of these we shall state those which are of special interest; Euclid does not state any one of them.

§ 41. Prop. 5. *To circumscribe a circle about a given triangle.*

The one solution which always exists contains the following:—

The three straight lines which bisect the sides of a triangle at right angles meet in a point, and this point is the centre of the circle circumscribed about the triangle.

Euclid adds in a corollary the following property:—

The centre of the circle circumscribed about a triangle lies within, on a side of, or without the triangle, according as the triangle is acute-angled, right-angled or obtuse-angled.

§ 42. Whilst it is always possible to draw a circle which is inscribed in or circumscribed about a given triangle, this is not the case with quadrilaterals or polygons of more sides. Of those for which this is possible the regular polygons, *i.e.* polygons which have all their sides and angles equal, are the most interesting. In each of them a circle may be inscribed, and another may be circumscribed about it.

Euclid does not use the word regular, but he describes the polygons in question as *equiangular* and *equilateral*. We shall use the name regular polygon. The regular triangle is equilateral, the regular quadrilateral is the square.

Euclid considers the regular polygons of 4, 5, 6 and 15 sides. For each of the first three he solves the problems—(1) to inscribe such a polygon in a given circle; (2) to circumscribe it about a given circle; (3) to inscribe a circle in, and (4) to circumscribe a circle about, such a polygon.

For the regular triangle the problems are not repeated, because more general problems have been solved.

Props. 6, 7, 8 and 9 solve these problems for the square.

The general problem of inscribing in a given circle a regular polygon of n sides depends upon the problem of dividing the circumference of a circle into n equal parts, or what comes to the same thing, of drawing from the centre of the circle n radii such that the angles between consecutive radii are equal, that is, to divide the space about the centre into n equal angles. Thus, if it is required to inscribe a square in a circle, we have to draw four lines from the centre, making the four angles equal. This is done by drawing two diameters at right angles to one another. The ends of these diameters are the vertices of the required square. If, on the other hand, tangents be drawn at these ends, we obtain a square circumscribed about the circle.

§ 43. To construct a *regular pentagon*, we find it convenient first to construct a *regular decagon*. This requires to divide the space about the centre into ten equal angles. Each will be $\frac{1}{10}$th of a right angle, or $\frac{1}{5}$th of two right angles. If we suppose the decagon constructed, and if we join the centre to the end of one side, we get an isosceles triangle, where the angle at the centre equals $\frac{1}{5}$th of two right angles; hence each of the angles at the base will be $\frac{2}{5}$ths of two right angles, as all three angles together equal two right angles. Thus we have to construct an isosceles triangle, having the angle at the vertex equal to half an angle at the base. This is solved in Prop. 10, by aid of the problem in Prop. 11 of the second book. If we make the sides of this triangle equal to the radius of the given circle, then the base will be the side of the regular decagon inscribed in the circle. This side being known the decagon can be constructed, and if the vertices are joined alternately, leaving out half their number, we obtain the regular pentagon. (Prop. 11.)

Euclid does not proceed thus. He wants the pentagon before the decagon. This, however, does not change the real nature of his solution, nor does his solution become simpler by not mentioning the decagon.

Once the regular pentagon is inscribed, it is easy to circumscribe another by drawing tangents at the vertices of the inscribed pentagon. This is shown in Prop. 12.

Props. 13 and 14 teach how a circle may be inscribed in or circumscribed about any given regular pentagon.

§ 44. The *regular hexagon* is more easily constructed, as shown in Prop. 15. The result is that the side of the regular hexagon inscribed in a circle is equal to the radius of the circle.

For this polygon the other three problems mentioned are not solved.

§ 45. The book closes with Prop. 16. To inscribe a regular quindecagon in a given circle. If we inscribe a regular pentagon and a regular hexagon in the circle, having one vertex in common, then the arc from the common vertex to the next vertex of the pentagon is $\frac{1}{5}$th of the circumference, and to the next vertex of the hexagon is $\frac{1}{6}$th of the circumference. The difference between these arcs is, therefore, $\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$th of the circumference. The latter may, therefore, be divided into thirty, and hence also in fifteen equal parts, and the regular quindecagon be described.

§ 46. We conclude with a few theorems about regular polygons which are not given by Euclid.

The straight lines perpendicular to and bisecting the sides of any regular polygon meet in a point. The straight lines bisecting the angles in the regular polygon meet in the same point. This point is the centre of the circles circumscribed about and inscribed in the regular polygon.

We can bisect any given arc (Prop. 30, III.). Hence we can divide a circumference into 2n equal parts as soon as it has been divided into n equal parts, or as soon as a regular polygon of n sides has been constructed. Hence—

If a regular polygon of n sides has been constructed, then a regular polygon of 2n sides, of 4n, of 8n sides, &c., may also be constructed. Euclid shows how to construct regular polygons of 3, 4, 5 and 15 sides. It follows that we can construct regular polygons of

3,	6,	12,	24	sides
4,	8,	16,	32	”
5,	10,	20,	40	”
15,	30,	60,	120	”

The construction of any new regular polygon not included in one of these series will give rise to a new series. Till the beginning of the 19th century nothing was added to the knowledge of regular polygons as given by Euclid. Then Gauss, in his celebrated *Arithmetic*, proved that every regular polygon of $2^n + 1$ sides may be constructed if this number $2^n + 1$ be prime, and that no others except those with $2^m (2^n + 1)$ sides can be constructed by elementary methods. This shows that regular polygons of 7, 9, 13 sides cannot thus be constructed, but that a regular polygon of 17 sides is possible; for $17 = 2^4 + 1$. The next polygon is one of 257 sides. The construction becomes already rather complicated for 17 sides.

Book V.

§ 47. The fifth book of the *Elements* is not exclusively geometrical. It contains the theory of ratios and proportion of quantities in general. The treatment, as here given, is admirable, and in every respect superior to the algebraical method by which Euclid’s theory is now generally replaced. We shall treat the subject in order to show why the usual algebraical treatment of proportion is not really sound. We begin by quoting those definitions at the beginning of Book V. which are most important. These definitions have given rise to much discussion.

The only definitions which are essential for the fifth book are Defs. 1, 2, 4, 5, 6 and 7. Of the remainder 3, 8 and 9 are more than useless, and probably not Euclid's, but additions of later editors, of whom Theon of Alexandria was the most prominent. Defs. 10 and 11 belong rather to the sixth book, whilst all the others are merely nominal. The really important ones are 4, 5, 6 and 7.

§ 48. To define a magnitude is not attempted by Euclid. The first two definitions state what is meant by a "part," that is, a submultiple or measure, and by a "multiple" of a given magnitude. The meaning of Def. 4 is that two given quantities can have a ratio to one another only in case that they are comparable as to their magnitude, that is, if they are of the same kind.

Def. 3, which is probably due to Theon, professes to define a ratio, but is as meaningless as it is uncalled for, for all that is wanted is given in Defs. 5 and 7.

In Def. 5 it is explained what is meant by saying that two magnitudes have the same ratio to one another as two other magnitudes, and in Def. 7 what we have to understand by a 683
greater or a less ratio. The 6th definition is only nominal, explaining the meaning of the word *proportional*.

Euclid represents magnitudes by lines, and often denotes them either by single letters or, like lines, by two letters. We shall use only single letters for the purpose. If a and b denote two magnitudes of the same kind, their ratio will be denoted by $a : b$; if c and d are two other magnitudes of the same kind, but possibly of a different kind from a and b, then if c and d have the same ratio to one another as a and b, this will be expressed by writing—

$$a : b :: c : d.$$

Further, if m is a (whole) number, ma shall denote the multiple of a which is obtained by taking it m times.

§ 49. The whole theory of ratios is based on Def. 5.

Def. 5. *The first of four magnitudes is said to have the same ratio to the second that the third has to the fourth when, any equimultiples whatever of the first and the third being taken, and any equimultiples whatever of the second and the fourth, if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; and if the multiple of the first is equal to that of the second, the multiple of the third is also equal to that of the fourth; and if the multiple of the first is greater than that of the second, the multiple of the third is also greater than that of the fourth.*

It will be well to show at once in an example how this definition can be used, by proving the first part of the first proposition in the sixth book. *Triangles of the same altitude are to one another as their bases*, or if a and b are the bases, and α and β the areas, of two triangles which have the same altitude, then $a : b :: \alpha : \beta$.

To prove this, we have, according to Definition 5, to show—

$$\begin{aligned}&\text{if } ma > nb, \text{ then } m\alpha > n\beta,\\ &\text{if } ma = nb, \text{ then } m\alpha = n\beta,\\ &\text{if } ma < nb, \text{ then } m\alpha < n\beta.\end{aligned}$$

That this is true is in our case easily seen. We may suppose that the triangles have a common vertex, and their bases in the same line. We set off the base a along the line containing the bases m times; we then join the different parts of division to the vertex, and get m triangles all equal to α. The triangle on ma as base equals, therefore, $m\alpha$. If we proceed in the same manner with the base b, setting it off n times, we find that the area of the triangle on the base nb equals $n\beta$, the vertex of all triangles being the same. But if two triangles have the same altitude, then their areas are equal if the bases are equal; hence $m\alpha = n\beta$ if $ma = nb$, and if their bases are unequal, then that has the greater area which is on the greater base; in other words, $m\alpha$ is greater than, equal to, or less than $n\beta$, according as ma is greater than, equal to, or less than nb, which was to be proved.

§ 50. It will be seen that even in this example it does not become evident what a ratio really is. It is still an open question whether ratios are magnitudes which we can compare. We do not know whether the ratio of two lines is a magnitude of the same kind as the ratio of two areas. Though we might say that Def. 5 defines *equal* ratios, still we do not know whether they are equal in the sense of the axiom, that two things which are equal to a third are equal to one another. That this is the case requires a proof, and until this proof is given we shall use the :: instead of the sign = , which, however, we shall afterwards introduce.

As soon as it has been established that all ratios are like magnitudes, it becomes easy to show that, in some cases at least, they are numbers. This step was never made by Greek mathematicians. They distinguished always most carefully between continuous magnitudes and the discrete series of numbers. In modern times it has become the custom to ignore this difference.

If, in determining the ratio of two lines, a common measure can be found, which is contained m times in the first, and n times in the second, then the ratio of the two lines equals the ratio of the two numbers m : n. This is shown by Euclid in Prop. 5, X. But the ratio of two numbers is, as a rule, a fraction, and the Greeks did not, as we do, consider fractions as numbers. Far less had they any notion of introducing irrational numbers, which are neither whole nor fractional, as we are obliged to do if we wish to say that all ratios are numbers. The incommensurable numbers which are thus introduced as ratios of incommensurable quantities are nowadays as familiar to us as fractions; but a proof is generally omitted that we may apply to them the rules which have been established for rational numbers only. Euclid's treatment of ratios avoids this difficulty. His definitions hold for commensurable as well as for incommensurable quantities. Even the notion of incommensurable quantities is avoided in Book V. But he proves that the more elementary rules of algebra hold for ratios. We shall state all his propositions in that algebraical form to which we are now accustomed. This may, of course, be done without changing the character of Euclid's method.

§. 51. Using the notation explained above we express the first propositions as follows:—

Prop. 1. If

$$a = ma', b = mb', c = mc',$$

then

$$a + b + c = m(a' + b' + c').$$

Prop. 2. If

$$a = mb, \text{ and } c = md,$$
$$e = nb, \text{ and } f = nd,$$

then a + e is the same multiple of b as c + f is of d, viz.:—

$$a + e = (m + n)b, \text{ and } c + f = (m + n)d.$$

Prop. 3. If $a = mb$, $c = md$, then is na the same multiple of b that nc is of d, viz. $na = nmb$, $nc = nmd$.

Prop. 4. If

$$a : b :: c : d,$$

then

$$ma : nb :: mc : nd.$$

Prop. 5. If

$$a = mb, \text{ and } c = md,$$

then

$$a - c = m(b - d).$$

Prop. 6. If

$$a = mb, c = md,$$

then are $a - nb$ and $c - nd$ either equal to, or equimultiples of, b and d, viz. $a - nb = (m - n)b$ and $c - nd = (m - n)d$, where $m - n$ may be unity.

All these propositions relate to *equimultiples*. Now follow propositions about ratios which are compared as to their magnitude.

§ 52. Prop. 7. If $a = b$, then $a : c :: b : c$ and $c : a :: c : b$.

The proof is simply this. As $a = b$ we know that $ma = mb$; therefore if

$$ma > nc, \text{ then } mb > nc,$$

if

$$ma = nc, \text{ then } mb = nc,$$

if

$$ma < nc, \text{ then } mb < nc,$$

therefore the first proportion holds by Definition 5.

Prop. 8. If

$$a > b, \text{ then } a : c > b : c,$$

and

$$c : a < c : b.$$

The proof depends on Definition 7.

Prop. 9 (converse to Prop. 7). If

$$a : c :: b : c,$$

or if

$$c : a :: c : b, \text{ then } a = b.$$

Prop. 10 (converse to Prop. 8). If

$$a : c > b : c, \text{ then } a > b,$$

and if

$$c : a < c : b, \text{ then } a < b.$$

Prop. 11. If

$$a : b :: c : d,$$

and

$$a : b :: e : f,$$

then

$$c : d :: e : f.$$

In words, *if too ratios are equal to a third, they are equal to one another*. After these propositions have been proved, we have a right to consider a ratio as a *magnitude*, for only now can we consider a ratio as something for which the axiom about magnitudes holds: things which are equal to a third are equal to one another.

We shall indicate this by writing in future the sign = instead of ::. The remaining propositions, which explain themselves, may then be stated as follows:

§ 53. Prop. 12. If

$$a : b = c : d = e : f,$$

then

$$a + c + e : b + d + f = a : b.$$

Prop. 13. If

$$a : b = c : d \text{ and } c : d > e : f,$$

then

$$a : b > e : f.$$

Prop. 14. If

$$a : b = c : d, \text{ and } a > c, \text{ then } b > d.$$

Prop. 15. Magnitudes have the same ratio to one another that their equimultiples have—

$$ma : mb = a : b.$$

Prop. 16. If a, b, c, d are magnitudes of the same kind, and if

$$a : b = c : d,$$

then

$$a : c = b : d.$$

Prop. 17. If

$$a + b : b = c + d : d,$$

then

$$a : b = c : d.$$

Prop. 18 (converse to 17). If

$$a : b = c : d$$

then

$$a + b : b = c + d : d.$$

Prop. 19. If a, b, c, d are quantities of the same kind, and if

$$a : b = c : d,$$

then

$$a - c : b - d = a : b.$$

§ 54. Prop. 20. *If there be three magnitudes, and another three, which have the same ratio, taken two and two, then if the first be greater than the third, the fourth shall be greater than the sixth: and if equal, equal; and if less, less.*

If we understand by

$$a : b : c : d : e : \ldots = a' : b' : c' : d' : e' : \ldots$$

that the ratio of any two consecutive magnitudes on the first side equals that of the corresponding magnitudes on the second side, we may write this theorem in symbols, thus:—

If a, b, c be quantities of one, and d, e, f magnitudes of the same or any other kind, such that

$$a : b : c = d : e : f,$$

and if

$$a > c, \text{ then } d > f,$$

but if

$$a = c, \text{ then } d = f,$$

and if

$$a < c, \text{ then } d < f.$$

Prop. 21. If

$$a : b = e : f \text{ and } b : c = d : e,$$

or if

$$a : b : c = 1/f : 1/e : 1/d,$$

and if 684

$$a > c, \text{ then } d > f,$$

but if

$$a = c, \text{ then } d = f,$$

and if

$$a < c, \text{ then } d < f.$$

By aid of these two propositions the following two are proved.

§ 55. Prop. 22. *If there be any number of magnitudes, and as many others, which have the same ratio, taken two and two in order, the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.*

We may state it more generally, thus:

If

$$a : b : c : d : e : \ldots = a' : b' : c' : d' : e' : \ldots ,$$

then not only have two consecutive, but any two magnitudes on the first side, the same ratio as the corresponding magnitudes on the other. For instance—

$$a : c = a' : c'; b : e = b' : e', \&c.$$

Prop. 23 we state only in symbols, viz.:—

$$a : b : c : d : e : \ldots = 1/a' : 1/b' : 1/c' : 1/d' : 1/e' \ldots,$$

then

$$a : c = c' : a',$$
$$b : e = e' : b',$$

and so on.

Prop. 24 comes to this: If $a : b = c : d$ and $e : b = f : d$, then

$$a + e : b = c + f : d.$$

Some of the proportions which are considered in the above propositions have special names. These we have omitted, as being of no use, since algebra has enabled us to bring the different operations contained in the propositions under a common point of view.

§ 56. The last proposition in the fifth book is of a different character.

Prop. 25. *If four magnitudes of the same kind be proportional, the greatest and least of them together shall be greater than the other two together.* In symbols—

If a, b, c, d be magnitudes of the same kind, and if $a : b = c : d$, and if a is the greatest, hence d the least, then $a + d > b + c$.

§ 57. We return once again to the question. What is a ratio? We have seen that we may treat ratios as magnitudes, and that all ratios are magnitudes of the same kind, for we may compare any two as to their magnitude. It will presently be shown that ratios of lines may be considered as *quotients* of lines, so that a ratio appears as answer to the question, How often is one line contained in another? But the answer to this question is given by a number, at least in some cases, and in all cases if we admit incommensurable numbers. Considered from this point of view, we may say the fifth book of the *Elements* shows that some of the simpler algebraical operations hold for incommensurable numbers. In the ordinary algebraical treatment of numbers this proof is altogether omitted, or given by a process of limits which does not seem to be natural to the subject.

Book VI.

§ 58. The sixth book contains the theory of similar figures. After a few definitions explaining terms, the first proposition gives the first application of the theory of proportion.

Prop. 1. *Triangles and parallelograms of the same altitude are to one another as their bases.*

The proof has already been considered in § 49.

From this follows easily the important theorem

Prop. 2. *If a straight line be drawn parallel to one of the sides of a triangle it shall cut the other sides, or those sides produced, proportionally; and if the sides or the sides produced be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.*

§ 59. The next proposition, together with one added by Simson as Prop. A, may be expressed more conveniently if we introduce a modern phraseology, viz. if in a line AB we assume a point C between A and B, we shall say that C divides AB internally in the ratio AC : CB; but if C be taken in the line AB produced, we shall say that AB is divided externally in the ratio AC : CB.

The two propositions then come to this:

Prop. 3. *The bisector of an angle in a triangle divides the opposite side internally in a ratio equal to the ratio of the two sides including that angle;* and conversely, *if a line through the vertex of a triangle divide the base internally in the ratio of the two other sides, then that line bisects the angle at the vertex*.

Simson’s Prop. A. *The line which bisects an exterior angle of a triangle divides the opposite side externally in the ratio of the other sides;* and conversely, *if a line through the vertex of a triangle divide the base externally in the ratio of the sides, then it bisects an exterior angle at the vertex of the triangle*.

If we combine both we have—

The two lines which bisect the interior and exterior angles at one vertex of a triangle divide the opposite side internally and externally in the same ratio, viz. in the ratio of the other two sides.

§ 60. The next four propositions contain the theory of similar triangles, of which four cases are considered. They may be stated together.

Two triangles are similar,—

1. (Prop. 4). *If the triangles are equiangular:*

2. (Prop. 5). *If the sides of the one are proportional to those of the other*;

3. (Prop. 6). *If two sides in one are proportional to two sides in the other, and if the angles contained by these sides are equal*;

4. (Prop. 7). *If two sides in one are proportional to two sides in the other, if the angles opposite homologous sides are equal, and if the angles opposite the other homologous sides are both acute, both right or both obtuse; homologous sides being in each case those which are opposite equal angles*.

An important application of these theorems is at once made to a right-angled triangle, viz.:—

Prop. 8. *In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another*.

Corollary.—From this it is manifest that the perpendicular drawn from the right angle of a right-angled triangle to the base is a mean proportional between the segments of the base, and also that each of the sides is a mean proportional between the base and the segment of the base adjacent to that side.

§ 61. There follow four propositions containing problems, in language slightly different from Euclid’s, viz.:—

Prop. 9. *To divide a straight line into a given number of equal parts*.

Prop. 10. *To divide a straight line in a given ratio*.

Prop. 11. *To find a third proportional to two given straight lines*.

Prop. 12. *To find a fourth proportional to three given straight lines*.

Prop. 13. *To find a mean proportional between two given straight lines*.

The last three may be written as equations with one unknown quantity—viz. if we call the given straight lines a, b, c, and the required line x, we have to find a line x so that

Prop. 11.

$$a : b = b : x;$$

Prop. 12.

$$a : b = c : x;$$

Prop. 13.

$$a : x = x : b.$$

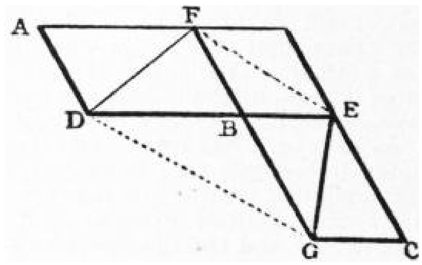
We shall see presently how these may be written without the signs of ratios.

§ 62. Euclid considers next proportions connected with parallelograms and triangles which are equal in area.

Prop. 14. *Equal parallelograms which have one angle of the one equal to one angle of the other have their sides about the equal angles reciprocally proportional; and parallelograms which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another*.

Prop. 15. *Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another*.

The latter proposition is really the same as the former, for if, as in the accompanying diagram, in the figure belonging to the former the two equal parallelograms AB and BC be bisected by the lines DF and EG, and if EF be drawn, we get the figure belonging to the latter.



It is worth noticing that the lines FE and DG are parallel. We may state therefore the theorem—

If two triangles are equal in area, and have one angle in the one vertically opposite to one angle in the other, then the two straight lines which join the remaining two vertices of the one to those of the other triangle are parallel.

§ 63. A most important theorem is

Prop. 16. If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

In symbols, if a, b, c, d are the four lines, and

if

$$a : b = c : d,$$

then

$$ad = bc;$$

and conversely, if

$$ad = bc,$$

then

$$a : b = c : d,$$

where ad and bc denote (as in § 20), the areas of the rectangles contained by a and d and by b and c respectively.

This allows us to transform every proportion between four lines into an equation between two products.

It shows further that the operation of forming a product of two lines, and the operation of forming their ratio are each the inverse of the other.

If we now define a quotient a/b of two lines as the *number* which multiplied into b gives a, so that

$$\frac{a}{b} b = a,$$

we see that from the equality of two quotients 685

$$\frac{a}{b} = \frac{c}{d}$$

follows, if we multiply both sides by bd,

$$\frac{a}{b} b \cdot d = \frac{c}{d} d \cdot b,$$

$$ad = cb.$$

But from this it follows, according to the last theorem, that

$$a : b = c : d.$$

Hence we conclude that the quotient a/b and the ratio a : b are different forms of the same magnitude, only with this important difference that the quotient a/b would have a meaning only if a and b have a common measure, until we introduce incommensurable numbers, while the ratio a : b has always a meaning, and thus gives rise to the introduction of incommensurable numbers.

Thus it is really the theory of ratios in the fifth book which enables us to extend the geometrical calculus given before in connexion with Book II. It will also be seen that if we write the ratios in Book V. as quotients, or rather as fractions, then most of the theorems state properties of quotients or of fractions.

§ 64. Prop. 17. *If three straight lines are proportional the rectangle contained by the extremes is equal to the square on the mean;* and conversely, is only a special case of 16.

After the problem, Prop. 18, *On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure*, there follows another fundamental theorem:

Prop. 19. *Similar triangles are to one another in the duplicate ratio of their homologous sides.* In other words, the areas of similar triangles are to one another as the squares on homologous sides. This is generalized in:

Prop. 20. *Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons are to one another in the duplicate ratio of their homologous sides.*

§ 65. Prop. 21. *Rectilineal figures which are similar to the same rectilineal figure are also similar to each other*, is an immediate consequence of the definition of similar figures. As similar figures may be said to be equal in "shape" but not in "size," we may state it also thus:

"Figures which are equal in shape to a third are equal in shape to each other."

Prop. 22. *If four straight lines be proportionals, the similar rectilineal figures similarly described on them shall also be proportionals; and if the similar rectilineal figures similarly described on four straight lines be proportionals, those straight lines shall be proportionals.*

This is essentially the same as the following:—

If

$$a : b = c : d,$$

then

$$a^2 : b^2 = c^2 : d^2.$$

§ 66. Now follows a proposition which has been much discussed with regard to Euclid's exact meaning in saying that a ratio is *compounded* of two other ratios, viz.:

Prop. 23. *Parallelograms which are equiangular to one another, have to one another the ratio which is compounded of the ratios of their sides.*

The proof of the proposition makes its meaning clear. In symbols the ratio $a : c$ is compounded of the two ratios $a : b$ and $b : c$, and if $a : b = a' : b'$, $b : c = b'' : c''$, then $a : c$ is compounded of $a' : b'$ and $b'' : c''$.

If we consider the ratios as numbers, we may say that the one ratio is the product of those of which it is compounded, or in symbols,

$$\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} = \frac{a'}{b'} \cdot \frac{b''}{c''}, \text{ if } \frac{a}{b} = \frac{a'}{b'} \text{ and } \frac{b}{c} = \frac{b''}{c''}.$$

The theorem in Prop. 23 is the foundation of all mensuration of areas. From it we see at once that two rectangles have the ratio of their areas compounded of the ratios of their sides.

If A is the area of a rectangle contained by a and b, and B that of a rectangle contained by c and d, so that $A = ab$, $B = cd$, then $A : B = ab : cd$, and this is, the theorem says, compounded of the ratios $a : c$ and $b : d$. In forms of quotients,

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}.$$

This shows how to multiply quotients in our geometrical calculus.

Further, *Two triangles have the ratios of their areas compounded of the ratios of their bases and their altitude.* For a triangle is equal in area to half a parallelogram which has the same base and the same altitude.

§ 67. To bring these theorems to the form in which they are usually given, we assume a straight line u as our unit of length (generally an inch, a foot, a mile, &c.), and determine the number α which expresses how often u is contained in a line a, so that α denotes the ratio $a : u$ whether commensurable or not, and that $a = \alpha u$. We call this number α the numerical value of a. If in the same manner β be the numerical value of a line b we have

$$a : b = \alpha : \beta;$$

in words: *The ratio of two lines (and of two like quantities in general) is equal to that of their numerical values.*

This is easily proved by observing that $a = \alpha u$, $b = \beta u$, therefore $a : b = \alpha u : \beta u$, and this may without difficulty be shown to equal $\alpha : \beta$.

If now a, b be base and altitude of one, a′, b′ those of another parallelogram, α, β and α′, β′ their numerical values respectively, and A, A′ their areas, then

$$\frac{A}{A'} = \frac{a}{a'} \cdot \frac{b}{b'} = \frac{\alpha}{\alpha'} \cdot \frac{\beta}{\beta'} = \frac{\alpha\beta}{\alpha'\beta'}\,.$$

In words: *The areas of two parallelograms are to each other as the products of the numerical values of their bases and altitudes.*

If especially the second parallelogram is the unit square, *i.e.* a square on the unit of length, then $\alpha' = \beta' = 1$, $A' = u^2$, and we have

$$\frac{A}{A'} = \alpha\beta \text{ or } A = \alpha\beta\cdot u^2.$$

This gives the theorem: The number of unit squares contained in a parallelogram equals the product of the numerical values of base and altitude, and similarly the number of unit squares contained in a triangle equals half the product of the numerical values of base and altitude.

This is often stated by saying that the area of a parallelogram is equal to the product of the base and the altitude, meaning by this product the product of the numerical values, and not the product as defined above in § 20.

§ 68. Propositions 24 and 26 relate to parallelograms about diagonals, such as are considered in Book I., 43. They are—

Prop. 24. *Parallelograms about the diameter of any parallelogram are similar to the whole parallelogram and to one another*; and its converse (Prop. 26), *If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.*

Between these is inserted a problem.

Prop. 25. *To describe a rectilineal figure which shall be similar to one given rectilinear figure, and equal to another given rectilineal figure*.

§ 69. Prop. 27 contains a theorem relating to the theory of maxima and minima. We may state it thus:

Prop. 27. *If a parallelogram be divided into two by a straight line cutting the base, and if on half the base another parallelogram be constructed similar to one of those parts, then this third parallelogram is greater than the other part.*

Of far greater interest than this general theorem is a special case of it, where the parallelograms are changed into rectangles, and where one of the parts into which the parallelogram is divided is made a square; for then the theorem changes into one which is easily recognized to be identical with the following:—

Of all rectangles which have the same perimeter the square has the greatest area.

This may also be stated thus:—

Of all rectangles which have the same area the square has the least perimeter.

§ 70. The next three propositions contain problems which may be said to be solutions of quadratic equations. The first two are, like the last, involved in somewhat obscure language. We transcribe them as follows:

Problem.—To describe on a given base a parallelogram, and to divide it either internally (Prop. 28) or externally (Prop. 29) from a point on the base into two parallelograms, of which the one has a given size (is equal in area to a given figure), whilst the other has a given shape (is similar to a given parallelogram).

If we express this again in symbols, calling the given base a, the one part x, and the altitude y, we have to determine x and y in the first case from the equations

$$(a - x)y = k^2,$$

$$\frac{x}{y} = \frac{p}{q}\,,$$

k^2 being the given size of the first, and p and q the base and altitude of the parallelogram which determine the shape of the second of the required parallelograms.

If we substitute the value of y, we get

$$(a - x)x = \frac{pk^2}{q}\,,$$

or,

$$ax - x^2 = b^2,$$

where a and b are known quantities, taking $b^2 = pk^2/q$.

The second case (Prop. 29) gives rise, in the same manner, to the quadratic

$$ax + x^2 = b^2.$$

The next problem—

Prop. 30. *To cut a given straight line in extreme and mean ratio*, leads to the equation

$$ax + x^2 = a^2.$$

This is, therefore, only a special case of the last, and is, besides, an old acquaintance, being essentially the same problem as that proposed in II. 11. 686

Prop. 30 may therefore be solved in two ways, either by aid of Prop. 29 or by aid of II. 11. Euclid gives both solutions.

§ 71. Prop. 31 (Theorem). *In any right-angled triangle, any rectilineal figure described on the side subtending the right angle is equal to the similar and similarly-described figures on the sides containing the right angle*,—is a pretty generalization of the theorem of Pythagoras (I. 47).

Leaving out the next proposition, which is of little interest, we come to the last in this book.

Prop. 33. *In equal circles angles, whether at the centres or the circumferences, have the same ratio which the arcs on which they stand have to one another; so also have the sectors*.

Of this, the part relating to angles at the centre is of special importance; it enables us to measure angles by arcs.

With this closes that part of the *Elements* which is devoted to the study of figures in a plane.

Book XI.

§ 72. In this book figures are considered which are not confined to a plane, viz. first relations between lines and planes in space, and afterwards properties of solids.

Of new definitions we mention those which relate to the perpendicularity and the inclination of lines and planes.

Def. 3. *A straight line is perpendicular, or at right angles, to a plane when it makes right angles with every straight line meeting it in that plane*.

The definition of perpendicular planes (Def. 4) offers no difficulty. Euclid defines the inclination of lines to planes and of planes to planes (Defs. 5 and 6) by aid of plane angles, included by straight lines, with which we have been made familiar in the first books.

The other important definitions are those of parallel planes, which never meet (Def. 8), and of solid angles formed by three or more planes meeting in a point (Def. 9).

To these we add the definition of a line parallel to a plane as a line which does not meet the plane.

§ 73. Before we investigate the contents of Book XI., it will be well to recapitulate shortly what we know of planes and lines from the definitions and axioms of the first book. There a plane has been defined as a surface which has the property that every straight line which joins two points in it lies altogether in it. This is equivalent to saying that a straight line which has two points in a plane has all points in the plane. Hence, a straight line which does not lie in the plane cannot have more than one point in common with the plane. This is virtually the same as Euclid's Prop. 1, viz.:—

Prop. 1. *One part of a straight line cannot be in a plane and another part without it*.

It also follows, as was pointed out in § 3, in discussing the definitions of Book I., that a plane is determined already by one straight line and a point without it, viz. if all lines be drawn through the point, and cutting the line, they will form a plane.

This may be stated thus:—

A plane is determined—

1st, *By a straight line and a point which does not lie on it;*

2nd, *By three points which do not lie in a straight line*; for if two of these points be joined by a straight line we have case 1;

3rd, *By two intersecting straight lines*; for the point of intersection and two other points, one in each line, give case 2;

4th, *By two parallel lines* (Def. 35, I.).

The third case of this theorem is Euclid’s

Prop. 2. *Two straight lines which cut one another are in one plane, and three straight lines which meet one another are in one plane*.

And the fourth is Euclid’s

Prop. 7. *If two straight lines be parallel, the straight line drawn from any point in one to any point in the other is in the same plane with the parallels*. From the definition of a plane further follows

Prop. 3. *If two planes cut one another, their common section is a straight line*.

§ 74. Whilst these propositions are virtually contained in the definition of a plane, the next gives us a new and fundamental property of space, showing at the same time that it is possible to have a straight line perpendicular to a plane, according to Def. 3. It states—

Prop. 4. *If a straight line is perpendicular to two straight lines in a plane which it meets, then it is perpendicular to all lines in the plane which it meets, and hence it is perpendicular to the plane*.

Def. 3 may be stated thus: If a straight line is perpendicular to a plane, then it is perpendicular to every line in the plane which it meets. The converse to this would be

All straight lines which meet a given straight line in the same point, and are perpendicular to it, lie in a plane which is perpendicular to that line.

This Euclid states thus:

Prop. 5. *If three straight lines meet all at one point, and a straight line stands at right angles to each of them at that point, the three straight lines shall be in one and the same plane*.

§ 75. There follow theorems relating to the theory of parallel lines in space, viz.:—

Prop. 6. *Any two lines which are perpendicular to the same plane are parallel to each other;* and conversely

Prop. 8. *If of two parallel straight lines one is perpendicular to a plane, the other is so also.*

Prop. 7. *If two straight lines are parallel, the straight line which joins any point in one to any point in the other is in the same plane as the parallels.* (See above, § 73.)

Prop. 9. *Two straight lines which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another*; where the words, “and not in the same plane with it,” may be omitted, for they exclude the case of three parallels in a plane, which has been proved before; and

Prop. 10. *If two angles in different planes have the two limits of the one parallel to those of the other, then the angles are equal.* That their planes are parallel is shown later on in Prop. 15.

This theorem is not necessarily true, for the angles in question may be supplementary; but then the one angle will be equal to that which is adjacent and supplementary to the other, and this latter angle will also have its limits parallel to those of the first.

From this theorem it follows that if we take any two straight lines in space which do not meet, and if we draw through any point P in space two lines parallel to them, then the angle included by these lines will always be the same, whatever the position of the point P may be. This angle has in modern times been called the angle between the given lines:—

By the angles between two not intersecting lines we understand the angles which two intersecting lines include that are parallel respectively to the two given lines.

§ 76. It is now possible to solve the following two problems:—

To draw a straight line perpendicular to a given plane from a given point which lies

1. *Not in the plane* (Prop. 11).

2. *In the plane* (Prop. 12).

The second case is easily reduced to the first—viz. if by aid of the first we have drawn any perpendicular to the plane from some point without it, we need only draw through the given point in the plane a line parallel to it, in order to have the required perpendicular given. The solution of the first part is of interest in itself. It depends upon a construction which may be expressed as a theorem.

If from a point A without a plane a perpendicular AB be drawn to the plane, and if from the foot B of this perpendicular another perpendicular BC be drawn to any straight line in the plane, then the straight line joining A to the foot C of this second perpendicular will also be perpendicular to the line in the plane.

The theory of perpendiculars to a plane is concluded by the theorem—

Prop. 13. *Through any point in space, whether in or without a plane, only one straight line can be drawn perpendicular to the plane.*

§ 77. The next four propositions treat of parallel planes. It is shown *that planes which have a common perpendicular are parallel* (Prop. 14); *that two planes are parallel if two intersecting straight lines in the one are parallel respectively to two straight lines in the other plane* (Prop. 15); *that parallel planes are cut by any plane in parallel straight lines* (Prop. 16); and lastly, *that any two straight lines are cut proportionally by a series of parallel planes* (Prop. 17).

This theory is made more complete by adding the following theorems, which are easy deductions from the last: *Two parallel planes have common perpendiculars* (converse to 14); and *Two planes which are parallel to a third plane are parallel to each other.*

It will be noted that Prop. 15 at once allows of the solution of the problem: "Through a given point to draw a plane parallel to a given plane." And it is also easily proved that this problem allows always of one, and only of one, solution.

§ 78. We come now to planes which are perpendicular to one another. Two theorems relate to them.

Prop. 18. *If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.*

Prop. 19. *If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.*

§ 79. If three planes pass through a common point, and if they bound each other, a solid angle of three faces, or a *trihedral* angle, is formed, and similarly by more planes a solid angle of more faces, or a *polyhedral* angle. These have many properties which are quite analogous to those of triangles and polygons in a plane. Euclid states some, viz.:—

Prop. 20. *If a solid angle be contained by three plane angles, any two of them are together greater than the third.*

But the next—

Prop. 21. *Every solid angle is contained by plane angles, which are together less than four right angles*—has no analogous theorem in the plane.

We may mention, however, that the theorems about triangles contained in the propositions of Book I., which do not depend upon the theory of parallels (that is all up to Prop. 27), have their corresponding theorems about trihedral angles. The latter are formed, if for "side of a triangle" we write "plane angle" or "face" of trihedral angle, and for "angle of triangle" we substitute "angle between two faces" where the planes containing the solid angle are called its *faces*. We get, for instance, from I. 4, the theorem, *If two trihedral angles have the angles of two faces in the one equal to the angles of two faces in the other, and have likewise the angles included by these faces equal, then the angles in the remaining faces are equal, and the angles between the other faces are equal each to each, viz. those which are opposite equal faces.* The solid angles themselves are not necessarily equal, for they may be only symmetrical like the right hand and the left. 687

The connexion indicated between triangles and trihedral angles will also be recognized in

Prop. 22. *If every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal, a triangle may be made of the straight lines that join the extremities of those equal straight lines.*

And Prop. 23 solves the problem, *To construct a trihedral angle having the angles of its faces equal to three given plane angles, any two of them being greater than the third.* It is, of course, analogous to the problem of constructing a triangle having its sides of given length.

Two other theorems of this kind are added by Simson in his edition of Euclid’s *Elements*.

§ 80. These are the principal properties of lines and planes in space, but before we go on to their applications it will be well to define the word *distance*. In geometry distance means always “shortest distance”; viz. the distance of a point from a straight line, or from a plane, is the length of the perpendicular from the point to the line or plane. The distance between two non-intersecting lines is the length of their common perpendicular, there being but one. The distance between two parallel lines or between two parallel planes is the length of the common perpendicular between the lines or the planes.

§ 81. *Parallelepipeds.*—The rest of the book is devoted to the study of the parallelepiped. In Prop. 24 the possibility of such a solid is proved, viz.:—

Prop. 24. *If a solid be contained by six planes two and two of which are parallel, the opposite planes are similar and equal parallelograms.*

Euclid calls this solid henceforth a parallelepiped, though he never defines the word. Either face of it may be taken as *base*, and its distance from the opposite face as *altitude*.

Prop. 25. *If a solid parallelepiped be cut by a plane parallel to two of its opposite planes, it divides the whole into two solids, the base of one of which shall be to the base of the other as the one solid is to the other.*

This theorem corresponds to the theorem (VI. 1) that parallelograms between the same parallels are to one another as their bases. A similar analogy is to be observed among a number of the remaining propositions.

§ 82. After solving a few problems we come to

Prop. 28. *If a solid parallelepiped be cut by a plane passing through the diagonals of two of the opposite planes, it shall be cut in two equal parts.*

In the proof of this, as of several other propositions, Euclid neglects the difference between solids which are symmetrical like the right hand and the left.

Prop. 31. *Solid parallelepipeds, which are upon equal bases, and of the same altitude, are equal to one another.*

Props. 29 and 30 contain special cases of this theorem leading up to the proof of the general theorem.

As consequences of this fundamental theorem we get

Prop. 32. *Solid parallelepipeds, which have the same altitude, are to one another as their bases;* and

Prop. 33. *Similar solid parallelepipeds are to one another in the triplicate ratio of their homologous sides.*

If we consider, as in § 67, the ratios of lines as numbers, we may also say—

The ratio of the volumes of similar parallelepipeds is equal to the ratio of the third powers of homologous sides.

Parallelepipeds which are not similar but equal are compared by aid of the theorem

Prop. 34. *The bases and altitudes of equal solid parallelepipeds are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the solid parallelepipeds are equal.*

§ 83. Of the following propositions the 37th and 40th are of special interest.

Prop. 37. *If four straight lines be proportionals, the similar solid parallelepipeds, similarly described from them, shall also be proportionals; and if the similar parallelepipeds similarly described from four straight lines be proportionals, the straight lines shall be proportionals.*

In symbols it says—

$$\text{If } a : b = c : d, \text{ then } a^3 : b^3 = c^3 : d^3.$$

Prop. 40 teaches how to compare the volumes of triangular prisms with those of parallelepipeds, by proving *that a triangular prism is equal in volume to a parallelepiped, which has its altitude and its base equal to the altitude and the base of the triangular prism.*

§ 84. From these propositions follow all results relating to the mensuration of volumes. We shall state these as we did in the case of areas. The starting-point is the “rectangular” parallelepiped, which has every edge perpendicular to the planes it meets, and which takes the place of the rectangle in the plane. If this has all its edges equal we obtain the “cube.”

If we take a certain line u as unit length, then the square on u is the unit of area, and the cube on u the unit of volume, that is to say, if we wish to measure a volume we have to determine how many unit cubes it contains.

A rectangular parallelepiped has, as a rule, the three edges unequal, which meet at a point. Every other edge is equal to one of them. If a, b, c be the three edges meeting at a point, then we may take the rectangle contained by two of them, say by b and c, as base and the third as altitude. Let V be its volume, V′ that of another rectangular parallelepiped which has the edges a′, b, c, hence the same base as the first. It follows then easily, from Prop. 25 or 32, that $V : V' = a : a'$; or in words,

Rectangular parallelepipeds on equal bases are proportional to their altitudes.

If we have two rectangular parallelepipeds, of which the first has the volume V and the edges a, b, c, and the second, the volume V′ and the edges a′, b′, c′, we may compare them by aid of two new ones which have respectively the edges a′, b, c and a′, b′, c, and the volumes V_1 and V_2. We then have

$$V : V_1 = a : a';\ V_1 : V_2 = b : b',\ V_2 : V' = c : c'.$$

Compounding these, we have

$$V : V' = (a : a')\,(b : b')\,(c : c'),$$

or

$$\frac{V}{V'} = \frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'}\,.$$

Hence, as a special case, making V′ equal to the unit cube U on u we get

$$\frac{V}{U} = \frac{a}{u} \cdot \frac{b}{u} \cdot \frac{c}{u} = \alpha\cdot\beta\cdot\gamma,$$

where α, β, γ are the numerical values of a, b, c; that is, *The number of unit cubes in a rectangular parallelepiped* is equal to the product of the numerical values of its three edges. This is generally expressed by saying the volume of a rectangular parallelepiped is measured by the product of its sides, or by the product of its base into its altitude, which in this case is the same.

Prop. 31 allows us to extend this to any parallelepipeds, and Props. 28 or 40, to triangular prisms.

The volume of any parallelepiped, or of any triangular prism, is measured by the product of base and altitude.

The consideration that any polygonal prism may be divided into a number of triangular prisms, which have the same altitude and the sum of their bases equal to the base of the polygonal prism, shows further that the same holds for any prism whatever.

Book XII.

§ 85. In the last part of Book XI. we have learnt how to compare the volumes of parallelepipeds and of prisms. In order to determine the volume of any solid bounded by plane faces we must determine the volume of pyramids, for every such solid may be decomposed into a number of pyramids.

As every pyramid may again be decomposed into triangular pyramids, it becomes only necessary to determine their volume. This is done by the

Theorem.—Every triangular pyramid is equal in volume to one third of a triangular prism having the same base and the same altitude as the pyramid.

This is an immediate consequence of Euclid’s

Prop. 7. *Every prism having a triangular base may be divided into three pyramids that have triangular bases, and are equal to one another.*

The proof of this theorem is difficult, because the three triangular pyramids into which the prism is divided are by no means equal in shape, and cannot be made to coincide. It has first to be proved that two triangular pyramids have equal volumes, if they have equal bases and equal altitudes. This Euclid does in the following manner. He first shows (Prop. 3) that a triangular pyramid may be divided into four parts, of which two are equal triangular pyramids similar to the whole pyramid, whilst the other two are equal triangular prisms, and further, that these two prisms together are greater than the two pyramids, hence more than half the given pyramid. He next shows (Prop. 4) that if two triangular pyramids are given, having equal bases and equal altitudes, and if each be divided as above, then the two triangular prisms in the one are equal to those in the other, and each of the remaining pyramids in the one has its base and altitude equal to the base and altitude of the remaining pyramids in the other. Hence to these pyramids the same process is again applicable. We are thus enabled to cut out of the two given pyramids equal parts, each greater than half the original pyramid. Of the remainder we can again cut out equal parts greater than half these remainders, and so on as far as we like. This process may be continued till the last remainder is smaller than any assignable quantity, however small. It follows, so we should conclude at present, that the two volumes must be equal, for they cannot differ by any assignable quantity.

To Greek mathematicians this conclusion offers far greater difficulties. They prove elaborately, by a *reductio ad absurdum*, that the volumes cannot be unequal. This proof must be read in the *Elements.* We must, however, state that we have in the above not proved Euclid's Prop. 5, but only a special case of it. Euclid does not suppose that the bases of the two pyramids to be compared are equal, and hence he proves that the volumes are as the bases. The reasoning of the proof becomes clearer in the special case, from which the general one may be easily deduced. 688

§ 86. Prop. 6 extends the result to pyramids with polygonal bases. From these results follow again the rules at present given for the mensuration of solids, viz. a pyramid is the third part of a triangular prism having the same base and the same altitude. But a triangular prism is equal in volume to a parallelepiped which has the same base and altitude. Hence if B is the base and h the altitude, we have

$$\begin{aligned} &\text{Volume of prism} &&= Bh, \\ &\text{Volume of pyramid} &&= \tfrac{1}{3}Bh, \end{aligned}$$

statements which have to be taken in the sense that B means the number of square units in the base, h the number of units of length in the altitude, or that B and h denote the numerical values of base and altitude.

§ 87. A method similar to that used in proving Prop. 5 leads to the following results relating to solids bounded by simple curved surfaces:—

Prop. 10. *Every cone is the third part of a cylinder which has the same base, and is of an equal altitude with it.*

Prop. 11. *Cones or cylinders of the same altitude are to one another as their bases.*

Prop. 12. *Similar cones or cylinders have to one another the triplicate ratio of that which the diameters of their bases have.*

Prop. 13. *If a cylinder be cut by a plane parallel to its opposite planes or bases, it divides the cylinder into two cylinders, one of which is to the other as the axis of the first to the axis of the other;* which may also be stated thus:—

Cylinders on the same base are proportional to their altitudes.

Prop. 14. *Cones or cylinders upon equal bases are to one another as their altitudes.*

Prop. 15. *The bases and altitudes of equal cones or cylinders are reciprocally proportional, and if the bases and altitudes be reciprocally proportional, the cones or cylinders are equal to one another.*

These theorems again lead to formulae in mensuration, if we compare a cylinder with a prism having its base and altitude equal to the base and altitude of the cylinder. This may be done by the method of exhaustion. We get, then, the result that their bases are equal, and have, if B denotes the numerical value of the base, and h that of the altitude,

$$\begin{aligned} &\text{Volume of cylinder} &&= Bh, \\ &\text{Volume of cone} &&= \tfrac{1}{3}Bh. \end{aligned}$$

§ 88. The remaining propositions relate to circles and spheres. Of the sphere only one property is proved, viz.:—

Prop. 18. *Spheres have to one another the triplicate ratio of that which their diameters have.* The mensuration of the sphere, like that of the circle, the cylinder and the cone, had not been settled in the time of Euclid. It was done by Archimedes.

Book XIII.

§ 89. The 13th and last book of Euclid’s *Elements* is devoted to the regular solids (see Polyhedron). It is shown that there are five of them, viz.:—

1. The regular *tetrahedron*, with 4 triangular faces and 4 vertices;
2. The *cube*, with 8 vertices and 6 square faces;
3. The *octahedron*, with 6 vertices and 8 triangular faces;
4. The *dodecahedron*, with 12 pentagonal faces, 3 at each of the 20 vertices;
5. The *icosahedron*, with 20 triangular faces, 5 at each of the 12 vertices.

It is shown how to inscribe these solids in a given sphere, and how to determine the lengths of their edges.

§ 90. The 13th book, and therefore the *Elements*, conclude with the scholium, “that no other regular solid exists besides the five ones enumerated.”

The proof is very simple. Each face is a regular polygon, hence the angles of the faces at any vertex must be angles in equal regular polygons, must be together less than four right angles (XI. 21), and must be three or more in number. Each angle in a regular triangle equals two-thirds of one right angle. Hence it is possible to form a solid angle with three, four or five regular triangles or faces. These give the solid angles of the tetrahedron, the octahedron and the icosahedron. The angle in a square (the regular quadrilateral) equals one right angle. Hence three will form a solid angle, that of the cube, and four will not. The angle in the regular pentagon equals $\frac{6}{5}$ of a right angle. Hence three of them equal $\frac{18}{5}$ (*i.e.* less than 4) right angles, and form the solid angle of the dodecahedron. Three regular polygons of six or more sides cannot form a solid angle. Therefore no other regular solids are possible.

(O. H.)

II. Projective Geometry

It is difficult, at the outset, to characterize projective geometry as compared with Euclidean. But a few examples will at least indicate the practical differences between the two.

In Euclid’s *Elements* almost all propositions refer to the *magnitude* of lines, angles, areas or volumes, and therefore to measurement. The statement that an angle is right, or that two straight lines are parallel, refers to measurement. On the other hand, the fact that a straight line does or does not cut a circle is independent of measurement, it being dependent only upon the mutual “position” of the line and the circle. This difference becomes clearer if we project any figure from one plane to another (see Projection). By this the length of lines, the magnitude of angles and areas, is altered, so that the projection, or shadow, of a square on a plane will not be a square; it will, however, be some quadrilateral. Again, the projection of a circle will not be a circle, but some other curve more or less resembling a circle. But one property may be stated at once—no straight line can cut the projection of a circle in more than two points, because no straight line can cut a circle in more than two points. There are, then, some properties of figures which do not alter by projection, whilst others do. To the latter belong nearly all properties relating to measurement, at least in the form in which they are generally given. The others are said to be projective properties, and their investigation forms the subject of projective geometry.

Different as are the kinds of properties investigated in the old and the new sciences, the methods followed differ in a still greater degree. In Euclid each proposition stands by itself; its connexion with others is never indicated; the leading ideas contained in its proof are not stated; general principles do not exist. In the modern methods, on the other hand, the greatest importance is attached to the leading thoughts which pervade the whole; and general principles, which bring whole groups of theorems under one aspect, are given rather than separate propositions. The whole tendency is towards generalization. A straight line is considered as given in its entirety, extending both ways to infinity, while Euclid never admits anything but finite quantities. The treatment of the infinite is in fact another fundamental difference between the two methods: Euclid avoids it; in modern geometry it is systematically introduced.

Of the different modern methods of geometry, we shall treat principally of the methods of projection and correspondence which have proved to be the most powerful. These have become independent of Euclidean Geometry, especially through the *Geometrie der Lage* of V. Staudt and the *Ausdehnungslehre* of Grassmann.

For the sake of brevity we shall presuppose a knowledge of Euclid’s *Elements*, although we shall use only a few of his propositions.

§ 1. *Geometrical Elements.* We consider space as filled with points, lines and planes, and these we call the elements out of which our figures are to be formed, calling any combination of these elements a “figure.”

By a line we mean a straight line in its entirety, extending both ways to infinity; and by a plane, a plane surface, extending in all directions to infinity.

We accept the three-dimensional space of experience—the space assumed by Euclid—which has for its properties (among others):—

Through any two points in space one and only one line may be drawn;

Through any three points which are not in a line, one and only one plane may be placed;

The intersection of two planes is a line;

A line which has two points in common with a plane lies in the plane, hence the intersection of a line and a plane is a single point; and

Three planes which do not meet in a line have one single point in common.

These results may be stated differently in the following form:—

I. A plane is determined—	A point is determined—
1. By three points which do not lie in a line;	1. By three planes which do not pass through a line;
2. By two intersecting lines;	2. By two intersecting lines
3. By a line and a point which does not lie in it.	3. By a plane and a line which does not lie in it.
A line is determined—	
1. By two points;	2. By two planes.

It will be observed that not only are planes determined by points, but also points by 689
planes; that therefore the planes may be considered as elements, like points; and also that in any one of the above statements we may interchange the words point and plane, and we obtain again a correct statement, provided that these statements themselves are true. As they stand, we ought, in several cases, to add “if they are not parallel,” or some such words, parallel lines and planes being evidently left altogether out of consideration. To correct this we have to reconsider the theory of parallels.

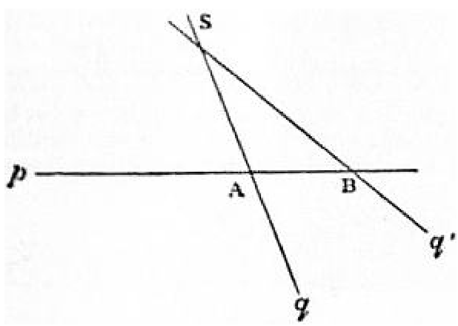


Fig. 1.

§ 2. *Parallels. Point at Infinity.*—Let us take in a plane a line p (fig. 1), a point S not in this line, and a line q drawn through S. Then this line q will meet the line p in a point A. If we turn the line q about S towards q’, its point of intersection with p will move along p towards B, passing, on continued turning, to a greater and greater distance, until it is moved out of our reach. If we turn q still farther, its continuation will meet p, but now at the other side of A. The point of intersection has disappeared to the right and reappeared to the left. There is one intermediate position where q is parallel to p—that is where it does not cut p. In every other position it cuts p in some finite point. If, on the other hand, we move the point A to an infinite distance in p, then the line q which passes through A will be a line which does not cut p at any finite point. Thus we are led to say: *Every* line through S which joins it to any point at an infinite distance in p is parallel to p. But by Euclid’s 12th axiom there is but one line parallel to p through S. The difficulty in which we are thus involved is due to the fact that we try to reason about infinity as if we, with our finite capabilities, could comprehend the infinite. To overcome this difficulty, we may say that all points at infinity in a line *appear* to us as one, and may be replaced by a single “ideal” point.

We may therefore now give the following definitions and axiom:—

Definition.—Lines which meet at infinity are called parallel.

Axiom.—All points at an infinite distance in a line may be considered as one single point.

Definition.—This ideal point is called the *point at infinity* in the line.

The axiom is equivalent to Euclid's Axiom 12, for it follows from either that through any point only one line may be drawn parallel to a given line.

This point at infinity in a line is reached whether we move a point in the one or in the opposite direction of a line to infinity. A line thus appears closed by this point, and we speak as if we could move a point along the line from one position A to another B in two ways, either through the point at infinity or through finite points only.

It must never be forgotten that this point at infinity is ideal; in fact, the whole notion of "infinity" is only a mathematical conception, and owes its introduction (as a method of research) to the working generalizations which it permits.

§ 3. *Line and Plane at Infinity.*—Having arrived at the notion of replacing all points at infinity in a line by one ideal point, there is no difficulty in replacing all points at infinity in a plane by one ideal line.

To make this clear, let us suppose that a line p, which cuts two fixed lines a and b in the points A and B, moves parallel to itself to a greater and greater distance. It will at last cut both a and b at their points at infinity, so that a line which joins the two points at infinity in two intersecting lines lies altogether at infinity. Every other line in the plane will meet it therefore at infinity, and thus it contains all points at infinity in the plane.

All points at infinity in a plane lie in a line, which is called the line at infinity *in the plane.*

It follows that parallel planes must be considered as planes having a common line at infinity, for any other plane cuts them in parallel lines which have a point at infinity in common.

If we next take two intersecting planes, then the point at infinity in their line of intersection lies in both planes, so that their lines at infinity meet. Hence every line at infinity meets every other line at infinity, and they are therefore all in one plane.

All points at infinity in space may be considered as lying in one ideal plane, which is called the plane at infinity.

§ 4. *Parallelism.*—We have now the following definitions:—

Parallel lines are lines which meet at infinity;

Parallel planes are planes which meet at infinity;

A line is parallel to a plane if it meets it at infinity.

Theorems like this—Lines (or planes) which are parallel to a third are parallel to each other—follow at once.

This view of parallels leads therefore to no contradiction of Euclid's *Elements.*

As immediate consequences we get the propositions:—

Every line meets a plane in one point, or it lies in it;

Every plane meets every other plane in a line;

Any two lines in the same plane meet.

§ 5. *Aggregates of Geometrical Elements.*—We have called points, lines and planes the elements of geometrical figures. We also say that an element of one kind contains one of the other if it lies in it or passes through it.

All the elements of one kind which are contained in one or two elements of a different kind form aggregates which have to be enumerated. They are the following:—

I. Of one dimension.

1. The *row*, or range, *of points* formed by all points in a line, which is called its base.
2. The *flat pencil* formed by all the lines through a point in a plane. Its base is the point in the plane.
3. The *axial pencil* formed by all planes through a line which is called its base or axis.

II. Of two dimensions.

1. The field of points and lines—that is, a plane with all its points and all its lines.
2. The pencil of lines and planes—that is, a point in space with all lines and all planes

through it.

III. Of three dimensions.

The space of points—that is, all points in space.

The space of planes—that is, all planes in space.

IV. Of four dimensions.

The space of lines, or all lines in space.

§ 6. *Meaning of "Dimensions."*—The word dimension in the above needs explanation. If in a plane we take a row p and a pencil with centre Q, then through every point in p one line in the pencil will pass, and every ray in Q will cut p in one point, so that we are entitled to say a row contains as many points as a flat pencil lines, and, we may add, as an axial pencil planes, because an axial pencil is cut by a plane in a flat pencil.

The number of elements in the row, in the flat pencil, and in the axial pencil is, of course, infinite and indefinite too, but the same in all. This number may be denoted by ∞. Then a plane contains ∞^2 points and as many lines. To see this, take a flat pencil in a plane. It contains ∞ lines, and each line contains ∞ points, whilst each point in the plane lies on one of these lines. Similarly, in a plane each line cuts a fixed line in a point. But this line is cut at each point by ∞ lines and contains ∞ points; hence there are ∞^2 lines in a plane.

A pencil in space contains as many lines as a plane contains points and as many planes as a plane contains lines, for any plane cuts the pencil in a field of points and lines. Hence a pencil contains ∞^2 lines and ∞^2 planes. *The field and the pencil are of two dimensions.*

To count the number of points in space we observe that each point lies on some line in a pencil. But the pencil contains ∞^2 lines, and each line ∞ points; hence space contains ∞^3 points. Each plane cuts any fixed plane in a line. But a plane contains ∞^2 lines, and through each pass ∞ planes; therefore space contains ∞^3 planes.

Hence space contains as many planes as points, but it contains an infinite number of times more lines than points or planes. To count them, notice that every line cuts a fixed plane in one point. But ∞^2 lines pass through each point, and there are ∞^2 points in the plane. Hence there are ∞^4 lines in space. *The space of points and planes is of three dimensions, but the space of lines is of four dimensions.*

A field of points or lines contains an infinite number of rows and flat pencils; a pencil contains an infinite number of flat pencils and of axial pencils; space contains a triple infinite number of pencils and of fields, ∞^4 rows and axial pencils and ∞^5 flat pencils—or, in other words, each point is a centre of ∞^2 flat pencils.

§ 7. The above enumeration allows a classification of figures. Figures in a row consist of groups of points only, and figures in the flat or axial pencil consist of groups of lines or planes. In the plane we may draw polygons; and in the pencil or in the point, solid angles, and so on.

We may also distinguish the different measurements We have—

In the row, length of segment;
In the flat pencil, angles;
In the axial pencil, dihedral angles between two planes;
In the plane, areas;
In the pencil, solid angles;
In the space of points or planes, volumes.

Segments of a Line

§ 8. Any two points A and B in space determine on the line through them a finite part, which may be considered as being described by a point moving from A to B. This we shall denote by AB, and distinguish it from BA, which is supposed as being described by a point moving from B to A, and hence in a direction or in a "sense" opposite to AB. Such a finite line, which has a definite sense, we shall call a "segment," so that AB and BA denote different segments, which are said to be equal in length but of opposite sense. The one sense is often called positive and the other negative.

In introducing the word "sense" for direction in a line, we have the word direction reserved for direction of the line itself, so that different lines have different directions, unless they be parallel, whilst in each line we have a positive and negative sense. 690

We may also say, with Clifford, that AB denotes the "step" of going from A to B.

§ 9. If we have three points A, B, C in a line (fig. 2), the step AB will bring us from A to B, and the step BC from B to C. Hence both steps are equivalent to the one step AC. This is expressed by saying that AC is the "sum" of AB and BC; in symbols—

$$AB + BC = AC,$$

where account is to be taken of the sense.

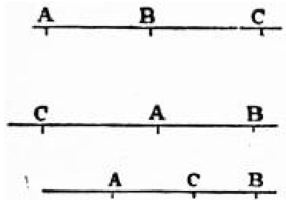


FIG. 2.

This equation is true whatever be the position of the three points on the line. As a special case we have

$$AB + BA = 0, \tag{1}$$

and similarly

$$AB + BC + CA = 0, \tag{2}$$

which again is true for any three points in a line.

We further write

$$AB = -BA.$$

where − denotes negative sense.

We can then, just as in algebra, change subtraction of segments into addition by changing the sense, so that AB − CB is the same as AB + (−CB) or AB + BC. A figure will at once show the truth of this. The sense is, in fact, in every respect equivalent to the "sign" of a number in algebra.

§ 10. Of the many formulae which exist between points in a line we shall have to use only one more, which connects the segments between any four points A, B, C, D in a line. We have

$$BC = BD + DC, CA = CD + DA, AB = AD + DB;$$

or multiplying these by AD, BD, CD respectively, we get

$$\begin{gathered} BC \cdot AD = BD \cdot AD + DC \cdot AD = BD \cdot AD - CD \cdot AD \\ CA \cdot BD = CD \cdot BD + DA \cdot BD = CD \cdot BD - AD \cdot BD \\ AB \cdot CD = AD \cdot CD + DB \cdot CD = AD \cdot CD - BD \cdot CD. \end{gathered}$$

It will be seen that the sum of the right-hand sides vanishes, hence that

$$BC \cdot AD + CA \cdot BD + AB \cdot CD = 0 \tag{3}$$

for any four points on a line.

§ 11. If C is any point in the line AB, then we say that C divides the segment AB in the ratio AC/CB, account being taken of the sense of the two segments AC and CB. If C lies between A and B the ratio is positive, as AC and CB have the same sense. But if C lies without the segment AB, *i.e.* if C divides AB externally, then the ratio is negative. To see how the value of this ratio changes with C, we will move C along the whole line (fig. 3), whilst A and B remain fixed. If C lies at the point A, then AC = 0, hence the ratio AC : CB vanishes. As C moves towards B, AC increases and CB decreases, so that our ratio increases. At the middle point M of AB it assumes the value +1, and then increases till it reaches an infinitely large value, when C arrives at B. On passing beyond B the ratio becomes negative. If C is at P we have AC = AP = AB + BP, hence

FIG. 3.

$$\frac{AC}{CB} = \frac{AB}{PB} + \frac{BP}{PB} = -\frac{AB}{BP} - 1.$$

In the last expression the ratio AB : BP is positive, has its greatest value ∞ when C coincides with B, and vanishes when BC becomes infinite. Hence, as C moves from B to the right to the point at infinity, the ratio AC : CB varies from −∞ to −1.

If, on the other hand, C is to the left of A, say at Q, we have AC = AQ = AB + BQ = AB −

QB, hence AC/CB = AB/QB − 1.

Here AB < QB, hence the ratio AB : QB is positive and always less than one, so that the whole is negative and < 1. If C is at the point at infinity it is −1, and then increases as C moves to the right, till for C at A we get the ratio = 0. Hence—

"As C moves along the line from an infinite distance to the left to an infinite distance at the right, the ratio always increases; it starts with the value −1, reaches 0 at A, +1 at M, ∞ at B, now changes sign to −∞, and increases till at an infinite distance it reaches again the value −1. *It assumes therefore all possible values from -∞ to +∞, and each value only once, so that not only does every position of* C *determine a definite value of the ratio* AC : CB, *but also, conversely, to every positive or negative value of this ratio belongs one single point in the line* AB.

[Relations between segments of lines are interesting as showing an application of algebra to geometry. The genesis of such relations from algebraic identities is very simple. For example, if a, b, c, x be any four quantities, then

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}=\frac{x}{(x-a)(x-b)(x-c)};$$

this may be proved, cumbrously, by multiplying up, or, simply, by decomposing the right-hand member of the identity into partial fractions. Now take a line ABCDX, and let AB = a, AC = b, AD = c, AX = x. Then obviously (a − b) = AB − AC = −BC, paying regard to signs; (a − c) = AB − AD = DB, and so on. Substituting these values in the identity we obtain the following relation connecting the segments formed by five points on a line:—

$$\frac{AB}{BC\cdot BD\cdot BX}+\frac{AC}{CD\cdot CB\cdot CX}+\frac{AD}{DB\cdot DC\cdot DX}=\frac{AX}{BX\cdot CX\cdot DX}.$$

Conversely, if a metrical relation be given, its validity may be tested by reducing to an algebraic equation, which is an identity if the relation be true. For example, if ABCDX be five collinear points, prove

$$\frac{AD\cdot AX}{AB\cdot AC}+\frac{BD\cdot BX}{BC\cdot BA}+\frac{CD\cdot CX}{CA\cdot CB}=1.$$

Clearing of fractions by multiplying throughout by AB · BC · CA, we have to prove

$$-AD\cdot AX\cdot BC-BD\cdot BX\cdot CA-CD\cdot CX\cdot AB=AB\cdot BC\cdot CA.$$

Take A as origin and let AB = a, AC = b, AD = c, AX = x. Substituting for the segments in terms of a, b, c, x, we obtain on simplification

$$a^2b-ab^2=-ab^2+a^2b, \text{ an obvious identity.}$$

An alternative method of testing a relation is illustrated in the following example:— If A, B, C, D, E, F be six collinear points, then

$$\frac{AE\cdot AF}{AB\cdot AC\cdot AD}+\frac{BE\cdot BF}{BC\cdot BD\cdot BA}+\frac{CE\cdot CF}{CD\cdot CA\cdot CB}+\frac{DE\cdot DF}{DA\cdot DB\cdot DC}=0.$$

Clearing of fractions by multiplying throughout by AB · BC · CD · DA, and reducing to a common origin O (calling OA = a, OB = b, &c.), an equation containing the second and lower powers of OA (= a), &c., is obtained. Calling OA = x, it is found that x = b, x = c, x = d are solutions. Hence the quadratic has three roots; consequently it is an identity.

The relations connecting five points which we have instanced above may be readily deduced from the six-point relation; the first by taking D at infinity, and the second by taking F at infinity, and then making the obvious permutations of the points.]

Projection and Cross-ratios

§ 12. If we join a point A to a point S, then the point where the line SA cuts a fixed plane π is called the projection of A on the plane π from S as centre of projection. If we have two planes π and π′ and a point S, we may project every point A in π to the other plane. If A′ is the projection of A, then A is also the projection of A′, so that the relations are reciprocal. To every figure in π we get as its projection a corresponding figure in π′.

We shall determine such properties of figures as remain true for the projection, and which are called projective properties. For this purpose it will be sufficient to consider at first only constructions in one plane.

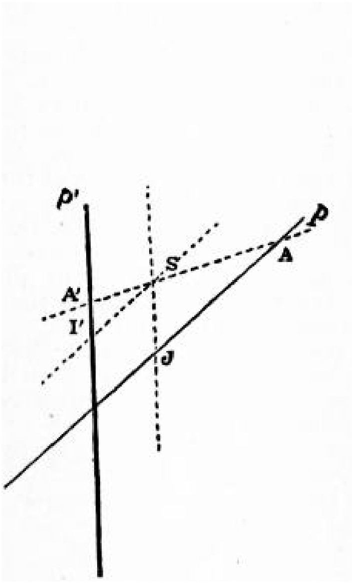


FIG. 4.

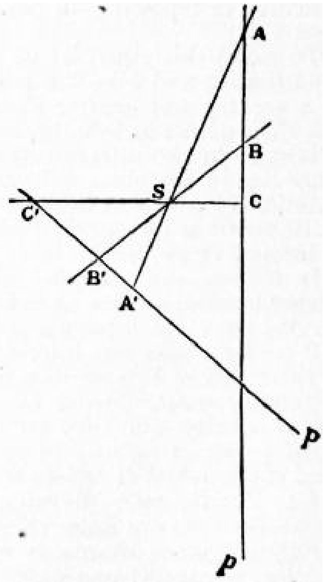


FIG. 5.

Let us suppose we have given in a plane two lines p and p′ and a centre S (fig. 4); we may then project the points in p from S to p′. Let A′, B′ ... be the projections of A, B ..., the point at infinity in p which we shall denote by I will be projected into a finite point I′ in p′, viz. into 691
the point where the parallel to p through S cuts p′. Similarly one point J in p will be projected into the point J′ at infinity in p′. This point J is of course the point where the parallel to p′ through S cuts p. We thus see that every point in p is projected into a single point in p′.

Fig. 5 shows that a segment AB will be projected into a segment A′B′ which is not equal to it, at least not as a rule; and also that the ratio AC : CB is not equal to the ratio A′C′ : C′B′ formed by the projections. These ratios will become equal only if p and p′ are parallel, for in this case the triangle SAB is similar to the triangle SA′B′. Between three points in a line and their projections there exists therefore in general no relation. But between four points a relation does exist.

§ 13. Let A, B, C, D be four points in p, A′, B′, C, D′ their projections in p′, then the ratio of the two ratios AC : CB and AD : DB into which C and D divide the segment AB is equal to the corresponding expression between A′, B′, C′, D′. In symbols we have

$$\frac{AC}{CB} : \frac{AD}{DB} = \frac{A'C'}{C'B'} : \frac{A'D'}{D'B'} .$$

This is easily proved by aid of similar triangles.

Through the points A and B on p draw parallels to p′, which cut the projecting rays in C_2, D_2, B_2 and A_1, C_1, D_1, as indicated in fig. 6. The two triangles ACC_2 and BCC_1 will be similar, as will also be the triangles ADD_2 and BDD_1.

The proof is left to the reader.

This result is of fundamental importance.

The expression AC/CB : AD/DB has been called by Chasles the “anharmonic ratio of the four points A, B, C, D.” Professor Clifford proposed the shorter name of “cross-ratio.” We shall adopt the latter. We have then the

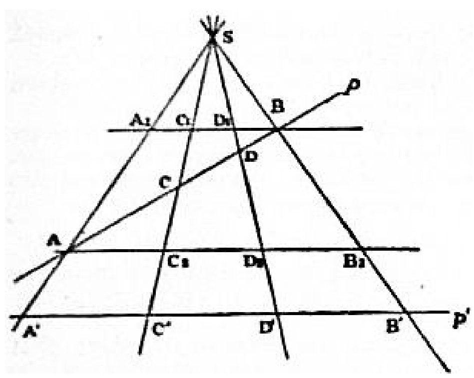


FIG. 6.

FUNDAMENTAL THEOREM.—*The cross-ratio of four points in a line is equal to the cross-ratio of their projections on any other line which lies in the same plane with it.*

§ 14. Before we draw conclusions from this result, we must investigate the meaning of a cross-ratio somewhat more fully.

If four points A, B, C, D are given, and we wish to form their cross-ratio, we have first to divide them into two groups of two, the points in each group being taken in a definite order. Thus, let A, B be the first, C, D the second pair, A and C being the first points in each pair. The cross-ratio is then the ratio AC : CB divided by AD : DB. This will be denoted by (AB, CD), so that

$$(AB, CD) = \frac{AC}{CB} : \frac{AD}{DB} .$$

This is easily remembered. In order to write it out, make first the two lines for the fractions, and put above and below these the letters A and B in their places, thus, A/*B : A/*B; and then fill up, crosswise, the first by C and the other by D.

§ 15. If we take the points in a different order, the value of the cross-ratio will change. We can do this in twenty-four different ways by forming all permutations of the letters. But of these twenty-four cross-ratios groups of four are equal, so that there are really only six different ones, and these six are reciprocals in pairs.

We have the following rules:—

I. If in a cross-ratio the two groups be interchanged, its value remains unaltered, *i.e.*

$$(AB, CD) = (CD, AB) = (BA, DC) = (DC, BA).$$

II. If in a cross-ratio the two points belonging to one of the two groups be interchanged, the cross-ratio changes into its reciprocal, *i.e.*

$$(AB, CD) = 1/(AB, DC) = 1/(BA, CD) = 1/(CD, BA) = 1/(DC, AB).$$

From I. and II. we see that eight cross-ratios are associated with (AB, CD).

III. If in a cross-ratio the two middle letters be interchanged, the cross-ratio α changes into its complement $1 - \alpha$, *i.e.* (AB, CD) = 1 − (AC, BD).

[§ 16. If λ = (AB, CD), μ = (AC, DB), ν = (AD, BC), then λ, μ, ν and their reciprocals $1/\lambda$, $1/\mu$, $1/\nu$ are the values of the total number of twenty-four cross-ratios. Moreover, λ, μ, ν are connected by the relations

$$\lambda + 1/\mu = \mu + 1/\nu = \nu + 1/\lambda = -\lambda\mu\nu = 1;$$

this proposition may be proved by substituting for λ, μ, ν and reducing to a common origin. There are therefore four equations between three unknowns; hence if one cross-ratio be given, the remaining twenty-three are determinate. Moreover, two of the quantities λ, μ, ν are positive, and the remaining one negative.

The following scheme shows the twenty-four cross-ratios expressed in terms of λ, μ, ν.]

(AB, CD) (BA, DC) (CD, AB) (DC, BA)	λ	$1 - \mu$	$1/(1 - \nu)$	(AD, BC) (BC, AD) (CB, DA) (DA, CB)	$(\lambda - 1)/\lambda$	$\mu/(\mu - 1)$	ν
(AC, DB) (BD, CA) (CA, BD) (DB, AC)	$1/(1 - \lambda)$	$1/\mu$	$(\nu - 1)/\nu$	(AC, BD) (BD, AC) (CA, DB) (DB, CA)	$1 - \lambda$	μ	$\nu/(\nu - 1)$
(AB, DC) (BA, CD) (CD, BA) (DC, AB)	$1/\lambda$	$1/(1 - \mu)$	$1 - \nu$	(AD, CB) (BC, DA) (CB, AD) (DA, BC)	$\lambda/(\lambda - 1)$	$(\mu - 1)/\mu$	$1/\nu$

§ 17. If one of the points of which a cross-ratio is formed is the point at infinity in the line, the cross-ratio changes into a simple ratio. It is convenient to let the point at infinity occupy the last place in the symbolic expression for the cross-ratio. Thus if I is a point at infinity, we have (AB, CI) = −AC/CB, because AI : IB = −1.

Every common ratio of three points in a line may thus be expressed as a cross-ratio, by adding the point at infinity to the group of points.

Harmonic Ranges

§ 18. If the points have special positions, the cross-ratios may have such a value that, of the six different ones, two and two become equal. If the first two shall be equal, we get $\lambda = 1/\lambda$, or $\lambda^2 = 1$, $\lambda = \pm 1$.

If we take $\lambda = +1$, we have (AB, CD) = 1, or AC/CB = AD/DB; that is, the points C and D coincide, provided that A and B are different.

If we take $\lambda = -1$, so that (AB, CD) = −1, we have AC/CB = −AD/DB. *Hence C and D divide AB internally and externally in the same ratio.*

The four points are in this case said to be *harmonic points*, and C *and* D *are said to be* *harmonic conjugates with regard to* A *and* B.

But we have also (CD, AB) = −1, so that A and B are harmonic conjugates with regard to C and D.

The principal property of harmonic points is that their cross-ratio remains unaltered if we interchange the two points belonging to one pair, viz.

$$(\mathrm{AB, CD}) = (\mathrm{AB, DC}) = (\mathrm{BA, CD}).$$

For four harmonic points the six cross-ratios become equal two and two:

$$\lambda = -1,\ 1 - \lambda = 2,\ \frac{\lambda}{\lambda - 1} = \tfrac{1}{2},\ \frac{1}{\lambda} = -1,\ \frac{1}{1 - \lambda} = \tfrac{1}{2},\ \frac{\lambda - 1}{\lambda} = 2.$$

Hence if we get four points whose cross-ratio is 2 or ½, then they are harmonic, but not arranged so that conjugates are paired. If this is the case the cross-ratio = −1.

§ 19. If we equate any two of the above six values of the cross-ratios, we get either $\lambda = 1$, 0, ∞, or $\lambda = -1$, 2, ½, or else λ becomes a root of the equation $\lambda^2 - \lambda + 1 = 0$, that is, an imaginary cube root of −1. In this case the six values become three and three equal, so that only two different values remain. This case, though important in the theory of cubic curves, is for our purposes of no interest, whilst harmonic points are all-important.

§ 20. From the definition of harmonic points, and by aid of § 11, the following properties are easily deduced.

If C and D are harmonic conjugates with regard to A and B, then one of them lies in, the other without AB; it is impossible to move from A to B without passing either through C or through D; the one blocks the finite way, the other the way through infinity. This is expressed by saying A and B are "separated" by C and D.

For every position of C there will be one and only one point D which is its harmonic conjugate with regard to any point pair A, B.

If A and B are different points, and if C coincides with A or B, D does. But if A and B coincide, one of the points C or D, lying between them, coincides with them, and the other may be anywhere in the line. It follows that, "*if of four harmonic conjugates two coincide, then a third coincides with them, and the fourth may be any point in the line.*"

If C is the middle point between A and B, then D is the point at infinity; for AC : CB = +1, hence AD : DB must be equal to −1. *The harmonic conjugate of the point at infinity in a line with regard to two points* A, B *is the middle point of* AB.

This important property gives a first example how metric properties are connected with projective ones.

[§ 21. *Harmonic properties of the complete quadrilateral and quadrangle.*

692

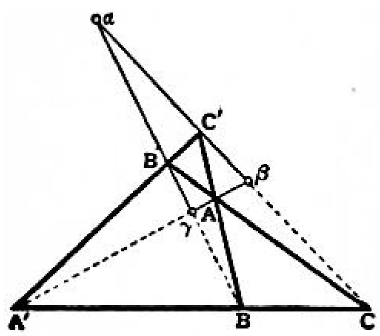


Fig. 7.

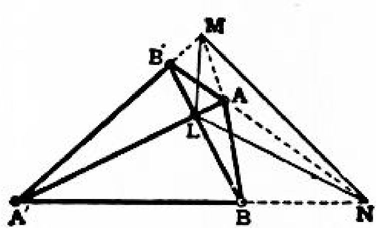


Fig. 8.

A figure formed by four lines in a plane is called a *complete quadrilateral*, or, shorter, a *four-side*. The four sides meet in six points, named the "vertices," which may be joined by three lines (other than the sides), named the "diagonals" or "harmonic lines." The diagonals enclose the "harmonic triangle of the quadrilateral." In fig. 7, A′B′C′, B′AC, C′AB, CBA′ are the sides, A, A′, B, B′, C, C′ the vertices, AA′, BB′, CC′ the harmonic lines, and αβγ the harmonic triangle of the quadrilateral. A figure formed by four coplanar points is named a *complete quadrangle*, or, shorter, a *four-point*. The four points may be joined by six lines, named the "sides," which intersect in three other points, termed the "diagonal or harmonic points." The harmonic points are the vertices of the "harmonic triangle of the complete quadrangle." In fig. 8, AA′, BB′ are the points, AA′, BB′, A′B′, B′A, AB, BA′ are the sides, L, M, N are the diagonal points, and LMN is the harmonic triangle of the quadrangle.

The harmonic property of the complete quadrilateral is: Any diagonal or harmonic line is harmonically divided by the other two; and of a complete quadrangle: The angle at any harmonic point is divided harmonically by the joins to the other harmonic points. To prove the first theorem, we have to prove (AA′, βγ), (BB′, γα), (CC′, βα) are harmonic. Consider the cross-ratio (CC′, αβ). Then projecting from A on BB′ we have A(CC′, αβ) = A(B′B, αγ). Projecting from A′ on BB′, A′(CC′, αβ) = A′(BB′, αγ). Hence (B′B, αγ) = (BB′, αγ), *i.e.* the cross-ratio (BB′, αγ) equals that of its reciprocal; hence the range is harmonic.

The second theorem states that the pencils L(BA, NM), M(B′A, LN), N(BA, LM) are harmonic. Deferring the subject of harmonic pencils to the next section, it will suffice to state here that any transversal intersects an harmonic pencil in an harmonic range. Consider the pencil L(BA, NM), then it is sufficient to prove (BA′, NM′) is harmonic. This follows from the previous theorem by considering A′B as a diagonal of the quadrilateral ALB′M.]

This property of the complete quadrilateral allows the solution of the problem:

To construct the harmonic conjugate D *to a point* C *with regard to two given points* A *and* B.

Through A draw any two lines, and through C one cutting the former two in G and H. Join these points to B, cutting the former two lines in E and F. The point D where EF cuts AB will be the harmonic conjugate required.

This remarkable construction requires nothing but the drawing of lines, and is therefore independent of measurement. In a similar manner the harmonic conjugate of the line VA for two lines VC, VD is constructed with the aid of the property of the complete quadrangle.

§ 22. *Harmonic Pencils.*—The theory of cross-ratios may be extended from points in a row to lines in a flat pencil and to planes in an axial pencil. We have seen (§ 13) that if the lines which join four points A, B, C, D to any point S be cut by any other line in A′, B′, C′, D′, then (AB, CD) = (A′B′, C′D′). In other words, four lines in a flat pencil are cut by every other line in four points whose cross-ratio is constant.

Definition.—By the cross-ratio of four rays in a flat pencil is meant the cross-ratio of the four points in which the rays are cut by any line. If a, b, c, d be the lines, then this cross-ratio is denoted by (ab, cd).

Definition.—By the cross-ratio of four planes in an axial pencil is understood the cross-ratio of the four points in which any line cuts the planes, or, what is the same thing, the cross-ratio of the four rays in which any plane cuts the four planes.

In order that this definition may have a meaning, it has to be proved that all lines cut the pencil in points which have the same cross-ratio. This is seen at once for two intersecting lines, as their plane cuts the axial pencil in a flat pencil, which is itself cut by the two lines. The cross-ratio of the four points on one line is therefore equal to that on the other, and equal to that of the four rays in the flat pencil.

If two non-intersecting lines p and q cut the four planes in A, B, C, D and A′, B′, C′, D′, draw a line r to meet both p and q, and let this line cut the planes in A″, B″, C″, D″. Then (AB, CD) = (A′B′, C′D′), for each is equal to (A″B″, C″D″).

§ 23. We may now also extend the notion of harmonic elements, viz.

Definition.—Four rays in a flat pencil and four planes in an axial pencil are said to be harmonic if their cross-ratio equals -1, that is, if they are cut by a line in four harmonic points.

If we understand by a “median line” of a triangle a line which joins a vertex to the middle point of the opposite side, and by a “median line” of a parallelogram a line joining middle points of opposite sides, we get as special cases of the last theorem:

The diagonals and median lines of a parallelogram form an harmonic pencil; and

At a vertex of any triangle, the two sides, the median line, and the line parallel to the base form an harmonic pencil.

Taking the parallelogram a rectangle, or the triangle isosceles, we get:

Any two lines and the bisections of their angles form an harmonic pencil. Or:

In an harmonic pencil, if two conjugate rays are perpendicular, then the other two are equally inclined to them; and, conversely, *if one ray bisects the angle between conjugate rays, it is perpendicular to its conjugate*.

This connects perpendicularity and bisection of angles with projective properties.

§ 24. We add a few theorems and problems which are easily proved or solved by aid of harmonics.

An harmonic pencil is cut by a line parallel to one of its rays in three equidistant points.

Through a given point to draw a line such that the segment determined on it by a given angle is bisected at that point.

Having given two parallel lines, to bisect on either any given segment without using a pair of compasses.

Having given in a line a segment and its middle point, to draw through any given point in the plane a line parallel to the given line.

To draw a line which joins a given point to the intersection of two given lines which meet off the drawing paper (by aid of § 21).

Correspondence. Homographic and Perspective Ranges

§ 25. Two rows, p and p′, which are one the projection of the other (as in fig. 5), stand in a definite relation to each other, characterized by the following properties.

1. *To each point in either corresponds one point in the other*; that is, those points are said to correspond which are projections of one another.

2. *The cross-ratio of any four points in one equals that of the corresponding points in the other.*

3. *The lines joining corresponding points all pass through the same point.*

If we suppose corresponding points marked, and the rows brought into any other position, then the lines joining corresponding points will no longer meet in a common point, and hence the third of the above properties will not hold any longer; but we have still a correspondence between the points in the two rows possessing the first two properties. Such a correspondence has been called a *one-one correspondence*, whilst the two rows between which such correspondence has been established are said to be *projective* or *homographic*. Two rows which are each the projection of the other are therefore *projective*. We shall presently see, also, that any two projective rows may always be placed in such a position that one appears as the projection of the other. If they are in such a position the rows are said to be in *perspective position*, or simply to be in *perspective*.

§ 26. The notion of a one-one correspondence between rows may be extended to flat and axial pencils, viz. a flat pencil will be said to be projective to a flat pencil if to each ray in the first corresponds one ray in the second, and if the cross-ratio of four rays in one equals that of the corresponding rays in the second.

Similarly an axial pencil may be projective to an axial pencil. But a flat pencil may also be projective to an axial pencil, or either pencil may be projective to a row. The definition is the same in each case: there is a one-one correspondence between the elements, and four elements have the same cross-ratio as the corresponding ones.

§ 27. There is also in each case a special position which is called *perspective*, viz.

1. Two projective rows are perspective if they lie in the same plane, and if the one row is a projection of the other.

2. Two projective flat pencils are perspective—(1) if they lie in the same plane, and have a row as a common section; (2) if they lie in the same pencil (in space), and are both sections of the same axial pencil; (3) if they are in space and have a row as common section, or are both sections of the same axial pencil, one of the conditions involving the other.

3. Two projective axial pencils, if their axes meet, and if they have a flat pencil as a common section.

4. A row and a projective flat pencil, if the row is a section of the pencil, each point lying in its corresponding line.

5. A row and a projective axial pencil, if the row is a section of the pencil, each point lying in its corresponding line.

6. A flat and a projective axial pencil, if the former is a section of the other, each ray lying in its corresponding plane.

That in each case the correspondence established by the position indicated is such as has been called projective follows at once from the definition. It is not so evident that the perspective position may always be obtained. We shall show in § 30 this for the first three cases. First, however, we shall give a few theorems which relate to the general correspondence, not to the perspective position.

§ 28. *Two rows or pencils, flat or axial, which are projective to a third are projective to each other*; this follows at once from the definitions.

§ 29. *If two rows, or two pencils, either flat or axial, or a row and a pencil, be projective, we may assume to any three elements in the one the three corresponding elements in the other, and then the correspondence is uniquely determined.*

For if in two projective rows we assume that the points A, B, C in the first correspond to the given points A′, B′, C′ in the second, then to any fourth point D in the first will correspond a point D′ in the second, so that

$$(AB, CD) = (A'B', C'D').$$

But there is only one point, D′, which makes the cross-ratio (A′B′, C′D′) equal to the given number (AB, CD).

The same reasoning holds in the other cases.

§ 30. If two rows are perspective, then the lines joining corresponding points all meet in a point, the centre of projection; and the point in which the two bases of the rows intersect as a point in the first row coincides with its corresponding point in the second.

This follows from the definition. The converse also holds, viz.

If two projective rows have such a position that one point in the one coincides with its corresponding point in the other, then they are perspective, that is, the lines joining corresponding points all pass through a common point, and form a flat pencil.

For let A, B, C, D ... be points in the one, and A′, B′, C′, D′ ... the corresponding points in the other row, and let A be made to coincide with its corresponding point A′. Let S be the point where the lines BB′ and CC′ meet, and let us join S to the point D in the first row. This line will cut the second row in a point D″, so that A, B, C, D are projected from S into the points A, B′, C′, D″. The cross-ratio (AB, CD) is therefore equal to (AB′, C′D″), and by hypothesis it is equal to (A′B′, C′D′). Hence (A′B′, C′D″) = (A′B′, C′D′), that is, D″ is the same point as D′.

§ 31. If two projected flat pencils in the same plane are in perspective, then the intersections of corresponding lines form a row, and the line joining the two centres as a line in the first pencil corresponds to the same line as a line in the second. And conversely,

If two projective pencils in the same plane, but with different centres, have one line in the one coincident with its corresponding line in the other, then the two pencils are perspective, that is, the intersection of corresponding lines lie in a line.

The proof is the same as in § 30.

§ 32. If two projective flat pencils in the same point (pencil in space), but not in the same plane, are perspective, then the planes joining corresponding rays all pass through a line (they form an axial pencil), and the line common to the two pencils (in which their planes intersect) corresponds to itself. And conversely:—

If two flat pencils which have a common centre, but do not lie in a common plane, are placed so that one ray in the one coincides with its corresponding ray in the other, then they are perspective, that is, the planes joining corresponding lines all pass through a line.

§ 33. If two projective axial pencils are perspective, then the intersection of corresponding planes lie in a plane, and the plane common to the two pencils (in which the two axes lie) corresponds to itself. And conversely:—

If two projective axial pencils are placed in such a position that a plane in the one coincides with its corresponding plane, then the two pencils are perspective, that is, corresponding planes meet in lines which lie in a plane.

The proof again is the same as in § 30.

§ 34. These theorems relating to perspective position become illusory if the projective rows of pencils have a common base. We then have:—

In two projective rows on the same line—and also in two projective and concentric flat pencils in the same plane, or in two projective axial pencils with a common axis—every element in the one coincides with its corresponding element in the other as soon as three elements in the one coincide with their corresponding elements in the other.

Proof (in case of two rows).—Between four elements A, B, C, D and their corresponding elements A′, B′, C′, D′ exists the relation (ABCD) = (A′B′C′D′). If now A′, B′, C′ coincide respectively with A, B, C, we get (AB, CD) = (AB, CD′), hence D and D′ coincide.

The last theorem may also be stated thus:—

In two projective rows or pencils, which have a common base but are not identical, not more than two elements in the one can coincide with their corresponding elements in the other.

Thus two projective rows on the same line cannot have more than two pairs of coincident points unless every point coincides with its corresponding point.

It is easy to construct two projective rows on the same line, which have two pairs of corresponding points coincident. Let the points A, B, C as points belonging to the one row correspond to A, B, and C′ as points in the second. Then A and B coincide with their corresponding points, but C does not. It is, however, not necessary that two such rows have twice a point coincident with its corresponding point; it is possible that this happens only once or not at all. Of this we shall see examples later.

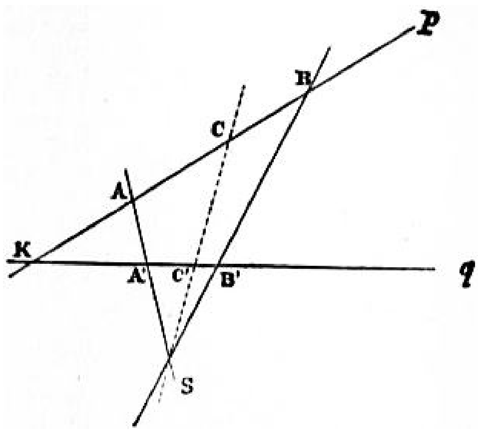


Fig. 9.

§ 35. If two projective rows or pencils are in perspective position, we know at once which element in one corresponds to any given element in the other. If p and q (fig. 9) are two projective rows, so that K corresponds to itself, and if we know that to A and B in p correspond A′ and B′ in q, then the point S, where AA′ meets BB′, is the centre of projection, and hence, in order to find the point C′ corresponding to C, we have only to join C to S; the point C′, where this line cuts q, is the point required.

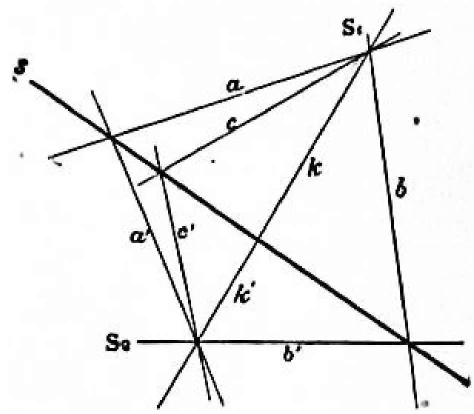


Fig. 10.

If two flat pencils, S_1 and S_2, in a plane are perspective (fig. 10), we need only to know two pairs, a, a′ and b, b′, of corresponding rays in order to find the axis s of projection. This being known, a ray c′ in S_2, corresponding to a given ray c in S_1, is found by joining S_2 to the point where c cuts the axis s.

A similar construction holds in the other cases of perspective figures.

On this depends the solution of the following general problem.

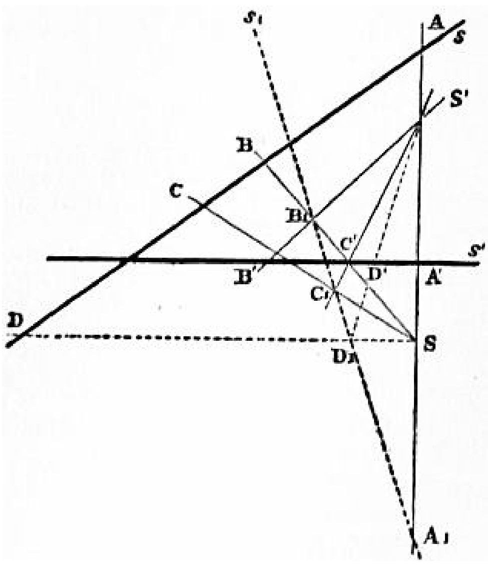


Fig. 11.

§ 36. Three pairs of corresponding elements in two projective rows or pencils being given, to determine for any element in one the corresponding element in the other.

We solve this in the two cases of two projective rows and of two projective flat pencils in a plane.

Problem I.—Let A, B, C be three points in a row s, A′, B′, C′ the corresponding points in a projective row s′, both being in a plane; it is required to find for any point D in s the corresponding point D′ in s′.

Problem II.—Let a, b, c be three rays in a pencil S, a′, b′, c′ the corresponding rays in a projective pencil S′, both being in the same plane; it is required to find for any ray d in S the corresponding ray d′ in S′.

The solution is made to depend on the construction of an auxiliary row or pencil which is perspective to both the given ones. This is found as follows:—

Solution of Problem I.—On the line joining two corresponding points, say AA′ (fig. 11), take any two points, S and S′, as centres of auxiliary pencils. Join the intersection B_1 of SB and S′B′ to the intersection C_1 of SC and S′C′ by the line s_1. Then a row on s_1 will be perspective to s with S as centre of projection, and to s′ with S′ as centre. To find now the point D′ on s′ corresponding to a point D on s we have only to determine the point D_1, where the line SD cuts s_1, and to draw $S'D_1$; the point where this line cuts s′ will be the required point D′.

Proof.—The rows s and s′ are both perspective to the row s_1, hence they are projective to one another. To A, B, C, D on s correspond A_1, B_1, C_1, D_1 on s_1, and to these correspond A′, B′, C′, D′ on s′; so that D and D′ are corresponding points as required.

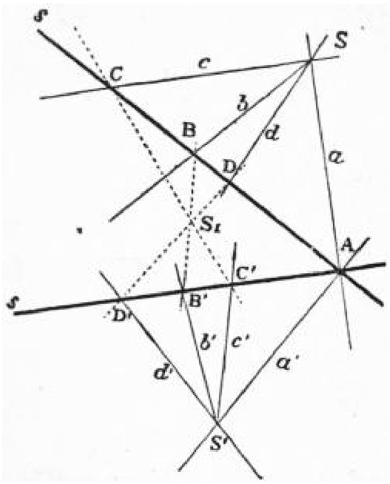


Fig. 12.

Solution of Problem II.—Through the intersection 694
A of two corresponding rays a and a′ (fig. 12), take two lines, s and s′, as bases of auxiliary rows. Let S_1 be the point where the line b_1, which joins B and B′, cuts the line c_1, which joins C and C′. Then a pencil S_1 will be perspective to S with s as axis of projection. To find the ray d′ in S′ corresponding to a given ray d in S, cut d by s at D; project this point from S_1 to D′ on s′ and join D′ to S′. This will be the required ray.

Proof.—That the pencil S_1 is perspective to S and also to S′ follows from construction. To the lines a_1, b_1, c_1, d_1 in S_1 correspond the lines a, b, c, d in S and the lines a′, b′, c′, d′ in S′, so that d and d′ are corresponding rays.

In the first solution the two centres, S, S′, are *any* two points on a line joining any two corresponding points, so that the solution of the problem allows of a great many different constructions. *But whatever construction be used, the point* D′, *corresponding to* D, *must be always the same*, according to the theorem in § 29. This gives rise to a number of theorems, into which, however, we shall not enter. The same remarks hold for the second problem.

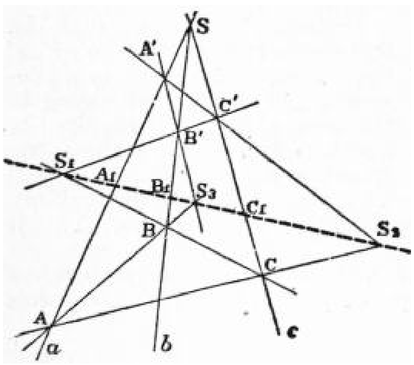


Fig. 13.

§ 37. *Homological Triangles.*—As a further application of the theorems about perspective rows and pencils we shall prove the following important theorem.

Theorem.—If ABC and A′B′C′ (fig. 13) be two triangles, such that the lines AA′, BB′, CC′ meet in a point S, then the intersections of BC and B′C′, of CA and C′A′, and of AB and A′B′ will lie in a line. Such triangles are said to be homological, or in perspective. The triangles are “co-axial” in virtue of the property that the meets of corresponding sides are collinear and copolar, since the lines joining corresponding vertices are concurrent.

Proof.—Let a, b, c denote the lines AA′, BB′, CC′, which meet at S. Then these may be taken as bases of projective rows, so that A, A′, S on a correspond to B, B′, S on b, and to C, C′, S on c. As the point S is common to all, any two of these rows will be perspective.

If	S_1 be the centre of projection of rows				b and c,
	S_2	”	”	”	c and a,
	S_3	”	”	”	a and b,

and if the line S_1S_2 cuts a in A_1, and b in B_1, and c in C_1, then A_1, B_1 will be corresponding points in a and b, both corresponding to C_1 in c. But a and b are perspective, therefore the line A_1B_1, that is S_1S_2, joining corresponding points must pass through the centre of projection S_3 of a and b. In other words, S_1, S_2, S_3 lie in a line. This is Desargues’ celebrated theorem if we state it thus:—

Theorem of Desargues.—If each of two triangles has one vertex on each of three concurrent lines, then the intersections of corresponding sides lie in a line, those sides being called corresponding which are opposite to vertices on the same line.

The converse theorem holds also, viz.

Theorem.—If the sides of one triangle meet those of another in three points which lie in a line, then the vertices lie on three lines which meet in a point.

The proof is almost the same as before.

§ 38. *Metrical Relations between Projective Rows.*—Every row contains one point which is distinguished from all others, viz. the point at infinity. In two projective rows, to the point I at infinity in one corresponds a point I' in the other, and to the point J' at infinity in the second corresponds a point J in the first. The points I' and J are in general finite. If now A and B are any two points in the one, A', B' the corresponding points in the other row, then

$$(AB, JI) = (A'B', J'I'),$$

or

$$AJ/JB : AI/IB = A'J'/J'B' : A'I'/I'B'.$$

But, by § 17,

$$AI/IB = A'J'/J'B' = -1;$$

therefore the last equation changes into

$$AJ \cdot A'I' = BJ \cdot B'I',$$

that is to say—

Theorem.—The product of the distances of any two corresponding points in two projective rows from the points which correspond to the points at infinity in the other is constant, viz. $AJ \cdot A'I' = k$. Steiner has called this number k the *Power of the correspondence*.

[The relation $AJ \cdot A'I' = k$ shows that if J, I' be given then the point A' corresponding to a specified point A is readily found; hence A, A' generate homographic ranges of which I and J' correspond to the points at infinity on the ranges. If we take any two origins O, O', on the ranges and reduce the expression $AJ \cdot A'I' = k$ to its algebraic equivalent, we derive an equation of the form $\alpha xx' + \beta x + \gamma x' + \delta = 0$. Conversely, if a relation of this nature holds, then points corresponding to solutions in x, x' form homographic ranges.]

§ 39. *Similar Rows.*—If the points at infinity in two projective rows correspond so that I' and J are at infinity, this result loses its meaning. But if A, B, C be any three points in one, A', B', C' the corresponding ones on the other row, we have

$$(AB, CI) = (A'B', C'I'),$$

which reduces to

$$AC/CB = A'C'/C'B' \text{ or } AC/A'C' = BC/B'C',$$

that is, corresponding segments are proportional. Conversely, if corresponding segments are proportional, then to the point at infinity in one corresponds the point at infinity in the other. If we call such rows *similar*, we may state the result thus—

Theorem.—Two projective rows are similar if to the point at infinity in one corresponds the point at infinity in the other, and conversely, if two rows are similar then they are projective, and the points at infinity are corresponding points.

From this the well-known propositions follow:—

Two lines are cut proportionally (in similar rows) by a series of parallels. The rows are perspective, with centre of projection at infinity.

If two similar rows are placed parallel, then the lines joining homologous points pass through a common point.

§ 40. If two flat pencils be projective, then there exists in either, one single pair of lines at right angles to one another, such that the corresponding lines in the other pencil are again at right angles.

To prove this, we place the pencils in perspective position (fig. 14) by making one ray coincident with its corresponding ray. Corresponding rays meet then on a line p. And now we draw the circle which has its centre O on p, and which passes through the centres S and S' of the two pencils. This circle cuts p in two points H and K. The two pairs of rays, h, k, and h', k', joining these points to S and S' will be pairs of corresponding rays at right angles. The construction gives in general but one circle, but if the line p is the perpendicular bisector of SS′, there exists an infinite number, and *to every right angle in the one pencil corresponds a right angle in the other*.

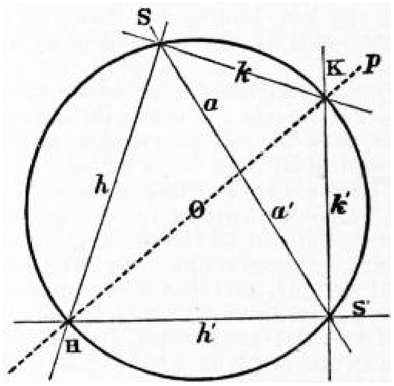


Fig. 14.

Principle of Duality

§ 41. It has been stated in § 1 that not only points, but also planes and lines, are taken as elements out of which figures are built up. We shall now see that the construction of one figure which possesses certain properties gives rise in many cases to the construction of another figure, by replacing, according to definite rules, elements of one kind by those of another. The new figure thus obtained will then possess properties which may be stated as soon as those of the original figure are known.

We obtain thus a principle, known as the *principle of duality* or of *reciprocity*, which enables us to construct to any figure not containing any measurement in its construction a *reciprocal* figure, as it is called, and to deduce from any theorem a *reciprocal* theorem, for which no further proof is needed.

It is convenient to print reciprocal propositions on opposite sides of a page broken into two columns, and this plan will occasionally be adopted.

We begin by repeating in this form a few of our former statements:—

Two points determine a line.	Two planes determine a line.
Three points which are not in a line determine a plane.	Three planes which do not pass through a line determine a point.
A line and a point without it determine a plane.	A line and a plane not through it determine a point.
Two lines in a plane determine a point.	Two lines through a point determine a plane.

These propositions show that it will be possible, when any figure is given, to construct a second figure by taking planes instead of points, and points instead of planes, but lines where we had lines.

For instance, if in the first figure we take a plane and three points in it, we have to take in 695
the second figure a point and three planes through it. The three points in the first, together with the three lines joining them two and two, form a triangle; the three planes in the second and their three lines of intersection form a trihedral angle. A triangle and a trihedral angle are therefore reciprocal figures.

Similarly, to any figure in a plane consisting of points and lines will correspond a figure consisting of planes and lines passing through a point S, and hence belonging to the pencil which has S as centre.

The figure reciprocal to four points in space which do not lie in a plane will consist of four planes which do not meet in a point. In this case each figure forms a tetrahedron.

§ 42. As other examples we have the following:—

To a row	is reciprocal	an axial pencil,
to a flat pencil	”	a flat pencil,
to a field of points and lines	”	a pencil of planes and lines,
to the space of points	”	the space of planes.

For the row consists of a line and all the points in it, reciprocal to it therefore will be a line with all planes through it, that is, an axial pencil; and so for the other cases.

This correspondence of reciprocity breaks down, however, if we take figures which contain measurement in their construction. For instance, there is no figure reciprocal to two planes at *right angles*, because there is no segment in a row which has a magnitude as definite as a right angle.

We add a few examples of reciprocal propositions which are easily proved.

Theorem.—If A, B, C, D are any four points in space, and if the lines AB and CD meet, then all four points lie in a plane, hence also AC and BD, as well as AD and BC, meet.	*Theorem.*—If α, β, γ, δ are four planes in space, and if the lines $\alpha\beta$ and $\gamma\delta$ meet, then all four planes lie in a point (pencil), hence also $\alpha\gamma$ and $\beta\delta$, as well as $\alpha\delta$ and $\beta\gamma$, meet.

Theorem.—*If of any number of lines every one meets every other, whilst all do not*

lie in a point, then all lie in a plane.	*lie in a plane, then all lie in a point* (*pencil*).

§ 43. Reciprocal figures as explained lie both in space of three dimensions. If the one is confined to a plane (is formed of elements which lie in a plane), then the reciprocal figure is confined to a pencil (is formed of elements which pass through a point).

But there is also a more special principle of duality, according to which figures are reciprocal which lie both in a plane or both in a pencil. In the plane we take points and lines as reciprocal elements, for they have this fundamental property in common, that two elements of one kind determine one of the other. In the pencil, on the other hand, lines and planes have to be taken as reciprocal, and here it holds again that two lines or planes determine one plane or line.

Thus, to one plane figure we can construct one reciprocal figure in the plane, and to each one reciprocal figure in a pencil. We mention a few of these. At first we explain a few names:—

A figure consisting of n points in a plane will be called an n-point.	A figure consisting of n lines in a plane will be called an n-side.
A figure consisting of n planes in a pencil will be called an n-flat.	A figure consisting of n lines in a pencil will be called an n-edge.

It will be understood that an n-side is different from a polygon of n sides. The latter has sides of finite length and n vertices, the former has sides all of infinite extension, and every point where two of the sides meet will be a vertex. A similar difference exists between a solid angle and an n-edge or an n-flat. We notice particularly—

A four-point has six sides, of which two and two are opposite, and three diagonal points, which are intersections of opposite sides.	A four-side has six vertices, of which two and two are opposite, and three diagonals, which join opposite vertices.
A four-flat has six edges, of which two and two are opposite, and three diagonal planes, which pass through opposite edges.	A four-edge has six faces, of which two and two are opposite, and three diagonal edges, which are intersections of opposite faces.

A four-side is usually called a complete quadrilateral, and a four-point a complete quadrangle. The above notation, however, seems better adapted for the statement of reciprocal propositions.

§ 44.

If a point moves in a plane it describes a plane curve.	If a line moves in a plane it envelopes a plane curve (fig. 15).
If a plane moves in a pencil it envelopes a cone.	If a line moves in a pencil it describes a cone.

A curve thus appears as generated either by points, and then we call it a "locus," or by lines, and then we call it an "envelope." In the same manner a cone, which means here a surface, appears either as the locus of lines passing through a fixed point, the "vertex" of the cone, or as the envelope of planes passing through the same point.

To a surface as locus of points corresponds, in the same manner, a surface as envelope of planes; and to a curve in space as locus of points corresponds a developable surface as envelope of planes.

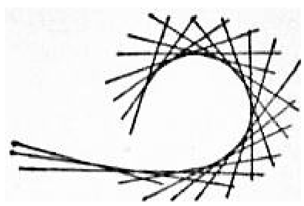


Fig. 15.

It will be seen from the above that we may, by aid of the principle of duality, construct for every figure a reciprocal figure, and that to any property of the one a reciprocal property of the other will exist, as long as we consider only properties which depend upon nothing but the positions and intersections of the different elements and not upon measurement.

For such propositions it will therefore be unnecessary to prove more than one of two reciprocal theorems.

Generation of Curves and Cones of Second Order or Second Class

§ 45. *Conics.*—If we have two projective pencils in a plane, corresponding rays will meet, and their point of intersection will constitute some locus which we have to investigate. Reciprocally, if two projective rows in a plane are given, then the lines which join corresponding points will envelope some curve. We prove first:—

Theorem.—If two projective flat pencils lie in a plane, but are neither in perspective nor concentric, then the locus of intersections of corresponding rays is a curve of the second order, that is, no line contains more than two points of the locus.

Proof.—We draw any line t. This cuts each of the pencils in a row, so that we have on t two rows, and these are projective because the pencils are projective. If corresponding rays of the two pencils meet on the line t, their intersection will be a point in the one row which coincides with its corresponding point in the other. But two projective rows on the same base cannot have more than two points of one coincident with their corresponding points in the other (§ 34).

Theorem.—If two projective rows lie in a plane, but are neither in perspective nor on a common base, then the envelope of lines joining corresponding points is a curve of the second class, that is, through no point pass more than two of the enveloping lines.

Proof.—We take any point T and join it to all points in each row. This gives two concentric pencils, which are projective because the rows are projective. If a line joining corresponding points in the two rows passes through T, it will be a line in the one pencil which coincides with its corresponding line in the other. But two projective concentric flat pencils in the same plane cannot have more than two lines of one coincident with their corresponding line in the other (§ 34).

It will be seen that the proofs are reciprocal, so that the one may be copied from the other by simply interchanging the words point and line, locus and envelope, row and pencil, and so on. We shall therefore in future prove seldom more than one of two reciprocal theorems, and often state one theorem only, the reader being recommended to go through the reciprocal proof by himself, and to supply the reciprocal theorems when not given.

§ 46. We state the theorems in the pencil reciprocal to the last, without proving them:—

Theorem.—If two projective flat pencils are concentric, but are neither perspective nor coplanar, then the envelope of the planes joining corresponding rays is a cone of the second class; that is, no line through the common centre contains more than two of the enveloping planes.

Theorem.—If two projective axial pencils lie in the same pencil (their axes meet in a point), but are neither perspective nor co-axial, then the locus of lines joining corresponding planes is a cone of the second order; that is, no plane in the pencil contains more than two of these lines.

§ 47. Of theorems about cones of second order and cones of second class we shall state only very few. We point out, however, the following connexion between the curves and cones under consideration:

The lines which join any point in space to the points on a curve of the second order form a cone of the second order.

The planes which join any point in space to the lines enveloping a curve of the second class envelope themselves a cone of the second class.

Every plane section of a cone of the second order is a curve of the second order.

Every plane section of a cone of the second class is a curve of the second class.

By its aid, or by the principle of duality, it will be easy to obtain theorems about them from the theorems about the curves.

We prove the first. A curve of the second order is generated by two projective pencils. These pencils, when joined to the point in space, give rise to two projective axial pencils, which generate the cone in question as the locus of the lines where corresponding planes meet.

§48.

Theorem.—The curve of second order which is generated by two projective flat pencils passes through the centres of the two pencils.

Proof.—If S and S′ are the two pencils, then to the ray SS′ or p′ in the pencil S′ corresponds in the pencil S a ray p, which is different from p′, for the pencils are not perspective. But p and p′ meet at S, so that S is a point on the curve, and similarly S′.

Theorem.—The envelope of second class which is generated by two projective rows contains the bases of these rows as enveloping lines or tangents.

Proof.—If s and s′ are the two rows, then to the point ss′ or P′ as a point in s′ corresponds in s a point P, which is not coincident with P′, for the rows are not perspective. But P and P′ are joined by s, so that s is one of the enveloping lines, and similarly s′.

It follows that every line in one of the two pencils cuts the curve in two points, viz. once at the centre S of the pencil, and once where it cuts its corresponding ray in the other pencil. These two points, however, coincide, if the line is cut by its corresponding line at S itself. The line p in S, which corresponds to the line SS′ in S′, is therefore the only line through S which has but one point in common with the curve, or which cuts the curve in two coincident points. Such a line is called a *tangent* to the curve, touching the latter at the point S, which is called the "point of contact."

In the same manner we get in the reciprocal investigation the result that through every point in one of the rows, say in s, two tangents may be drawn to the curve, the one being s, the other the line joining the point to its corresponding point in s′. There is, however, one point P in s for which these two lines coincide. Such a point in one of the tangents is called the "point of contact" of the tangent. We thus get—

Theorem.—To the line joining the centres of the projective pencils as a line in one pencil corresponds in the other the tangent at its centre.

Theorem.—To the point of intersection of the bases of two projective rows as a point in one row corresponds in the other the *point of contact* of its base.

§ 49. Two projective pencils are determined if three pairs of corresponding lines are given. Hence if a_1, b_1, c_1 are three lines in a pencil S_1, and a_2, b_2, c_2 the corresponding lines in a projective pencil S_2, the correspondence and therefore the curve of the second order generated by the points of intersection of corresponding rays is determined. Of this curve we know the two centres S_1 and S_2, and the three points a_1a_2, b_1b_2, c_1c_2, hence five points in all. This and the reciprocal considerations enable us to solve the following two problems:

Problem.—To construct a curve of the second order, of which five points S_1, S_2, A, B, C are given.

Problem.—To construct a curve of the second class, of which five tangents u_1, u_2, a, b, c are given.

In order to solve the left-hand problem, we take two of the given points, say S_1 and S_2, as centres of pencils. These we make projective by taking the rays a_1, b_1, c_1, which join S_1 to A, B, C respectively, as corresponding to the rays a_2, b_2, c_2, which join S_2 to A, B, C respectively, so that three rays meet their corresponding rays at the given points A, B, C. This determines the correspondence of the pencils which will generate a curve of the second order passing through A, B, C and through the centres S_1 and S_2, hence through the five given points. To find more points on the curve we have to construct for any ray in S_1 the corresponding ray in S_2. This has been done in § 36. But we repeat the construction in order to deduce further properties from it. We also solve the right-hand problem. Here we select two, viz. u_1, u_2 of the five given lines, u_1, u_2, a, b, c, as bases of two rows, and the points A_1, B_1, C_1 where a, b, c cut u_1 as corresponding to the points A_2, B_2, C_2 where a, b, c cut u_2.

We get then the following solutions of the two problems:

Solution.—Through the point A draw any two lines, u_1 and u_2 (fig. 16), the first u_1 to cut the pencil S_1 in a row AB_1C_1, the other u_2 to cut the pencil S_2 in a row AB_2C_2. These two rows will be perspective, as the point A corresponds to itself, and the centre of projection will be the point S, where the lines B_1B_2 and C_1C_2 meet. To find now for any ray d_1 in S_1 its corresponding ray d_2 in S_2, we determine the point D_1 where d_1 cuts u_1, project this point from S to D_2 on u_2 and join S_2 to D_2. This will be the required

Solution.—In the line a take any two points S_1 and S_2 as centres of pencils (fig. 17), the first S_1 $(A_1B_1C_1)$ to project the row u_1, the other S_2 $(A_2B_2C_2)$ to project the row u_2. These two pencils will be perspective, the line S_1A_1 being the same as the corresponding line S_2A_2, and the axis of projection will be the line u, which joins the intersection B of S_1B_1 and S_2B_2 to the intersection C of S_1C_1 and S_2C_2. To find now for any point D_1 in u_1 the corresponding point D_2 in u_2, we draw S_1D_1 and project the point D where ray d_2 which cuts d_1 at some point D on the curve.

this line cuts u from S_2 to u_2. This will give the required point D_2, and the line d joining D_1 to D_2 will be a new tangent to the curve.

§ 50. These constructions prove, when rightly interpreted, very important properties of the curves in question.

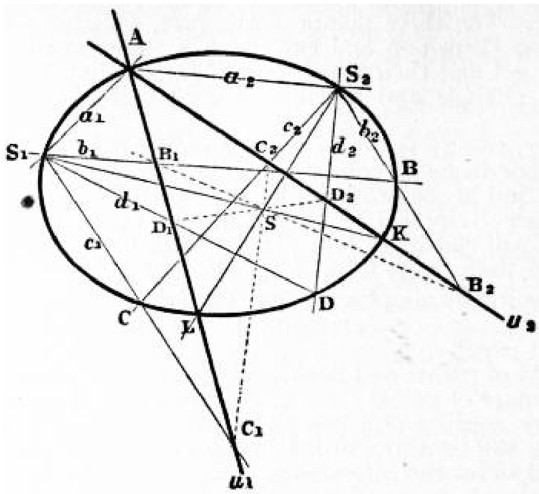


FIG. 16.

If in fig. 16 we draw in the pencil S_1 the ray k_1 which passes through the auxiliary centre S, it will be found that the corresponding ray k_2 cuts it on u_2. Hence—

Theorem.—In the above construction the bases of the auxiliary rows u_1 and u_2 cut the curve where they cut the rays S_2S and S_1S respectively.

Theorem.—In the above construction (fig. 17) the tangents to the curve from the centres of the auxiliary pencils S_1 and S_2 are the lines which pass through u_2u and u_1u respectively.

As A is any given point on the curve, and u_1 any line through it, we have solved the problems:

Problem.—To find the second point in which any line through a known point on the curve cuts the curve.

Problem.—To find the second tangent which can be drawn from any point in a given tangent to the curve.

If we determine in S_1 (fig. 16) the ray corresponding to the ray S_2S_1 in S_2, we get the tangent at S_1. Similarly, we can determine the point of contact of the tangents u_1 or u_2 in fig. 17.

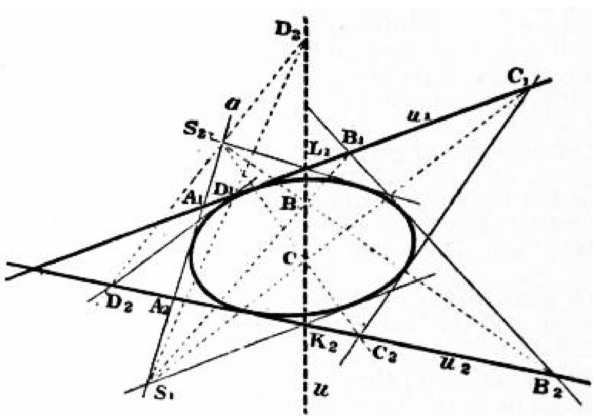


FIG. 17.

§ 51. If five points are given, of which not three are in a line, then we can, as has just been shown, always draw a curve of the second order through them; we select two of the points as centres of projective pencils, and then one such curve is determined. It will be presently shown that we get always the same curve if two other points are taken as centres of pencils, that therefore five points *determine* one curve of the second order, and reciprocally, that five tangents determine one curve of the second class. Six points taken at random will therefore not lie on a curve of the second order. In order that this may be the case a certain condition has to be satisfied, and this condition is easily obtained from the construction in § 49, fig. 16. If we consider the conic determined by the five points A, S_1, S_2, K, L, then the point D will be on the curve if, and only if, the points on D_1, S, D_2 be in a line.

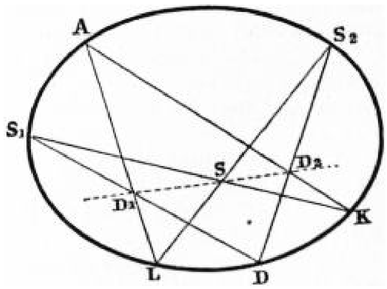


FIG. 18.

This may be stated differently if we take AKS_1DS_2L (figs. 16 and 18) as a hexagon inscribed in the conic, then AK and DS_2 will be opposite sides, so will be KS_1 and S_2L, as well as S_1D and LA. The first two meet in D_2, the others in S and D_1 respectively. We may therefore state the required condition, together with the reciprocal one, as follows:—

Pascal's Theorem.—If a hexagon be inscribed in a curve of the second order, then the intersections of opposite sides are three points in a line.	*Brianchon's Theorem.*—If a hexagon be circumscribed about a curve of the second class, then the lines joining opposite vertices are three lines meeting in a point. 697

These celebrated theorems, which are known by the names of their discoverers, are perhaps the most fruitful in the whole theory of conics. Before we go over to their applications we have to show that we obtain the same curve if we take, instead of S_1, S_2, any two other points on the curve as centres of projective pencils.

§ 52. We know that the curve depends only upon the correspondence between the pencils S_1 and S_2, and not upon the special construction used for finding new points on the curve. The point A (fig. 16 or 18), through which the two auxiliary rows u_1, u_2 were drawn, may therefore be changed to any other point on the curve. Let us now suppose the curve drawn, and keep the points S_1, S_2, K, L and D, and hence also the point S fixed, whilst we move A along the curve. Then the line AL will describe a pencil about L as centre, and the point D_1 a row on S_1D perspective to the pencil L. At the same time AK describes a pencil about K and D_2 a row perspective to it on S_2D. But by Pascal's theorem D_1 and D_2 will always lie in a line with S, so that the rows described by D_1 and D_2 are perspective. It follows that the pencils K and L will themselves be projective, corresponding rays meeting on the curve. This proves that we get the same curve whatever pair of the five given points we take as centres of projective pencils. Hence—

Only one curve of the second order can be drawn which passes through five given points.	Only one curve of the second class can be drawn which touches five given lines.

We have seen that if on a curve of the second order two points coincide at A, the line joining them becomes the tangent at A. If, therefore, a point on the curve and its tangent are given, this will be equivalent to having given two points on the curve. Similarly, if on the curve of second class a tangent and its point of contact are given, this will be equivalent to two given tangents.

We may therefore extend the last theorem:

Only one curve of the second order can be drawn, of which four points and the tangent at one of them, or three points and the tangents at two of them, are given.	Only one curve of the second class can be drawn, of which four tangents and the point of contact at one of them, or three tangents and the points of contact at two of them, are given.

§ 53. At the same time it has been proved:

If all points on a curve of the second order be joined to any two of them, then the two pencils thus formed are projective, those rays being corresponding which meet on the curve. Hence—	All tangents to a curve of second class are cut by any two of them in projective rows, those being corresponding points which lie on the same tangent. Hence—
The cross-ratio of four rays joining a point S on a curve of second order to four fixed points A, B, C, D in the curve is independent of the position of S, and is called the cross-ratio of the four points A,	The cross-ratio of the four points in which any tangent u is cut by four fixed tangents a, b, c, d is independent of the position of u, and is called the cross-ratio of the four tangents a, b, c, d.

B, C, D.

If this cross-ratio equals −1 the four points are said to be four harmonic points.

If this cross-ratio equals −1 the four tangents are said to be four harmonic tangents.

We have seen that a curve of second order, as generated by projective pencils, has at the centre of each pencil one tangent; and further, that any point on the curve may be taken as centre of such pencil. Hence—

A curve of second order has at every point one tangent.

A curve of second class has on every tangent a point of contact.

§ 54. We return to Pascal’s and Brianchon’s theorems and their applications, and shall, as before, state the results both for curves of the second order and curves of the second class, but prove them only for the former.

Pascal’s theorem may be used when five points are given to find more points on the curve, viz. it enables us to find the point where any line through one of the given points cuts the curve again. It is convenient, in making use of Pascal’s theorem, to number the points, to indicate the order in which they are to be taken in forming a hexagon, which, by the way, may be done in 60 different ways. It will be seen that 1 2 (leaving out 3) 4 5 are opposite sides, so are 2 3 and (leaving out 4) 5 6, and also 3 4 and (leaving out 5) 6 1.

If the points 1 2 3 4 5 are given, and we want a 6th point on a line drawn through 1, we know all the sides of the hexagon with the exception of 5 6, and this is found by Pascal’s theorem.

If this line should happen to pass through 1, then 6 and 1 coincide, or the line 6 1 is the tangent at 1. And always if two consecutive vertices of the hexagon approach nearer and nearer, then the side joining them will ultimately become a tangent.

We may therefore consider a pentagon inscribed in a curve of second order and the tangent at one of its vertices as a hexagon, and thus get the theorem:

Every pentagon inscribed in a curve of second order has the property that the intersections of two pairs of non-consecutive sides lie in a line with the point where the fifth side cuts the tangent at the opposite vertex.

Every pentagon circumscribed about a curve of the second class has the property that the lines which join two pairs of non-consecutive vertices meet on that line which joins the fifth vertex to the point of contact of the opposite side.

This enables us also to solve the following problems.

Given five points on a curve of second order to construct the tangent at any one of them.

Given five tangents to a curve of second class to construct the point of contact of any one of them.

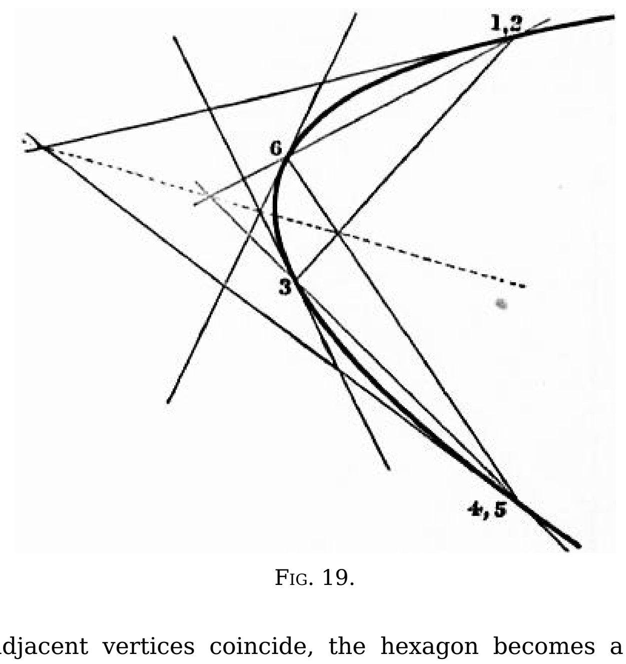


Fig. 19.

If two pairs of adjacent vertices coincide, the hexagon becomes a quadrilateral, with tangents at two vertices. These we take to be opposite, and get the following theorems:

If a quadrilateral be inscribed in a curve of second order, the intersections of opposite sides, and also the intersections

If a quadrilateral be circumscribed about a curve of second class, the lines joining opposite vertices, and also the lines of the tangents at opposite vertices, lie in a line (fig. 19).

joining points of contact of opposite sides, meet in a point.

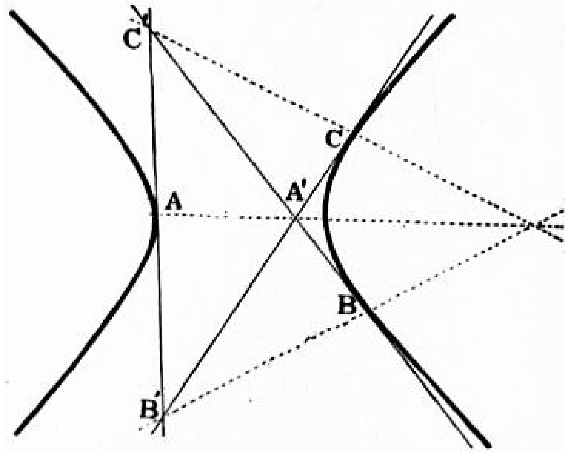


Fig. 20.

If we consider the hexagon made up of a triangle and the tangents at its vertices, we get—

If a triangle is inscribed in a curve of the second order, the points in which the sides are cut by the tangents at the opposite vertices meet in a point.

If a triangle be circumscribed about a curve of second class, the lines which join the vertices to the points of contact of the opposite sides meet in a point (fig. 20).

§ 55. Of these theorems, those about the quadrilateral give rise to a number of others. Four points A, B, C, D may in three different ways be formed into a quadrilateral, for we may take them in the order ABCD, or ACBD, or ACDB, so that either of the points B, C, D may be taken as the vertex opposite to A. Accordingly we may apply the theorem in three different ways.

Let A, B, C, D be four points on a curve of second order (fig. 21), and let us take them as forming a quadrilateral by taking the points in the order ABCD, so that A, C and also B, D are pairs of opposite vertices. Then P, Q will be the points where opposite sides meet, and E, 698
F the intersections of tangents at opposite vertices. The four points P, Q, E, F lie therefore in a line. The quadrilateral ACBD gives us in the same way the four points Q, R, G, H in a line, and the quadrilateral ABDC a line containing the four points R, P, I, K. These three lines form a triangle PQR.

The relation between the points and lines in this figure may be expressed more clearly if we consider ABCD as a four-point inscribed in a conic, and the tangents at these points as a four-side circumscribed about it,—viz. it will be seen that P, Q, R are the diagonal points of the four-point ABCD, whilst the sides of the triangle PQR are the diagonals of the circumscribing four-side. Hence the theorem—

Any four-point on a curve of the second order and the four-side formed by the tangents at these points stand in this relation that the diagonal points of the four-point lie in the diagonals of the four-side. And conversely,

If a four-point and a circumscribed four-side stand in the above relation, then a curve of the second order may be described which passes through the four points and touches there the four sides of these figures.

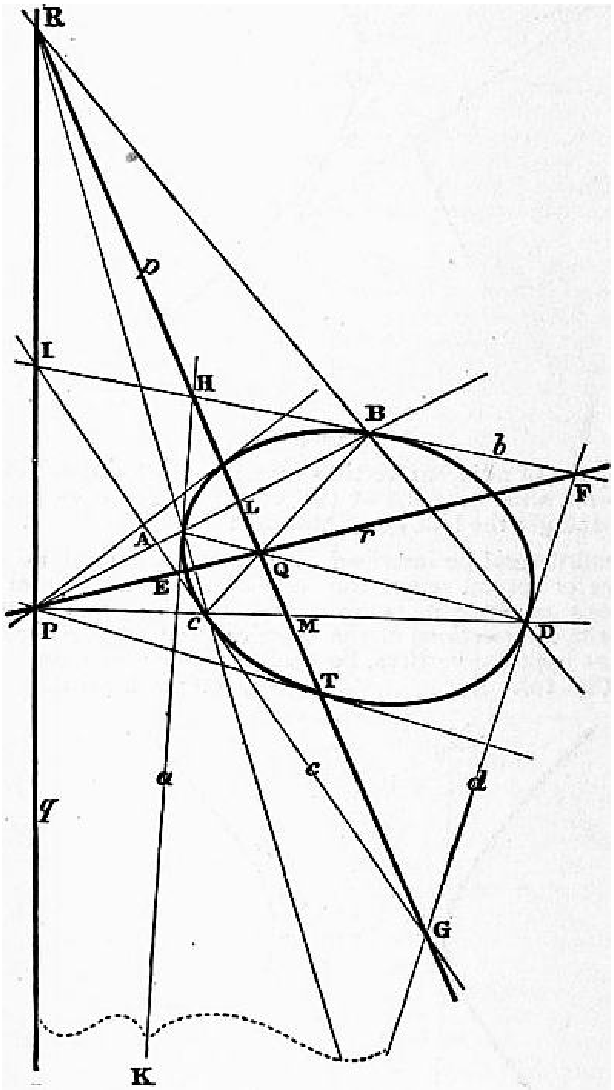


FIG. 21.

That the last part of the theorem is true follows from the fact that the four points A, B, C, D and the line a, as tangent at A, determine a curve of the second order, and the tangents to this curve at the other points B, C, D are given by the construction which leads to fig. 21.

The theorem reciprocal to the last is—

Any four-side circumscribed about a curve of second class and the four-point formed by the points of contact stand in this relation that the diagonals of the four-side pass through the diagonal points of the four-point. And conversely,

If a four-side and an inscribed four-point stand in the above relation, then a curve of the second class may be described which touches the sides of the four-side at the points of the four-point.

§ 56. The four-point and the four-side in the two reciprocal theorems are alike. Hence if we have a four-point ABCD and a four-side abcd related in the manner described, then not only may a curve of the second order be drawn, but also a curve of the second class, which both touch the lines a, b, c, d at the points A, B, C, D.

The curve of second order is already more than determined by the points A, B, C and the tangents a, b, c at A, B and C. The point D may therefore be *any* point on this curve, and d any tangent to the curve. On the other hand the curve of the second class is more than determined by the three tangents a, b, c and their points of contact A, B, C, so that d is any tangent to this curve. It follows that every tangent to the curve of second order is a tangent of a curve of the second class having the same point of contact. In other words, the curve of second order is a curve of second class, and *vice versa*. Hence the important theorems—

Every curve of second order is a curve of second class.	*Every curve of second class is a curve of second order.*

The curves of second order and of second class, having thus been proved to be identical, shall henceforth be called by the common name of *Conics*.

For these curves hold, therefore, all properties which have been proved for curves of second order or of second class. We may therefore now state Pascal's and Brianchon's theorem thus—

Pascal’s Theorem.—If a hexagon be inscribed in a conic, then the intersections of opposite sides lie in a line.

Brianchon’s Theorem.—If a hexagon be circumscribed about a conic, then the diagonals forming opposite centres meet in a point.

§ 57. If we suppose in fig. 21 that the point D together with the tangent d moves along the curve, whilst A, B, C and their tangents a, b, c remain fixed, then the ray DA will describe a pencil about A, the point Q a projective row on the fixed line BC, the point F the row b, and the ray EF a pencil about E. But EF passes always through Q. Hence the pencil described by AD is projective to the pencil described by EF, and therefore to the row described by F on b. At the same time the line BD describes a pencil about B projective to that described by AD (§ 53). Therefore the pencil BD and the row F on b are projective. Hence—

If on a conic a point A *be taken and the tangent a at this point, then the cross-ratio of the four rays which join* A *to any four points on the curve is equal to the cross-ratio of the points in which the tangents at these points cut the tangent at* A.

§ 58. There are theorems about cones of second order and second class in a pencil which are reciprocal to the above, according to § 43. We mention only a few of the more important ones.

The locus of intersections of corresponding planes in two projective axial pencils whose axes meet is a cone of the second order.

The envelope of planes which join corresponding lines in two projective flat pencils, not in the same plane, is a cone of the second class.

Cones of second order and cones of second class are identical.

Every plane cuts a cone of the second order in a conic.

A cone of second order is uniquely determined by five of its edges or by five of its tangent planes, or by four edges and the tangent plane at one of them, &c. &c.

Pascal’s Theorem.—If a solid angle of six faces be inscribed in a cone of the second order, then the intersections of opposite faces are three lines in a plane.

Brianchon’s Theorem.—If a solid angle of six edges be circumscribed about a cone of the second order, then the planes through opposite edges meet in a line.

Each of the other theorems about conics may be stated for cones of the second order.

§ 59. *Projective Definitions of the Conics.*—We now consider the shape of the conics. We know that any line in the plane of the conic, and hence that the line at infinity, either has no point in common with the curve, or one (counting for two coincident points) or two distinct points. If the line at infinity has no point on the curve the latter is altogether finite, and is called an *Ellipse* (fig. 21). If the line at infinity has only one point in common with the conic, the latter extends to infinity, and has the line at infinity a tangent. It is called a *Parabola* (fig. 22). If, lastly, the line at infinity cuts the curve in two points, it consists of two separate parts which each extend in two branches to the points at infinity where they meet. The curve is in this case called an *Hyperbola* (see fig. 20). The tangents at the two points at infinity are finite because the line at infinity is not a tangent. They are called *Asymptotes*. The branches of the hyperbola approach these lines indefinitely as a point on the curves moves to infinity.

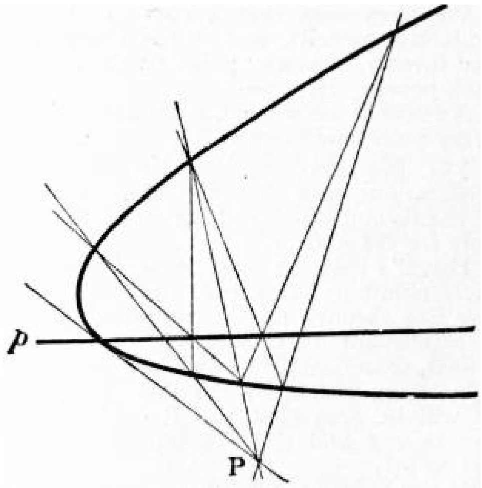


Fig. 22.

§ 60. That the circle belongs to the curves of the second order is seen at once if we state in a slightly different form the theorem that in a circle all angles at the circumference standing upon the same arc are equal. If two points S_1, S_2 on a circle be joined to any other two points A and B on the circle, then the angle included by the rays S_1A and S_1B is equal to that between the rays S_2A and S_2B, so that as A moves along the circumference the rays S_1A and S_2A describe equal and therefore projective pencils. The circle can thus be generated by two projective pencils, and is a curve of the second order.

If we join a point in space to all points on a circle, we get a (circular) cone of the second order (§ 43). Every plane section of this cone is a conic. This conic will be an ellipse, a parabola, or an hyperbola, according as the line at infinity in the plane has no, one or two points in common with the conic in which the plane at infinity cuts the cone. It follows that our curves of second order may be obtained as sections of a circular cone, and that they are identical with the "Conic Sections" of the Greek mathematicians.

§ 61. Any two tangents to a parabola are cut by all others in projective rows; but the line at infinity being one of the tangents, the points at infinity on the rows are corresponding points, and the rows therefore similar. Hence the theorem—

The tangents to a parabola cut each other proportionally.

Pole and Polar

§ 62. We return once again to fig. 21, which we obtained in § 55.

If a four-side be circumscribed about and a four-point inscribed in a conic, so that the vertices of the second are the points of contact of the sides of the first, then the triangle formed by the diagonals of the first is the same as that formed by the diagonal points of the other.

Such a triangle will be called a *polar-triangle* of the conic, so that PQR in fig. 21 is a polar-triangle. It has the property that on the side p opposite P meet the tangents at A and B, and also those at C and D. From the harmonic properties of four-points and four-sides it follows further that the points L, M, where it cuts the lines AB and CD, are harmonic conjugates with regard to AB and CD respectively.

If the point P is given, and we draw a line through it, cutting the conic in A and B, then the point Q harmonic conjugate to P with regard to AB, and the point H where the tangents at A and B meet, are determined. But they lie both on p, and therefore this line is determined. If we now draw a second line through P, cutting the conic in C and D, then the point M harmonic conjugate to P with regard to CD, and the point G where the tangents at C and D meet, must also lie on p. As the first line through P already determines p, the second may be any line through P. Now every two lines through P determine a four-point ABCD on the conic, and therefore a polar-triangle which has one vertex at P and its opposite side at p. This result, together with its reciprocal, gives the theorems—

All polar-triangles which have one vertex in common have also the opposite side in common.

All polar-triangles which have one side in common have also the opposite vertex in common.

§ 63. To any point P in the plane of, but not on, a conic corresponds thus one line p as the side opposite to P in all polar-triangles which have one vertex at P, and reciprocally to every line p corresponds one point P as the vertex opposite to p in all triangles which have p as one side.

We call the line p the *polar* of P, and the point P the *pole* of the line p with regard to the conic.

If a point lies on the conic, we call the tangent at that point its polar; and reciprocally we call the point of contact the pole of tangent.

§ 64. From these definitions and former results follow—

The polar of any point P not on the conic is a line p, which has the following properties:—	The pole of any line p not a tangent to the conic is a point P, which has the following properties:—
1. On every line through P which cuts the conic, the polar of P contains the harmonic conjugate of P with regard to those points on the conic.	1. Of all lines through a point on p from which two tangents may be drawn to the conic, the pole P contains the line which is harmonic conjugate to p, with regard to the two tangents.
2. If tangents can be drawn from P, their points of contact lie on p.	2. If p cuts the conic, the tangents at the intersections meet at P.
3. Tangents drawn at the points where any line through P cuts the conic meet on p; and conversely,	3. The point of contact of tangents drawn from any point on p to the conic lie in a line with P; and conversely,
4. If from any point on p, tangents be drawn, their points of contact will lie in a line with P.	4. Tangents drawn at points where any line through P cuts the conic meet on p.

5. Any four-point on the conic which has one diagonal point at P has the other two lying on p.	5. Any four-side circumscribed about a conic which has one diagonal on p has the other two meeting at P.

The truth of 2 follows from 1. If T be a point where p cuts the conic, then one of the points where PT cuts the conic, and which are harmonic conjugates with regard to PT, coincides with T; hence the other does—that is, PT touches the curve at T.

That 4 is true follows thus: If we draw from a point H on the polar one tangent a to the conic, join its point of contact A to the pole P, determine the second point of intersection B of this line with the conic, and draw the tangent at B, it will pass through H, and will therefore be the second tangent which may be drawn from H to the curve.

§ 65. The second property of the polar or pole gives rise to the theorem—

From a point in the plane of a conic, two, one or no tangents may be drawn to the conic, as its polar has two, one, or no points in common with the curve.	A line in the plane of a conic has two, one or no points in common with the conic, according as two, one or no tangents can be drawn from its pole to the conic.

Of any point in the plane of a conic we say that it was *without,* on or *within* the curve according as two, one or no tangents to the curve pass through it. The points on the conic separate those within the conic from those without. That this is true for a circle is known from elementary geometry. That it also holds for other conics follows from the fact that every conic may be considered as the projection of a circle, which will be proved later on.

The fifth property of pole and polar stated in § 64 shows how to find the polar of any point and the pole of any line by aid of the straight-edge only. Practically it is often convenient to draw three secants through the pole, and to determine only one of the diagonal points for two of the four-points formed by pairs of these lines and the conic (fig. 22).

These constructions also solve the problem—

From a point without a conic, to draw the two tangents to the conic by aid of the straight-edge only.

For we need only draw the polar of the point in order to find the points of contact.

§ 66. The property of a polar-triangle may now be stated thus—

In a polar-triangle each side is the polar of the opposite vertex, and each vertex is the pole of the opposite side.

If P is one vertex of a polar-triangle, then the other vertices, Q and R, lie on the polar p of P. One of these vertices we may choose arbitrarily. For if from any point Q on the polar a secant be drawn cutting the conic in A and D (fig. 23), and if the lines joining these points to P cut the conic again at B and C, then the line BC will pass through Q. Hence P and Q are two of the vertices on the polar-triangle which is determined by the four-point ABCD. The third vertex R lies also on the line p. It follows, therefore, also—

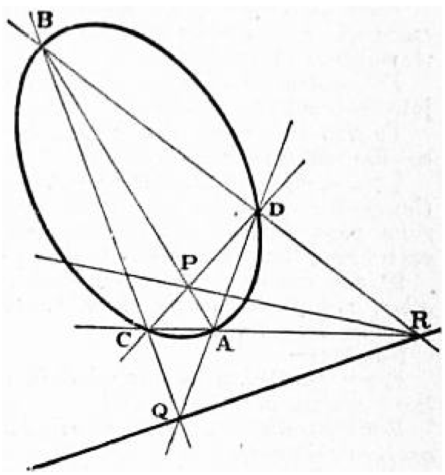


FIG. 23.

If Q *is a point on the polar of* P, *then* P *is a point on the polar of* Q; and reciprocally,

If q *is a line through the pole of* p, *then* p *is a line through the pole of* q.

This is a very important theorem. It may also be stated thus—

If a point moves along a line describing a row, its polar turns about the pole of the line describing a pencil.

This pencil is projective to the row, so that the cross-ratio of four poles in a row equals the cross-ratio of its four polars, which pass through the pole of the row.

To prove the last part, let us suppose that P, A and B in fig. 23 remain fixed, whilst Q moves along the polar p of P. This will make CD turn about P and move R along p, whilst QD and RD describe projective pencils about A and B. Hence Q and R describe projective rows, and hence PR, which is the polar of Q, describes a pencil projective to either.

§ 67. Two points, of which one, and therefore each, lies on the polar of the other, are said to be *conjugate with regard to the conic*; and two lines, of which one, and therefore each, passes through the pole of the other, are said to be *conjugate with regard to the conic*. Hence all points conjugate to a point P lie on the polar of P; all lines conjugate to a line p pass through the pole of p.

If the line joining two conjugate poles cuts the conic, then the poles are harmonic conjugates with regard to the points of intersection; hence one lies within the other without the conic, and all points conjugate to a point within a conic lie without it.

Of a polar-triangle any two vertices are conjugate poles, any two sides conjugate lines. If, therefore, one side cuts a conic, then one of the two vertices which lie on this side is within and the other without the conic. The vertex opposite this side lies also without, for it is the pole of a line which cuts the curve. In this case therefore one vertex lies within, the other two without. If, on the other hand, we begin with a side which does not cut the conic, then its pole lies within and the other vertices without. Hence—

Every polar-triangle has one and only one vertex within the conic.

We add, without a proof, the theorem—

The four points in which a conic is cut by two conjugate polars are four harmonic points in the conic.

§ 68. If two conics intersect in four points (they cannot have more points in common, § 52), there exists one and only one four-point which is inscribed in both, and therefore one polar-triangle common to both. 700

Theorem.—Two conics which intersect in four points have always one and only one common polar-triangle; and reciprocally,

Two conics which have four common tangents have always one and only one common polar-triangle.

Diameters and Axes of Conics

§ 69. *Diameters.*—The theorems about the harmonic properties of poles and polars contain, as special cases, a number of important metrical properties of conics. These are obtained if either the pole or the polar is moved to infinity,—it being remembered that the harmonic conjugate to a point at infinity, with regard to two points A, B, is the middle point of the segment AB. The most important properties are stated in the following theorems:—

The middle points of parallel chords of a conic lie in a line—viz. on the polar to the point at infinity on the parallel chords.

This line is called a *diameter*.

The polar of every point at infinity is a diameter.

The tangents at the end points of a diameter are parallel, and are parallel to the chords bisected by the diameter.

All diameters pass through a common point, the pole of the line at infinity.

All diameters of a parabola are parallel, the pole to the line at infinity being the point where the curve touches the line at infinity.

In case of the ellipse and hyperbola, the pole to the line at infinity is a finite point called the *centre* of the curve.

A centre of a conic bisects every chord through it.

The centre of an ellipse is within the curve, for the line at infinity does not cut the ellipse.

The centre of an hyperbola is without the curve, because the line at infinity cuts the curve. Hence also—

From the centre of an hyperbola two tangents can be drawn to the curve which have their point of contact at infinity. These are called *Asymptotes* (§ 59).

To construct a diameter of a conic, draw two parallel chords and join their middle points.

To find the centre of a conic, draw two diameters; their intersection will be the centre.

§ 70. *Conjugate Diameters.*—A polar-triangle with one vertex at the centre will have the opposite side at infinity. The other two sides pass through the centre, and are called *conjugate diameters*, each being the polar of the point at infinity on the other.

Of two conjugate diameters each bisects the chords parallel to the other, and if one cuts *the curve, the tangents at its ends are parallel to the other diameter.*

Further—

Every parallelogram inscribed in a conic has its sides parallel to two conjugate diameters; and

Every parallelogram circumscribed about a conic has as diagonals two conjugate diameters.

This will be seen by considering the parallelogram in the first case as an inscribed four-point, in the other as a circumscribed four-side, and determining in each case the corresponding polar-triangle. The first may also be enunciated thus—

The lines which join any point on an ellipse or an hyperbola to the ends of a diameter are parallel to two conjugate diameters.

§ 71. *If every diameter is perpendicular to its conjugate the conic is a circle.*

For the lines which join the ends of a diameter to any point on the curve include a right angle.

A conic which has more than one pair of conjugate diameters at right angles to each other is a circle.

Let AA′ and BB′ (fig. 24) be one pair of conjugate diameters at right angles to each other, CC and DD′ a second pair. If we draw through the end point A of one diameter a chord AP parallel to DD′, and join P to A′, then PA and PA′ are, according to § 70, parallel to two conjugate diameters. But PA is parallel to DD′, hence PA′ is parallel to CC, and therefore PA and PA′ are perpendicular. If we further draw the tangents to the conic at A and A′, these will be perpendicular to AA′, they being parallel to the conjugate diameter BB′. We know thus five points on the conic, viz. the points A and A′ with their tangents, and the point P. Through these a circle may be drawn having AA′ as diameter; and as through five points one conic only can be drawn, this circle must coincide with the given conic.

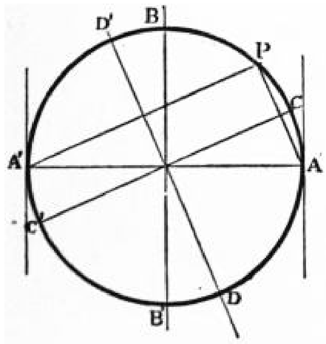


FIG. 24.

§ 72. *Axes.*—Conjugate diameters perpendicular to each other are called *axes*, and the points where they cut the curve *vertices* of the conic.

In a circle every diameter is an axis, every point on it is a vertex; and any two lines at right angles to each other may be taken as a pair of axes of any circle which has its centre at their intersection.

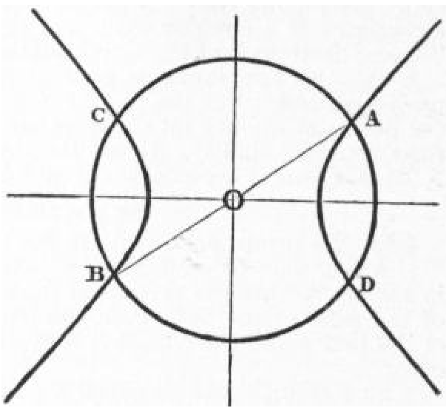


FIG. 25.

If we describe on a diameter AB of an ellipse or hyperbola a circle concentric to the conic, it will cut the latter in A and B (fig. 25). Each of the semicircles in which it is divided by AB will be partly within, partly without the curve, and must cut the latter therefore again in a point. The circle and the conic have thus four points A, B, C, D, and therefore one polar-triangle, in common (§ 68). Of this the centre is one vertex, for the line at infinity is the polar to this point, both with regard to the circle and the other conic. The other two sides are conjugate diameters of both, hence perpendicular to each other. This gives—

An ellipse as well as an hyperbola has one pair of axes.

This reasoning shows at the same time *how to construct the axis of an ellipse or of an hyperbola*.

A parabola has one axis, if we define an axis as a diameter perpendicular to the chords which it bisects. It is easily constructed. The line which bisects any two parallel chords is a diameter. Chords perpendicular to it will be bisected by a parallel diameter, and this is the axis.

§ 73. The first part of the right-hand theorem in § 64 may be stated thus: any two conjugate lines through a point P without a conic are harmonic conjugates with regard to the two tangents that may be drawn from P to the conic.

If we take instead of P the centre C of an hyperbola, then the conjugate lines become conjugate diameters, and the tangents asymptotes. Hence—

Any two conjugate diameters of an hyperbola are harmonic conjugates with regard to the asymptotes.

As the axes are conjugate diameters at right angles to one another, it follows (§ 23)—

The axes of an hyperbola bisect the angles between the asymptotes.

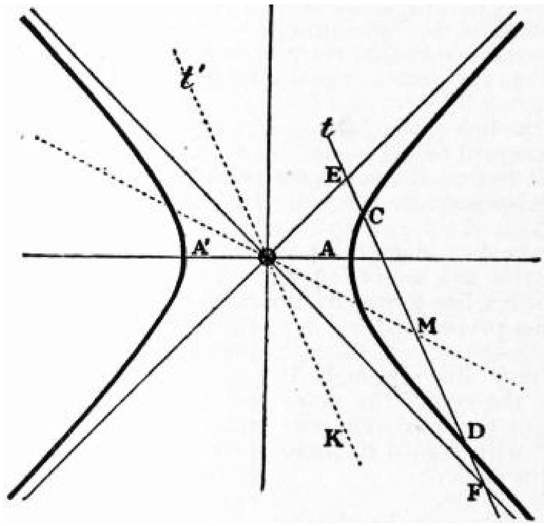


FIG. 26.

Let O be the centre of the hyperbola (fig. 26), t any secant which cuts the hyperbola in C, D and the asymptotes in E, F, then the line OM which bisects the chord CD is a diameter conjugate to the diameter OK which is parallel to the secant t, so that OK and OM are harmonic with regard to the asymptotes. The point M therefore bisects EF. But by construction M bisects CD. It follows that DF = EC, and ED = CF; or

On any secant of an hyperbola the segments between the curve and the asymptotes are equal.

If the chord is changed into a tangent, this gives—

The segment between the asymptotes on any tangent to an hyperbola is bisected by the point of contact.

The first part allows a simple solution of the problem to find any number of points on an hyperbola, of which the asymptotes and one point are given. This is equivalent to three points and the tangents at two of them. This construction requires measurement.

§ 74. For the parabola, too, follow some metrical properties. A diameter PM (fig. 27) bisects every chord conjugate to it, and the pole P of such a chord BC lies on the diameter. But a diameter cuts the parabola once at infinity. Hence—

The segment PM *which joins the middle point* M *of a chord of a parabola to the pole* P *of the chord is bisected by the parabola at* A.

§ 75. Two asymptotes and any two tangents to an hyperbola may be considered as a quadrilateral circumscribed about the hyperbola. But in such a quadrilateral the intersections of the diagonals and the points of contact of opposite sides lie in a line (§ 54). If therefore DEFG (fig. 28) is such a quadrilateral, then the diagonals DF and GE will meet on the line which joins the points of contact of the asymptotes, that is, on the line at infinity; hence they are parallel. From this the following theorem is a simple deduction: 701

All triangles formed by a tangent and the asymptotes of an hyperbola are equal in area.

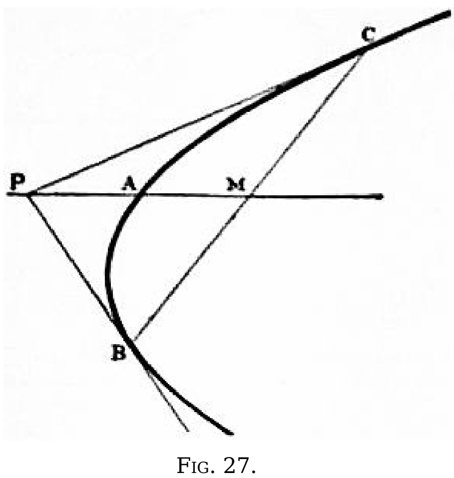


FIG. 27.

If we draw at a point P (fig. 28) on an hyperbola a tangent, the part HK between the asymptotes is bisected at P. The parallelogram PQOQ′ formed by the asymptotes and lines parallel to them through P will be half the triangle OHK, and will therefore be constant. If we now take the asymptotes OX and OY as oblique axes of co-ordinates, the lines OQ and QP will be the co-ordinates of P, and will satisfy the equation $xy = \text{const.} = a^2$.

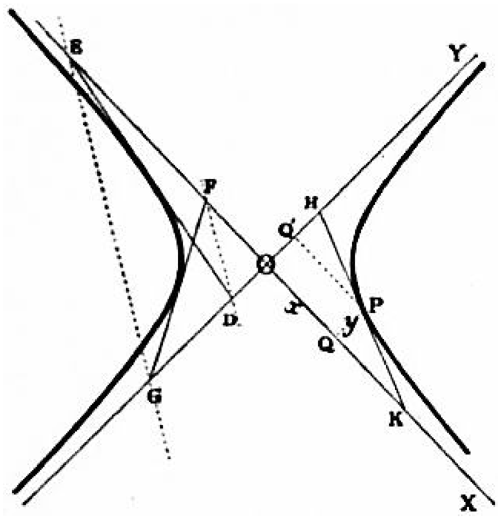


FIG. 28.

For the asymptotes as axes of co-ordinates the equation of the hyperbola is $xy = \text{const.}$

INVOLUTION

§ 76. If we have two projective rows, ABC on u and A′B′C′ on u′, and place their bases on the same line, then each point in this line counts twice, once as a point in the row u and once as a point in the row u′. In fig. 29 we denote the points as points in the one row by letters above the line A, B, C ..., and as points in the second row by A′, B′, C′ ... below the line. Let now A and B′ be the same point, then to A will correspond a point A′ in the second, and to B′ a point B in the first row. In general these points A′ and B will be different. It may, however, happen that they coincide. Then the correspondence is a peculiar one, as the following theorem shows:

FIG. 29.

If two projective rows lie on the same base, and if it happens that to one point in the base the same point corresponds, whether we consider the point as belonging to the first or to the second row, then the same will happen for every point in the base—that is to say, to every point in the line corresponds the same point in the first as in the second row.

In order to determine the correspondence, we may assume three pairs of corresponding points in two projective rows. Let then A′, B′, C′, in fig. 30, correspond to A, B, C, so that A and B′, and also B and A′, denote the same point. Let us further denote the point C′ when considered as a point in the first row by D; then it is to be proved that the point D′, which corresponds to D, is the same point as C. We know that the cross-ratio of four points is equal to that of the corresponding row. Hence

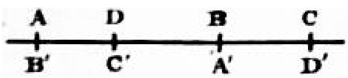


FIG. 30.

$$(AB, CD) = (A′B′, C′D′)$$

but replacing the dashed letters by those undashed ones which denote the same points, the second cross-ratio equals (BA, DD′), which, according to § 15, equals (AB, D′D); so that the equation becomes

$$(AB, CD) = (AB, D′D).$$

This requires that C and D′ coincide.

§ 77. Two projective rows on the same base, which have the above property, that to every point, whether it be considered as a point in the one or in the other row, corresponds the same point, are said to be in *involution*, or to form an *involution* of points on the line.

We mention, but without proving it, that any two projective rows may be placed so as to form an involution.

An involution may be said to consist of a row of pairs of points, to every point A corresponding a point A′, and to A′ again the point A. These points are said to be conjugate, or, better, one point is termed the "mate" of the other.

From the definition, according to which an involution may be considered as made up of two projective rows, follow at once the following important properties:

1. The cross-ratio of four points equals that of the four conjugate points.

2. If we call a point which coincides with its mate a "focus" or "double point" of the involution, we may say: An involution has either two foci, or one, or none, and is called respectively a hyperbolic, parabolic or elliptic involution (§ 34).

3. In an hyperbolic involution any two conjugate points are harmonic conjugates with regard to the two foci.

For if A, A′ be two conjugate points, F_1, F_2 the two foci, then to the points F_1, F_2, A, A′ in the one row correspond the points F_1, F_2, A′, A in the other, each focus corresponding to itself. Hence $(F_1F_2, AA') = (F_1F_2, A'A)$—that is, we may interchange the two points AA′ without altering the value of the cross-ratio, which is the characteristic property of harmonic conjugates (§ 18).

4. The point conjugate to the point at infinity is called the "centre" of the involution. Every involution has a centre, unless the point at infinity be a focus, in which case we may say that the centre is at infinity.

In an hyperbolic involution the centre is the middle point between the foci.

5. The product of the distances of two conjugate points A, A′ from the centre O is constant: $OA \cdot OA' = c$.

For let A, A′ and B, B′ be two pairs of conjugate points, the centre, I the point at infinity, then

$$(AB, OI) = (A'B', IO),$$

or

$$OA \cdot OA' = OB \cdot OB'.$$

In order to determine the distances of the foci from the centre, we write F for A and A′ and get

$$OF^2 = c;\ OF = \pm\sqrt{c}.$$

Hence if c is positive OF is real, and has two values, equal and opposite. The involution is hyperbolic.

If $c = 0$, $OF = 0$, and the two foci both coincide with the centre. If c is negative, $\sqrt{c}$ becomes imaginary, and there are no foci. Hence we may write—

In an hyperbolic involution,	$OA \cdot OA' = k^2$,
In a parabolic involution,	$OA \cdot OA' = 0$,
In an elliptic involution,	$OA \cdot OA' = -k^2$.

From these expressions it follows that conjugate points A, A′ in an hyperbolic involution lie on the same side of the centre, and in an elliptic involution on opposite sides of the centre, and that in a parabolic involution one coincides with the centre.

In the first case, for instance, $OA \cdot OA'$ is positive; hence OA and OA′ have the same sign.

It also follows that two segments, AA′ and BB′, between pairs of conjugate points have the following positions: in an hyperbolic involution they lie either one altogether within or altogether without each other; in a parabolic involution they have one point in common; and in an elliptic involution they overlap, each being partly within and partly without the other.

Proof.—We have $OA . OA' = OB \cdot OB' = k^2$ in case of an hyperbolic involution. Let A and B be the points in each pair which are nearer to the centre O. If now A, A′ and B, B′ lie on the same side of O, and if B is nearer to O than A, so that $OB < OA$, then $OB' > OA'$; hence B′ lies farther away from O than A′, or the segment AA′ lies within BB′. And so on for the other cases.

6. An involution is determined—

(α) By two pairs of conjugate points. Hence also

(β) By one pair of conjugate points and the centre;
(γ) By the two foci;
(δ) By one focus and one pair of conjugate points;
(ε) By one focus and the centre.

7. The condition that A, B, C and A′, B′, C′ may form an involution may be written in one of the forms—

$$(AB, CC') = (A'B', C'C),$$

or

$$(AB, CA') = (A'B', C'A),$$

or

$$(AB, C'A') = (A'B', CA),$$

for each expresses that in the two projective rows in which A, B, C and A′, B′, C′ are conjugate points two conjugate elements may be interchanged. 702

8. Any three pairs. A, A′, B, B′, C, C′, of conjugate points are connected by the relations:

$$\frac{AB' \cdot BC' \cdot CA'}{A'B \cdot B'C \cdot C'A} = \frac{AB' \cdot BC \cdot C'A'}{A'B \cdot B'C' \cdot CA} = \frac{AB \cdot B'C' \cdot CA'}{A'B' \cdot BC \cdot C'A} = \frac{AB \cdot B'C \cdot C'A'}{A'B' \cdot BC' \cdot CA} = -1.$$

These relations readily follow by working out the relations in (7) (above).

§ 78. *Involution of a quadrangle.—The sides of any four-point are cut by any line in six points in involution, opposite sides being cut in conjugate points.*

Let $A_1B_1C_1D_1$ (fig. 31) be the four-point. If its sides be cut by the line p in the points A, A′, B, B′, C, C′, if further, C_1D_1 cuts the line A_1B_1 in C_2, and if we project the row $A_1B_1C_2C$ to p once from D_1 and once from C_1, we get (A′B′, C′C) = (BA, C′C).

Interchanging in the last cross-ratio the letters in each pair we get (A′B′, C′C) = (AB, CC′). Hence by § 77 (7) the points are in involution.

The theorem may also be stated thus:

The three points in which any line cuts the sides of a triangle and the projections, from any point in the plane, of the vertices of the triangle on to the same line are six points in involution.

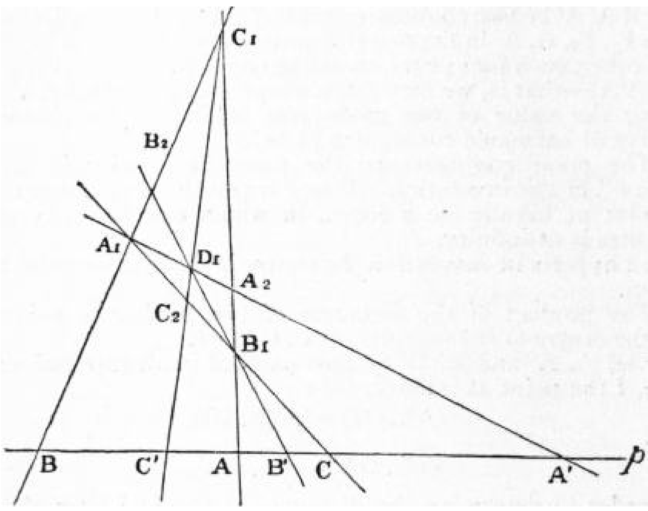


FIG. 31.

Or again—

The projections from any point on to any line of the six vertices of a four-side are six points in involution, the projections of opposite vertices being conjugate points.

This property gives a simple means to construct, by aid of the straight edge only, in an involution of which two pairs of conjugate points are given, to any point its conjugate.

§ 79. *Pencils in Involution.*—The theory of involution may at once be extended from the row to the flat and the axial pencil—viz. we say that there is an involution in a flat or in an axial pencil if any line cuts the pencil in an involution of points. An involution in a pencil consists of pairs of conjugate rays or planes; it has two, one or no *focal rays* (double lines) or *planes*, but nothing corresponding to a centre.

An involution in a flat pencil contains always one, and in general only one, pair of conjugate rays which are perpendicular to one another. For in two projective flat pencils exist always two corresponding right angles (§ 40).

Each involution in an axial pencil contains in the same manner one pair of conjugate planes at right angles to one another.

As a rule, there exists but one pair of conjugate lines or planes at right angles to each other. But it is possible that there are more, and then there is an infinite number of such pairs. An involution in a flat pencil, in which every ray is perpendicular to its conjugate ray, is said to be *circular*. That such involution is possible is easily seen thus: if in two concentric flat pencils each ray on one is made to correspond to that ray on the other which is perpendicular to it, then the two pencils are projective, for if we turn the one pencil through a right angle each ray in one coincides with its corresponding ray in the other. But these two projective pencils are in involution.

A circular involution has no focal rays, because no ray in a pencil coincides with the ray perpendicular to it.

§ 80. *Every elliptical involution in a row may be considered as a section of a circular involution.*

In an elliptical involution any two segments AA′ and BB′ lie partly within and partly without each other (fig. 32). Hence two circles described on AA′ and BB′ as diameters will intersect in two points E and E′. The line EE′ cuts the base of the involution at a point O, which has the property that OA . OA′ = OB · OB′, for each is equal to OE . OE′. The point O is therefore the centre of the involution. If we wish to construct to any point C the conjugate point C′, we may draw the circle through CEE′. This will cut the base in the required point C′ for OC · OC′ = OA · OA′. But EC and EC′ are at right angles. Hence the involution which is obtained by joining E or E′ to the points in the given involution is circular. This may also be expressed thus:

Every elliptical involution has the property that there are two definite points in the plane from which any two conjugate points are seen under a right angle.

At the same time the following problem has been solved:

To determine the centre and also the point corresponding to any given point in an elliptical involution of which two pairs of conjugate points are given.

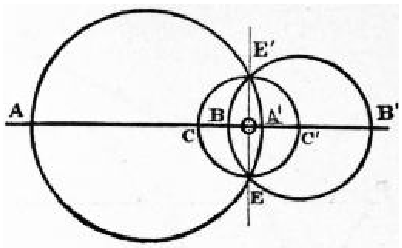


FIG. 32.

§ 81. *Involution Range on a Conic.*—By the aid of § 53, the points on a conic may be made to correspond to those on a line, so that the row of points on the conic is projective to a row of points on a line. We may also have two projective rows on the same conic, and these will be in involution as soon as one point on the conic has the same point corresponding to it all the same to whatever row it belongs. An involution of points on a conic will have the property (as follows from its definition, and from § 53) that the lines which join conjugate points of the involution to any point on the conic are conjugate lines of an involution in a pencil, and that a fixed tangent is cut by the tangents at conjugate points on the conic in points which are again conjugate points of an involution on the fixed tangent. For such involution on a conic the following theorem holds:

The lines which join corresponding points in an involution on a conic all pass through a fixed point; and reciprocally, the points of intersection of conjugate lines in an involution among tangents to a conic lie on a line.

We prove the first part only. The involution is determined by two pairs of conjugate points, say by A, A′ and B, B′ (fig. 33). Let AA′ and BB′ meet in P. If we join the points in involution to any point on the conic, and the conjugate points to another point on the conic, we obtain two projective pencils. We take A and A′ as centres of these pencils, so that the pencils A(A′BB′) and A′(AB′B) are projective, and in perspective position,

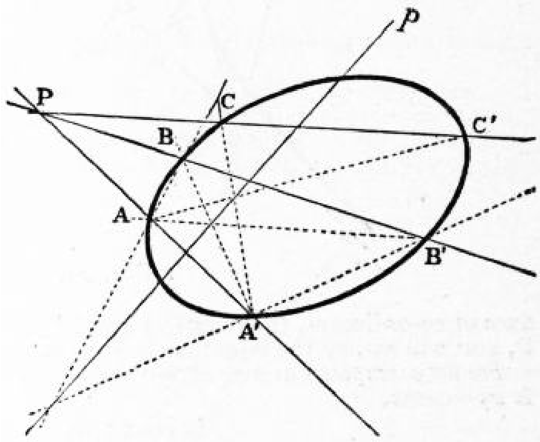


Fig. 33

because AA′ corresponds to A′A. Hence corresponding rays meet in a line, of which two points are found by joining AB′ to A′B and AB to A′B′. It follows that the *axis* of perspective is the polar of the point P, where AA′ and BB′ meet. If we now wish to construct to any other point C on the conic the corresponding point C′, we join C to A′ and the point where this line cuts p to A. The latter line cuts the conic again in C′. But we know from the theory of pole and polar that the line CC′ passes through P. The point of concurrence is called the "pole of the involution," and the line of collinearity of the meets is called the "axis of the involution."

Involution Determined by a Conic on a Line.—Foci

§ 82. The polars, with regard to a conic, of points in a row p form a pencil P projective to the row (§ 66). This pencil cuts the base of the row p in a projective row.

If A is a point in the given row, A′ the point where the polar of A cuts p, then A and A′ will be corresponding points. If we take A′ a point in the first row, then the polar of A′ will pass through A, so that A corresponds to A′—in other words, the rows are in involution. The conjugate points in this involution are conjugate points with regard to the conic. Conjugate points coincide only if the polar of a point A passes through A—that is, if A lies on the conic. Hence—

A conic determines on every line in its plane an involution, in which those points are conjugate which are also conjugate with regard to the conic.

If the line cuts the conic the involution is hyperbolic, the points of intersection being the foci.

If the line touches the conic the involution is parabolic, the two foci coinciding at the point of contact.

If the line does not cut the conic the involution is elliptic, having no foci.

If, on the other hand, we take a point P in the plane of a conic, we get to each line a 703
through P one conjugate line which joins P to the pole of a. These pairs of conjugate lines through P form an involution in the pencil at P. The focal rays of this involution are the tangents drawn from P to the conic. This gives the theorem reciprocal to the last, viz:—

A conic determines in every pencil in its plane an involution, corresponding lines being conjugate lines with regard to the conic.

If the point is without the conic the involution is hyperbolic, the tangents from the points being the focal rays.

If the point lies on the conic the involution is parabolic, the tangent at the point counting for coincident focal rays.

If the point is within the conic the involution is elliptic, having no focal rays.

It will further be seen that the involution determined by a conic on any line p is a section of the involution, which is determined by the conic at the pole P of p.

§ 83. *Foci.*—The centre of a pencil in which the conic determines a circular involution is called a "focus" of the conic.

In other words, a focus is such a point that every line through it is perpendicular to its conjugate line. The polar to a focus is called a *directrix* of the conic.

From the definition it follows that *every focus lies on an axis*, for the line joining a focus to the centre of the conic is a diameter to which the conjugate lines are perpendicular; and *every line joining two foci is an axis*, for the perpendiculars to this line through the foci are conjugate to it. These conjugate lines pass through the pole of the line, the pole lies therefore at infinity, and the line is a diameter, hence by the last property an axis.

It follows that all *foci lie on one axis*, for no line joining a point in one axis to a point in the other can be an axis.

As the conic determines in the pencil which has its centre at a focus a circular involution, no tangents can be drawn from the focus to the conic. Hence *each focus lies within a conic*; and *a directrix does not cut the conic*.

Further properties are found by the following considerations:

§ 84. Through a point P one line p can be drawn, which is with regard to a given conic conjugate to a given line q, viz. that line which joins the point P to the pole of the line q. If the line q is made to describe a pencil about a point Q, then the line p will describe a pencil about P. These two pencils will be projective, for the line p passes through the pole of q, and whilst q describes the pencil Q, its pole describes a projective row, and this row is perspective to the pencil P.

We now take the point P on an axis of the conic, draw any line p through it, and from the pole of p draw a perpendicular q to p. Let q cut the axis in Q. Then, in the pencils of conjugate lines, which have their centres at P and Q, the lines p and q are conjugate lines at right angles to one another. Besides, to the axis as a ray in either pencil will correspond in the other the perpendicular to the axis (§ 72). The conic generated by the intersection of corresponding lines in the two pencils is therefore the circle on PQ as diameter, *so that every line in P is perpendicular to its corresponding line in Q*.

To every point P on an axis of a conic corresponds thus a point Q, such that conjugate lines through P and Q are perpendicular.

We shall show that these *point-pairs* P, Q *form an involution*. To do this let us move P along the axis, and with it the line p, keeping the latter parallel to itself. Then P describes a row, p a perspective pencil (of parallels), and the pole of p a projective row. At the same time the line q describes a pencil of parallels perpendicular to p, and perspective to the row formed by the pole of p. The point Q, therefore, where q cuts the axis, describes a row projective to the row of points P. The two points P and Q describe thus two projective rows on the axis; and not only does P as a point in the first row correspond to Q, but also Q as a point in the first corresponds to P. The two rows therefore form an involution. *The centre of this involution, it is easily seen, is the centre of the conic.*

A focus of this involution has the property that any two conjugate lines through it are perpendicular; hence, it is a focus to the conic.

Such involution exists on each axis. But only one of these can have foci, because all foci lie on the same axis. The involution on one of the axes is elliptic, and appears (§ 80) therefore as the section of two circular involutions in two pencils whose centres lie in the other axis. These centres are foci, hence the one axis contains two foci, the other axis none; *or every central conic has two foci which lie on one axis equidistant from the centre*.

The axis which contains the foci is called the *principal axis*; in case of an hyperbola it is the axis which cuts the curve, because the foci lie within the conic.

In case of the parabola there is but one axis. The involution on this axis has its centre at infinity. One focus is therefore at infinity, the one focus only is finite. *A parabola has only one focus.*

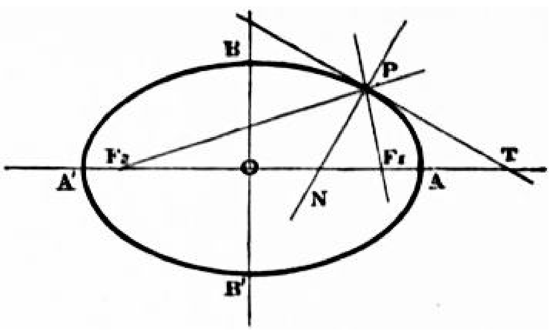


FIG. 34.

§ 85. If through any point P (fig. 34) on a conic the tangent PT and the normal PN (*i.e.* the perpendicular to the tangent through the point of contact) be drawn, these will be conjugate lines with regard to the conic, and at right angles to each other. They will therefore cut the principal axis in two points, which are conjugate in the involution considered in § 84; hence they are harmonic conjugates with regard to the foci. If therefore the two foci F_1 and F_2 be joined to P, these lines will be harmonic with regard to the tangent and normal. As the latter are perpendicular, they will bisect the angles between the other pair. Hence—

The lines joining any point on a conic to the two foci are equally inclined to the tangent *and normal at that point.*

In case of the parabola this becomes—

The line joining any point on a parabola to the focus and the diameter through the point, are equally inclined to the tangent and normal at that point.

From the definition of a focus it follows that—

The segment of a tangent between the directrix and the point of contact is seen from the focus belonging to the directrix under a right angle, because the lines joining the focus to the ends of this segment are conjugate with regard to the conic, and therefore perpendicular.

With equal ease the following theorem is proved:

The two lines which join the points of contact of two tangents each to one focus, but not both to the same, are seen from the intersection of the tangents under equal angles.

§ 86. Other focal properties of a conic are obtained by the following considerations:

Let F (fig. 35) be a focus to a conic, f the corresponding directrix, A and B the points of contact of two tangents meeting at T, and P the point where the line AB cuts the directrix. Then TF will be the polar of P (because polars of F and T meet at P). Hence TF and PF are conjugate lines through a focus, and therefore perpendicular. They are further harmonic conjugates with regard to FA and FB (§§ 64 and 13), so that they bisect the angles formed by these lines. This by the way proves—

The segments between the point of intersection of two tangents to a conic and their points of contact are seen from a focus under equal angles.

If we next draw through A and B lines parallel to TF, then the points A_1, B_1 where these cut the directrix will be harmonic conjugates with regard to P and the point where FT cuts the directrix. The lines FT and FP bisect therefore also the angles between FA_1 and FB_1. From this it follows easily that the triangles FAA_1 and FBB_1 are equiangular, and therefore similar, so that $FA : AA_1 = FB : BB_1$.

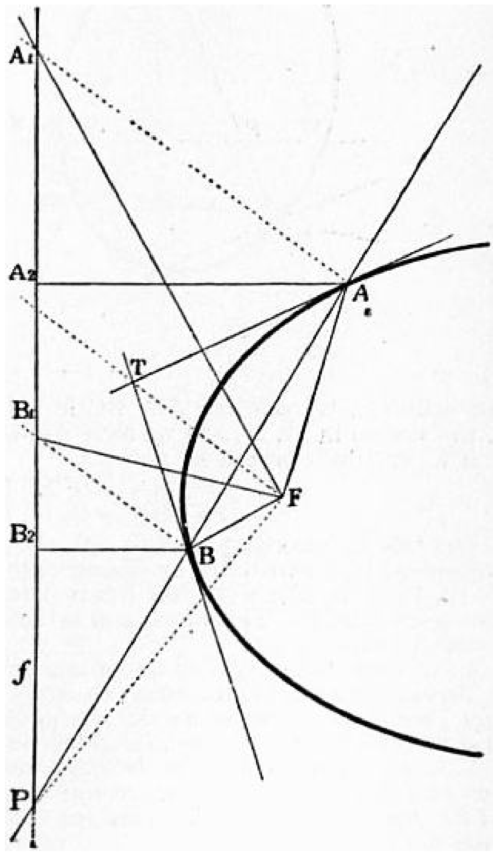


FIG. 35.

The triangles AA_1A_2 and BB_1B_2 formed by drawing perpendiculars from A and B to the directrix are also similar, so that $AA_1 : AA_2 = = BB_1 : BB_2$. This, combined with the above proportion, gives $FA : AA_2 = FB : BB_2$. Hence the theorem:

The ratio of the distances of any point on a conic from a focus and the corresponding directrix is constant.

To determine this ratio we consider its value for a vertex on the principal axis. In an ellipse the focus lies between the two vertices on this axis, hence the focus is nearer to a vertex than to the corresponding directrix. Similarly, in an hyperbola a vertex is nearer to the 704
directrix than to the focus. In a parabola the vertex lies halfway between directrix and focus.

It follows in an ellipse the ratio between the distance of a point from the focus to that from the directrix is less than unity, in the parabola it equals unity, and in the hyperbola it is greater than unity.

It is here the same which focus we take, because the two foci lie symmetrical to the axis of the conic. If now P is any point on the conic having the distances r_1 and r_2 from the foci and the distances d_1 and d_2 from the corresponding directrices, then $r_1/d_1 = r_2/d_2 = e$, where e is constant. Hence also $(r_1 \pm r_2) / (d_1 \pm d_2) = e$.

In the ellipse, which lies between the directrices, $d_1 + d_2$ is constant, therefore also r_1 +

r_2. In the hyperbola on the other hand $d_1 - d_2$ is constant, equal to the distance between the directrices, therefore in this case $r_1 - r_2$ is constant.

If we call the distances of a point on a conic from the focus its focal distances we have the theorem:

In an ellipse the sum of the focal distances is constant; and in an hyperbola the difference of the focal distances is constant.

This constant sum or difference equals in both cases the length of the principal axis.

Pencil of Conics

§ 87. Through four points A, B, C, D in a plane, of which no three lie in a line, an infinite number of conics may be drawn, viz. through these four points and any fifth one single conic. This system of conics is called a pencil of conics. Similarly, all conics touching four fixed lines form a system such that any fifth tangent determines one and only one conic. We have here the theorems:

The pairs of points in which any line is cut by a system of conics through four fixed points are in involution.	The pairs of tangents which can be drawn from a point to a system of conics touching four fixed lines are in involution.

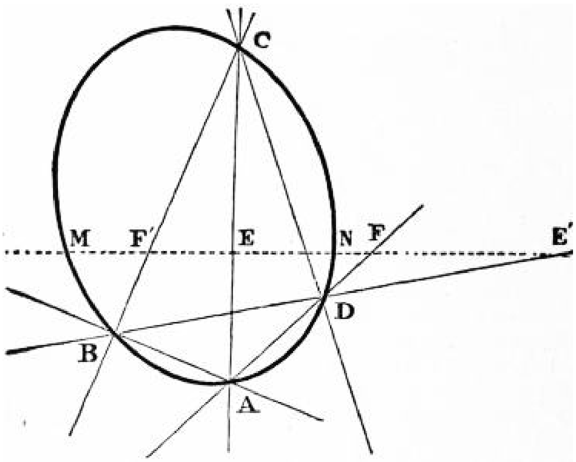


Fig. 36.

We prove the first theorem only. Let ABCD (fig. 36) be the four-point, then any line t will cut two opposite sides AC, BD in the points E, E′, the pair AD, BC in points F, F′, and any conic of the system in M, N, and we have A(CD, MN) = B(CD, MN).

If we cut these pencils by t we get

$$(EF, MN) = (F'E', MN)$$

or

$$(EF, MN) = (E'F', NM).$$

But this is, according to § 77 (7), the condition that M, N are corresponding points in the involution determined by the point pairs E, E′, F, F′ in which the line t cuts pairs of opposite sides of the four-point ABCD. This involution is independent of the particular conic chosen.

§ 88. There follow several important theorems:

Through four points two, one, or no conics may be drawn which touch any given line, according as the involution determined by the given four-point on the line has real, coincident or imaginary foci.

Two, one, or no conics may be drawn which touch four given lines and pass through a given point, according as the involution determined by the given four-side at the point has real, coincident or imaginary focal rays.

For the conic through four points which touches a given line has its point of contact at a focus of the involution determined by the four-point on the line.

As a special case we get, by taking the line at infinity:

Through four points of which none is at infinity either two or no parabolas may be drawn.

The problem of drawing a conic through four points and touching a given line is solved by determining the points of contact on the line, that is, by determining the foci of the involution in which the line cuts the sides of the four-point. The corresponding remark holds for the problem of drawing the conics which touch four lines and pass through a given point.

Ruled Quadric Surfaces

§ 89. We have considered hitherto projective rows which lie in the same plane, in which case lines joining corresponding points envelop a conic. We shall now consider projective rows whose bases do not meet. In this case, corresponding points will be joined by lines which do not lie in a plane, but on some surface, which like every surface generated by lines is called a *ruled* surface. This surface clearly contains the bases of the two rows.

If the points in either row be joined to the base of the other, we obtain two axial pencils which are also projective, those planes being corresponding which pass through corresponding points in the given rows. If A′, A be two corresponding points, α, α′ the planes in the axial pencils passing through them, then AA′ will be the line of intersection of the corresponding planes α, α′ and also the line joining corresponding points in the rows.

If we cut the whole figure by a plane this will cut the axial pencils in two projective flat pencils, and the curve of the second order generated by these will be the curve in which the plane cuts the surface. Hence

The locus of lines joining corresponding points in two projective rows which do not lie in the same plane is a surface which contains the bases of the rows, and which can also be generated by the lines of intersection of corresponding planes in two projective axial pencils. This surface is cut by every plane in a curve of the second order, hence either in a conic or in a line-pair. No line which does not lie altogether on the surface can have more than two points in common with the surface, which is therefore said to be of the second order or is called a ruled quadric surface.

That no line which does not lie on the surface can cut the surface in more than two points is seen at once if a plane be drawn through the line, for this will cut the surface in a conic. It follows also that a line which contains more than two points of the surface lies altogether on the surface.

§ 90. Through any point in space one line can always be drawn cutting two given lines which do not themselves meet.

If therefore three lines in space be given of which no two meet, then through every point in either one line may be drawn cutting the other two.

If a line moves so that it always cuts three given lines of which no two meet, then it generates a ruled quadric surface.

Let a, b, c be the given lines, and p, q, r ... lines cutting them in the points A, A′, A″ ...; B, B′, B″ ...; C, C′, C″ ... respectively; then the planes through a containing p, q, r, and the planes through b containing the same lines, may be taken as corresponding planes in two axial pencils which are projective, because both pencils cut the line c in the same row, C, C′, C″ ...; the surface can therefore be generated by projective axial pencils.

Of the lines p, q, r ... no two can meet, for otherwise the lines a, b, c which cut them would also lie in their plane. There is a single infinite number of them, for one passes through each point of a. These lines are said to form a set of lines on the surface.

If now three of the lines p, q, r be taken, then every line d cutting them will have three points in common with the surface, and will therefore lie altogether on it. This gives rise to a second set of lines on the surface. From what has been said the theorem follows:

A ruled quadric surface contains two sets of straight lines. Every line of one set cuts every line of the other, but no two lines of the same set meet.

Any two lines of the same set may be taken as bases of two projective rows, or of two projective pencils which generate the surface. They are cut by the lines of the other set in two projective rows.

The plane at infinity like every other plane cuts the surface either in a conic proper or in a line-pair. In the first case the surface is called an *Hyperboloid of one sheet*, in the second an *Hyperbolic Paraboloid*.

The latter may be generated by a line cutting three lines of which one lies at infinity, that is, cutting two lines and remaining parallel to a given plane.

Quadric Surfaces

§ 91. The conics, the cones of the second order, and the ruled quadric surfaces complete the figures which can be generated by projective rows or flat and axial pencils, that is, by those aggregates of elements which are of one dimension (§§ 5, 6). We shall now consider the simpler figures which are generated by aggregates of two dimensions. The space at our disposal will not, however, allow us to do more than indicate a few of the results.

§ 92. We establish a correspondence between the lines and planes in pencils in space, or reciprocally between the points and lines in two or more planes, but consider principally pencils.

In two pencils we may either make planes correspond to planes and lines to lines, or else planes to lines and lines to planes. If hereby the condition be satisfied that to a flat, or axial, pencil corresponds in the first case a projective flat, or axial, pencil, and in the second a projective axial, or flat, pencil, the pencils are said to be *projective* in the first case and *reciprocal* in the second.

For instance, two pencils which join two points S_1 and S_2 to the different points and lines in a given plane π are projective (and in perspective position), if those lines and planes be taken as corresponding which meet the plane π in the same point or in the same line. In this 705
case every plane through both centres S_1 and S_2 of the two pencils will correspond to itself. If these pencils are brought into any other position they will be projective (but not perspective).

The correspondence between two projective pencils is uniquely determined, if to four rays (or planes) in the one the corresponding rays (or planes) in the other are given, provided that no three rays of either set lie in a plane.

Let a, b, c, d be four rays in the one, a′, b′, c′, d′ the corresponding rays in the other pencil. We shall show that we can find for every ray e in the first a single corresponding ray e′ in the second. To the axial pencil a (b, c, d ...) formed by the planes which join a to b, c, d ..., respectively corresponds the axial pencil a′ (b′, c′, d′ ...), and this correspondence is determined. Hence, the plane a′e′ which corresponds to the plane ae is determined. Similarly the plane b′e′ may be found and both together determine the ray e′.

Similarly the correspondence between two reciprocal pencils is determined if for four rays in the one the corresponding planes in the other are given.

§ 93. We may now combine—

1. Two reciprocal pencils.

   Each ray cuts its corresponding plane in a point, the locus of these points is a quadric surface.

2. Two projective pencils.

   Each plane cuts its corresponding plane in a line, but a ray as a rule does not cut its corresponding ray. The locus of points where a ray cuts its corresponding ray is a twisted cubic. The lines where a plane cuts its corresponding plane are secants.

3. Three projective pencils.

   The locus of intersection of corresponding planes is a cubic surface.

Of these we consider only the first two cases.

§ 94. If two pencils are reciprocal, then to a plane in either corresponds a line in the other, to a flat pencil an axial pencil, and so on. Every line cuts its corresponding plane in a point. If S_1 and S_2 be the centres of the two pencils, and P be a point where a line a_1 in the first cuts its corresponding plane α_2, *then the line* b_2 *in the pencil* S_2 *which passes through* P *will meet its corresponding plane* β_1 *in* P. For b_2 is a line in the plane α_2. The corresponding plane β_1 must therefore pass through the line a_1, hence through P.

The points in which the lines in S_1 cut the planes corresponding to them in S_2 are therefore the same as the points in which the lines in S_2 cut the planes corresponding to them in S_1.

The locus of these points is a surface which is cut by a plane in a conic or in a line-pair and by a line in not more than two points unless it lies altogether on the surface. The surface itself is therefore called a quadric surface, or a surface of the second order.

To prove this we consider any line p in space.

The flat pencil in S_1 which lies in the plane drawn through p and the corresponding axial pencil in S_2 determine on p two projective rows, and those points in these which coincide with their corresponding points lie on the surface. But there exist only two, or one, or no such points, unless every point coincides with its corresponding point. In the latter case the line lies altogether on the surface.

This proves also that a plane cuts the surface in a curve of the second order, as no line can have more than two points in common with it. To show that this is a curve of the same kind as those considered before, we have to show that it can be generated by projective flat pencils. We prove first that this is true for any plane through the centre of one of the pencils, and afterwards that every point on the surface may be taken as the centre of such pencil. Let then α_1 be a plane through S_1. To the flat pencil in S_1 which it contains corresponds in S_2 a projective axial pencil with axis a_2 and this cuts α_1 in a second flat pencil. These two flat pencils in α_1 are projective, and, in general, neither concentric nor perspective. They generate therefore a conic. But if the line a_2 passes through S_1 the pencils will have S_1 as common centre, and may therefore have two, or one, or no lines united with their corresponding lines. The section of the surface by the plane α_1 will be accordingly a line-pair or a single line, or else the plane α_1 will have only the point S_1 in common with the surface.

Every line l_1 through S_1 cuts the surface in two points, viz. first in S_1 and then at the point where it cuts its corresponding plane. If now the corresponding plane passes through S_1, as in the case just considered, then the two points where l_1 cuts the surface coincide at S_1, and the line is called a tangent to the surface with S_1 as point of contact. Hence if l_1 be a tangent, it lies in that plane τ_1 which corresponds to the line S_2S_1 as a line in the pencil S_2. The section of this plane has just been considered. It follows that—

All tangents to quadric surface at the centre of one of the reciprocal pencils lie in a plane which is called the tangent plane to the surface at that point as point of contact.

To the line joining the centres of the two pencils as a line in one corresponds in the other the tangent plane at its centre.

The tangent plane to a quadric surface either cuts the surface in two lines, or it has only a single line, or else only a single point in common with the surface.

In the first case the point of contact is said to be hyperbolic, in the second parabolic, in the third elliptic.

§ 95. It remains to be proved that every point S on the surface may be taken as centre of one of the pencils which generate the surface. Let S be any point on the surface Φ′ generated by the reciprocal pencils S_1 and S_2. We have to establish a reciprocal correspondence between the pencils S and S_1, so that the surface generated by them is identical with Φ. To do this we draw two planes α_1 and β_1 through S_1, cutting the surface Φ in two conics which we also denote by α_1 and β_1. These conics meet at S_1, and at some other point T where the line of intersection of α_1 and β_1 cuts the surface.

In the pencil S we draw some plane σ which passes through T, but not through S_1 or S_2. It will cut the two conics first at T, and therefore each at some other point which we call A and B respectively. These we join to S by lines a and b, and now establish the required correspondence between the pencils S_1 and S as follows:—To S_1T shall correspond the plane σ, to the plane α_1 the line a, and to β_1 the line b, hence to the flat pencil in α_1 the axial pencil a. These pencils are made projective by aid of the conic in α_1.

In the same manner the flat pencil in β_1 is made projective to the axial pencil b by aid of the conic in β_1, corresponding elements being those which meet on the conic. This determines the correspondence, for we know for more than four rays in S_1 the corresponding planes in S. The two pencils S and S_1 thus made reciprocal generate a quadric surface Φ′, which passes through the point S and through the two conics α_1 and β_1.

The two surfaces Φ and Φ′ have therefore the points S and S_1 and the conics α_1 and β_1 in common. To show that they are identical, we draw a plane through S and S_2, cutting each of the conics α_1 and β_1 in two points, which will always be possible. This plane cuts Φ and Φ′ in two conics which have the point S and the points where it cuts α_1 and β_1 in common, that is five points in all. The conics therefore coincide.

This proves that all those points P on Φ′ lie on Φ which have the property that the plane SS_2P cuts the conics α_1, β_1 in two points each. If the plane SS_2P has not this property, then we draw a plane SS_1P. This cuts each surface in a conic, and these conics have in common the points S, S_1, one point on each of the conics α_1, β_1, and one point on one of the conics through S and S_2 which lie on both surfaces, hence five points. They are therefore coincident, and our theorem is proved.

§ 96. The following propositions follow:—

A quadric surface has at every point a tangent plane.

Every plane section of a quadric surface is a conic or a line-pair.

Every line which has three points in common with a quadric surface lies on the surface.

Every conic which has five points in common with a quadric surface lies on the surface.

Through two conics which lie in different planes, but have two points in common, and through one external point always one quadric surface may be drawn.

§ 97. *Every plane which cuts a quadric surface in a line-pair is a tangent plane.* For every line in this plane through the centre of the line-pair (the point of intersection of the two lines) cuts the surface in two coincident points and is therefore a tangent to the surface, *the centre of the line-pair being the point of contact*.

If a quadric surface contains a line, then every plane through this line cuts the surface in a line-pair (or in two coincident lines). For this plane cannot cut the surface in a conic. Hence:—

If a quadric surface contains one line p *then it contains an infinite number of lines, and through every point* Q *on the surface, one line* q *can be drawn which cuts* p. For the plane through the point Q and the line p cuts the surface in a line-pair which must pass through Q and of which p is one line.

No two such lines q *on the surface can meet.* For as both meet p their plane would contain p and therefore cut the surface in a triangle.

Every line which cuts three lines q *will be on the surface*; for it has three points in common with it.

Hence the quadric surfaces which contain lines are the same as the ruled quadric surfaces considered in §§ 89-93, but with one important exception. In the last investigation we have left out of consideration the possibility of a plane having only one line (two coincident lines) in common with a quadric surface.

§ 98. To investigate this case we suppose first that there is one point A on the surface through which two different lines a, b can be drawn, which lie altogether on the surface.

If P is any other point on the surface which lies neither on a nor b, then the plane through P and a will cut the surface in a second line a′ which passes through P and which cuts a. Similarly there is a line b′ through P which cuts b. These two lines a′ and b′ *may* coincide, but then they must coincide with PA.

If this happens for one point P, it happens for every other point Q. For if two different lines could be drawn through Q, then by the same reasoning the line PQ would be altogether on the surface, hence two lines would be drawn through P against the assumption. From this follows:—

If there is one point on a quadric surface through which one, but only one, line can be drawn on the surface, then through every point one line can be drawn, and all these lines 706
meet in a point. The surface is a cone of the second order.

If through one point on a quadric surface, two, and only two, lines can be drawn on the surface, then through every point two lines may be drawn, and the surface is ruled quadric surface.

If through one point on a quadric surface no line on the surface can be drawn, then the surface contains no lines.

Using the definitions at the end of § 95, we may also say:—

On a quadric surface the points are all hyperbolic, or all parabolic, or all elliptic.

As an example of a quadric surface with elliptical points, we mention the sphere which may be generated by two reciprocal pencils, where to each line in one corresponds the plane perpendicular to it in the other.

§ 99. *Poles and Polar Planes.*—The theory of poles and polars with regard to a conic is easily extended to quadric surfaces.

Let P be a point in space not on the surface, which we suppose not to be a cone. On every line through P which cuts the surface in two points we determine the harmonic conjugate Q of P with regard to the points of intersection. Through one of these lines we draw two planes α and β. The locus of the points Q in α is a line a, the polar of P with regard to the conic in which α cuts the surface. Similarly the locus of points Q in β is a line b. This cuts a, because the line of intersection of α and β contains but one point Q. The locus of all points Q therefore is a plane. *This plane is called the polar plane of the point* P, *with regard to the quadric surface. If* P *lies on the surface we take the tangent plane of P as its polar.*

The following propositions hold:—

1. *Every point has a polar plane*, which is constructed by drawing the polars of the point with regard to the conics in which two planes through the point cut the surface.

2. *If* Q *is a point in the polar of* P, *then* P *is a point in the polar of* Q, because this is true with regard to the conic in which a plane through PQ cuts the surface.

3. *Every plane is the polar plane of one point, which is called the Pole of the plane.*

The pole to a plane is found by constructing the polar planes of three points in the plane. Their intersection will be the pole.

4. *The points in which the polar plane of P cuts the surface are points of contact of tangents drawn from P to the surface*, as is easily seen. Hence:—

5. *The tangents drawn from a point P to a quadric surface form a cone of the second order*, for the polar plane of P cuts it in a conic.

6. *If the pole describes a line a, its polar plane will turn about another line* a′, as follows from 2. *These lines a and a′ are said to be conjugate with regard to the surface.*

§ 100. The pole of the line at infinity is called the *centre* of the surface. If it lies at the infinity, the plane at infinity is a tangent plane, and the surface is called a *paraboloid*.

The polar plane to any point at infinity passes through the centre, and is called a diametrical plane.

A line through the centre is called a diameter. It is bisected at the centre. The line conjugate to it lies at infinity.

If a point moves along a diameter its polar plane turns about the conjugate line at infinity; that is, *it moves parallel to itself, its centre moving on the first line.*

The middle points of parallel chords lie in a plane, viz. in the polar plane of the point at infinity through which the chords are drawn.

The centres of parallel sections lie in a diameter which is a line conjugate to the line at infinity in which the planes meet.

Twisted Cubics

§ 101. If two pencils with centres S_1 and S_2 are made projective, then to a ray in one corresponds a ray in the other, to a plane a plane, to a flat or axial pencil a projective flat or axial pencil, and so on.

There is a double infinite number of lines in a pencil. We shall see that a single infinite number of lines in one pencil meets its corresponding ray, and that the points of intersection form a curve in space.

Of the double infinite number of planes in the pencils each will meet its corresponding plane. This gives a system of a double infinite number of lines in space. We know (§ 5) that there is a quadruple infinite number of lines in space. From among these we may select those which satisfy one or more given conditions. The systems of lines thus obtained were first systematically investigated and classified by Plücker, in his *Geometrie des Raumes*. He uses the following names:—

A *treble infinite* number of lines, that is, all lines which satisfy one condition, are said to form a *complex of lines*; *e.g.* all lines cutting a given line, or all lines touching a surface.

A *double infinite* number of lines, that is, all lines which satisfy two conditions, or which are common to two complexes, are said to form a *congruence of lines*; *e.g.* all lines in a plane, or all lines cutting two curves, or all lines cutting a given curve twice.

A *single infinite* number of lines, that is, all lines which satisfy three conditions, or which belong to three complexes, form a *ruled surface*; *e.g.* one set of lines on a ruled quadric surface, or developable surfaces which are formed by the tangents to a curve.

It follows that all lines in which corresponding planes in two projective pencils meet form a congruence. We shall see this congruence consists of all lines which cut a twisted cubic twice, or of all *secants* to a twisted cubic.

§ 102. Let l_1 be the line S_1S_2 as a line in the pencil S_1. To it corresponds a line l_2 in S_2. *At* *each of the centres two corresponding lines meet.* The two axial pencils with l_1 and l_2 as axes are projective, and, as, their axes meet at S_2, the intersections of corresponding planes form a cone of the second order (§ 58), with S_2 as centre. If π_1 and π_2 be corresponding planes, then their intersection will be a line p_2 which passes through S_2. Corresponding to it in S_1 will be a line p_1 which lies in the plane π_1, and which therefore meets p_2 at some point P. Conversely, if p_2 be any line in S_2 which meets its corresponding line p_1 at a point P, then to the plane l_2p_2 will correspond the plane l_1p_1, that is, the plane S_1S_2P. These planes intersect in p_2, so that p_2 is a line on the quadric cone generated by the axial pencils l_1 and l_2. Hence:—

All lines in one pencil which meet their corresponding lines in the other form a cone of the second order which has its centre at the centre of the first pencil, and passes through the centre of the second.

From this follows that the points in which corresponding rays meet lie on two cones of the second order which have the ray joining their centres in common, and form therefore, together with the line S_1S_2 or l_1, the intersection of these cones. Any plane cuts each of the cones in a conic. These two conics have necessarily that point in common in which it cuts the line l_1, and therefore besides either one or three other points. It follows that the curve is of the third order as a plane may cut it in three, but not in more than three, points. Hence:—

The locus of points in which corresponding lines on two projective pencils meet is a curve of the third order or a "twisted cubic" k, which passes through the centres of the pencils, and which appears as the intersection of two cones of the second order, which have one line in common.

A line belonging to the congruence determined by the pencils is a secant of the cubic; it has two, or one, or no points in common with this cubic, and is called accordingly a secant proper, a tangent, or a secant improper of the cubic. A secant improper may be considered, to use the language of coordinate geometry, as a secant with imaginary points of intersection.

§ 103. If a_1 and a_2 be any two corresponding lines in the two pencils, then corresponding planes in the axial pencils having a_1 and a_2 as axes generate a ruled quadric surface. If P be any point on the cubic k, and if p_1, p_2 be the corresponding rays in S_1 and S_2 which meet at P, then to the plane a_1p_1 in S_1 corresponds a_2p_2 in S_2. These therefore meet in a line through P.

This may be stated thus:—

Those secants of the cubic which cut a ray a_1, *drawn through the centre* S_1 *of one pencil, form a ruled quadric surface which passes through both centres, and which contains the twisted cubic* k. *Of such surfaces an infinite number exists. Every ray through* S_1 *or* S_2 *which is not a secant determines one of them.*

If, however, the rays a_1 and a_2 are secants meeting at A, then the ruled quadric surface becomes a cone of the second order, having A as centre. Or *all lines of the congruence which pass through a point on the twisted cubic k form a cone of the second order*. In other words, the projection of a twisted cubic from any point in the curve on to any plane is a conic.

If a_1 is not a secant, but made to pass through any point Q in space, the ruled quadric surface determined by a_1 will pass through Q. *There will therefore be one line of the congruence passing through* Q, *and only one.* For if two such lines pass through Q, then the lines S_1Q and S_2Q will be corresponding lines; hence Q will be a point on the cubic k, and an infinite number of secants will pass through it. Hence:—

Through every point in space not on the twisted cubic one and only one secant to the cubic can be drawn.

§ 104. The fact that all the secants through a point on the cubic form a quadric cone shows that the centres of the projective pencils generating the cubic are not distinguished from any other points on the cubic. If we take any two points S, S′ on the cubic, and draw the secants through each of them, we obtain two quadric cones, which have the line SS′ in common, and which intersect besides along the cubic. If we make these two pencils having S and S′ as centres projective by taking four rays on the one cone as corresponding to the four rays on the other which meet the first on the cubic, the correspondence is determined. These two pencils will generate a cubic, and the two cones of secants having S and S′ as centres will be identical with the above cones, for each has five rays in common with one of the first, viz. the line SS′ and the four lines determined for the correspondence; therefore these two cones intersect in the original cubic. This gives the theorem:—

On a twisted cubic any two points may be taken as centres of projective pencils which *generate the cubic, corresponding planes being those which meet on the same secant.*

Of the two projective pencils at S and S′ we may keep the first fixed, and move the centre of the other along the curve. The pencils will hereby remain projective, and a plane α in S will be cut by its corresponding plane α′ always in the same secant a. Whilst S′ moves along the curve the plane α′ will turn about a, describing an axial pencil.

Authorities.—In this article we have given a purely geometrical theory of conics, cones of the second order, quadric surfaces, &c. In doing so we have followed, to a great extent, 707
Reye’s *Geometrie der Lage*, and to this excellent work those readers are referred who wish for a more exhaustive treatment of the subject. Other works especially valuable as showing the development of the subject are: Monge, *Géométrie descriptive*: Carnot, *Géométrie de position* (1803), containing a theory of transversals; Poncelet’s great work *Traité des propriétés projectives des figures* (1822); Möbins, *Barycentrischer Calcul* (1826); Steiner, *Abhängigkeit geometrischer Gestalten* (1832), containing the first full discussion of the projective relations between rows, pencils, &c.; Von Staudt, *Geometrie der Lage* (1847) and *Beiträge zur Geometrie der Lage* (1856-1860), in which a system of geometry is built up from the beginning without any reference to number, so that ultimately a number itself gets a geometrical definition, and in which imaginary elements are systematically introduced into pure geometry; Chasles, *Aperçu historique* (1837), in which the author gives a brilliant account of the progress of modern geometrical methods, pointing out the advantages of the different purely geometrical methods as compared with the analytical ones, but without taking as much account of the German as of the French authors; Id., *Rapport sur les progrès de la géométrie* (1870), a continuation of the *Aperçu*; Id., *Traité de géométrie supérieure* (1852); Cremona, *Introduzione ad una teoria geometrica delle curve piane* (1862) and its continuation *Preliminari di una teoria geometrica delle superficie* (German translations by Curtze). As more elementary books, we mention: Cremona, *Elements of Projective Geometry*, translated from the Italian by C. Leudesdorf (2nd ed., 1894); J.W. Russell, *Pure Geometry* (2nd ed., 1905).

(O. H.)

III. Descriptive Geometry

This branch of geometry is concerned with the methods for representing solids and other figures in three dimensions by drawings in one plane. The most important method is that which was invented by Monge towards the end of the 18th century. It is based on parallel projections to a plane by rays perpendicular to the plane. Such a projection is called orthographic (see Projection, § 18). If the plane is horizontal the projection is called the plan of the figure, and if the plane is vertical the elevation. In Monge’s method a figure is represented by its plan and elevation. It is therefore often called drawing in plan and elevation, and sometimes simply orthographic projection.

§ 1. We suppose then that we have two planes, one horizontal, the other vertical, and these we call the planes of plan and of elevation respectively, or the horizontal and the vertical plane, and denote them by the letters π_1 and π_2. Their line of intersection is called the axis, and will be denoted by xy.

If the surface of the drawing paper is taken as the plane of the plan, then the vertical plane will be the plane perpendicular to it through the axis xy. To bring this also into the plane of the drawing paper we turn it about the axis till it coincides with the horizontal plane. This process of turning one plane down till it coincides with another is called *rabatting* one to the other. Of course there is no necessity to have one of the two planes horizontal, but even when this is not the case it is convenient to retain the above names.

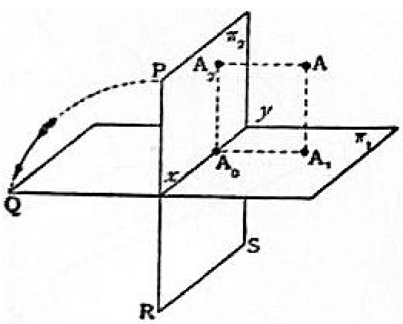


Fig. 37.

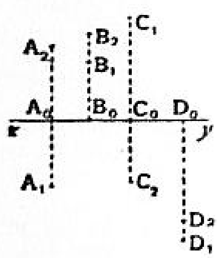


Fig. 38.

The whole arrangement will be better understood by referring to fig. 37. A point A in space is there projected by the perpendicular AA_1 and AA_2 to the planes π_1 and π_2 so that A_1 and

A_2 are the horizontal and vertical projections of A.

If we remember that a line is perpendicular to a plane that is perpendicular to every line in the plane if only it is perpendicular to any two intersecting lines in the plane, we see that the axis which is perpendicular both to AA_1 and to AA_2 is also perpendicular to A_1A_0 and to A_2A_0 because these four lines are all in the same plane. Hence, if the plane π_2 be turned about the axis till it coincides with the plane π_1, then A_2A_0 will be the continuation of A_1A_0. This position of the planes is represented in fig. 38, in which the line A_1A_2 is perpendicular to the axis x.

Conversely any two points A_1, A_2 in a line perpendicular to the axis will be the projections of some point in space when the plane π_2 is turned about the axis till it is perpendicular to the plane π_1, because in this position the two perpendiculars to the planes π_1 and π_2 through the points A_1 and A_2 will be in a plane and therefore meet at some point A.

Representation of Points.—We have thus the following method of representing in a single plane the position of points in space:—*we take in the plane a line xy as the axis, and then any pair of points A_1, A_2 in the plane on a line perpendicular to the axis represent a point A in space*. If the line A_1A_2 cuts the axis at A_0, and if at A_1 a perpendicular be erected to the plane, then the point A will be in it at a height $A_1A = A_0A_2$ above the plane. This gives the position of the point A relative to the plane π_1. In the same way, if in a perpendicular to π_2 through A_2 a point A be taken such that $A_2A = A_0A_1$, then this will give the point A relative to the plane π_2.

§ 2. The two planes π_1, π_2 in their original position divide space into four parts. These are called the four quadrants. We suppose that the plane π_2 is turned as indicated in fig. 37, so that the point P comes to Q and R to S, then the quadrant in which the point A lies is called the first, and we say that in the first quadrant a point lies above the horizontal and in front of the vertical plane. Now we go round the axis in the sense in which the plane π_2 is turned and come in succession to the second, third and fourth quadrant. In the second a point lies above the plane of the plan and behind the plane of elevation, and so on. In fig. 39, which represents a side view of the planes in fig. 37 the quadrants are marked, and in each a point with its projection is taken. Fig. 38 shows how these are represented when the plane π_2 is turned down. We see that

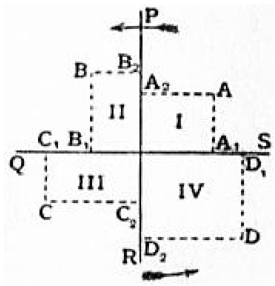


FIG. 39.

A point lies in the first quadrant if the plan lies below, the elevation above the axis; in the second if plan and elevation both lie above; in the third if the plan lies above, the elevation below; in the fourth if plan and elevation both lie below the axis.

If a point lies in the horizontal plane, its elevation lies in the axis and the plan coincides with the point itself. *If a point lies in the vertical plane*, its plan lies in the axis and the elevation coincides with the point itself. *If a point lies in the axis*, both its plan and elevation lie in the axis and coincide with it.

Of each of these propositions, which will easily be seen to be true, the converse holds also.

§ 3. *Representation of a Plane.*—As we are thus enabled to represent points in a plane, we can represent any finite figure by representing its separate points. It is, however, not possible to represent a plane in this way, for the projections of its points completely cover the planes π_1 and π_2, and no plane would appear different from any other. But any plane α cuts each of the planes π_1, π_2 in a line. These are called the traces of the plane. They cut each other in the axis at the point where the latter cuts the plane α.

A plane is determined by its two traces, which are two lines that meet on the axis, and, conversely, *any two lines which meet on the axis determine a plane*.

If the plane is parallel to the axis its traces are parallel to the axis. Of these one may be at infinity; then the plane will cut one of the planes of projection at infinity and will be parallel to it. Thus a plane parallel to the horizontal plane of the plan has only one finite trace, viz. that with the plane of elevation.

If the plane passes through the axis both its traces coincide with the axis. This is the only case in which the representation of the plane by its two traces fails. A third plane of projection is therefore introduced, which is best taken perpendicular to the other two. We call it simply the third plane and denote it by π_3.

As it is perpendicular to π_1, it may be taken as the plane of elevation, its line of intersection γ with π_1 being the axis, and be turned down to coincide with π_1. This is represented in fig. 40. OC is the axis xy whilst OA and OB are the traces of the third plane. They lie in one line γ. The plane is rabatted about γ to the horizontal plane. A plane α through the axis xy will then show in it a trace α_3. In fig. 40 the lines OC and OP will thus be the traces of a plane through the axis xy, which makes an angle POQ with the horizontal plane.

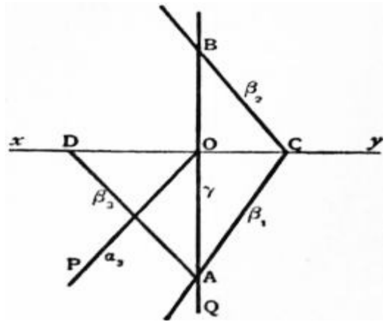


FIG. 40.

We can also find the trace which any other plane makes with π_3. In rabatting the plane π_3 its trace OB with the plane π_2 will come to the position OD. Hence a plane β having the traces CA and CB will have with the third plane the trace β_3, or AD if OD = OB.

It also follows immediately that— 708

If a plane α is perpendicular to the horizontal plane, then every point in it has its horizontal projection in the horizontal trace of the plane, as all the rays projecting these points lie in the plane itself.

Any plane which is perpendicular to the horizontal plane has its vertical trace perpendicular to the axis.

Any plane which is perpendicular to the vertical plane has its horizontal trace perpendicular to the axis and the vertical projections of all points in the plane lie in this trace.

§ 4. *Representation of a Line.*—A line is determined either by two points in it or by two planes through it. We get accordingly two representations of it either by projections or by traces.

First.—*A line a is represented by its projections* a_1 *and* a_2 *on the two planes* π_1 *and* π_2. These may be any two lines, for, bringing the planes π_1, π_2 into their original position, the planes through these lines perpendicular to π_1 and π_2 respectively will intersect in some line a which has a_1, a_2 as its projections.

Secondly.—*A line a is represented by its traces—that is, by the points in which it cuts the two planes* π_1, π_2. Any two points may be taken as the traces of a line in space, for it is determined when the planes are in their original position as the line joining the two traces. This representation becomes undetermined if the two traces coincide in the axis. In this case we again use a third plane, or else the projections of the line.

The fact that there are different methods of representing points and planes, and hence two methods of representing lines, suggests the principle of duality (section ii., *Projective Geometry*, § 41). It is worth while to keep this in mind. It is also worth remembering that traces of planes or lines always lie in the planes or lines which they represent. Projections do not as a rule do this excepting when the point or line projected lies in one of the planes of projection.

Having now shown how to represent points, planes and lines, we have to state the conditions which must hold in order that these elements may lie one in the other, or else that the figure formed by them may possess certain metrical properties. It will be found that the former are very much simpler than the latter.

Before we do this, however, we shall explain the notation used; for it is of great importance to have a systematic notation. We shall denote points in space by capitals A, B, C; planes in space by Greek letters α, β, γ; lines in space by small letters a, b, c; horizontal projections by suffixes 1, like A_1, a_1; vertical projections by suffixes 2, like A_2, a_2; traces by single and double dashes α' α'', a', a''. Hence P_1 will be the horizontal projection of a point P in space; a line a will have the projections a_1, a_2 and the traces a' and a''; a plane α has the traces α' and α''.

§ 5. *If a point lies in a line, the projections of the point lie in the projections of the line.*

If a line lies in a plane, the traces of the line lie in the traces of the plane.

These propositions follow at once from the definitions of the projections and of the traces.

If a point lies in two lines its projections must lie in the projections of both. Hence

If two lines, given by their projections, intersect, the intersection of their planes and the intersection of their elevations must lie in a line perpendicular to the axis, because they must be the projections of the point common to the two lines.

Similarly—*If two lines given by their traces lie in the same plane or intersect, then the lines joining their horizontal and vertical traces respectively must meet on the axis*, because they must be the traces of the plane through them.

§ 6. *To find the projections of a line which joins two points A, B given by their projections* A_1, A_2 *and* B_1, B_2, we join A_1, B_1 and A_2, B_2; these will be the projections required. For example, the traces of a line are two points in the line whose projections are known or at all events easily found. They are the traces themselves and the feet of the perpendiculars from them to the axis.

Hence *if* a′ a″ (fig. 41) *are the traces of a line a, and if the perpendiculars from them cut the axis in* P *and* Q *respectively, then the line* a′Q *will be the horizontal and* a″P *the vertical projection of the line*.

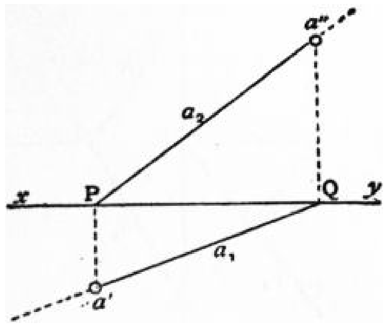


Fig. 41.

Conversely, if the projections a_1, a_2 of a line are given, and if these cut the axis in Q and P respectively, then *the perpendiculars* Pa′ *and* Qa″ *to the axis drawn through these points cut the projections* a_1 *and* a_2 *in the traces* a′ *and* a″.

To find the line of intersection of two planes, we observe that this line lies in both planes; its traces must therefore lie in the traces of both. Hence the points where the horizontal traces of the given planes meet will be the horizontal, and the point where the vertical traces meet the vertical trace of the line required.

§ 7. *To decide whether a point* A, *given by its projections, lies in a plane α, given by its traces*, we draw a line p by joining A to some point in the plane α and determine its traces. If these lie in the traces of the plane, then the line, and therefore the point A, lies in the plane; otherwise not. This is conveniently done by joining A_1 to some point p′ in the trace α′; this gives p_1; and the point where the perpendicular from p′ to the axis cuts the latter we join to A_2; this gives p_2. If the vertical trace of this line lies in the vertical trace of the plane, then, and then only, does the line p, and with it the point A, lie in the plane α.

§ 8. *Parallel planes have parallel traces*, because parallel planes are cut by any plane, hence also by π_1 and by π_2, in parallel lines.

Parallel lines have parallel projections, because points at infinity are projected to infinity.

If a line is parallel to a plane, then lines through the traces of the line and parallel to the traces of the plane must meet on the axis, because these lines are the traces of a plane parallel to the given plane.

§ 9. *To draw a plane through two intersecting lines or through two parallel lines*, we determine the traces of the lines; the lines joining their horizontal and vertical traces respectively will be the horizontal and vertical traces of the plane. They will meet, at a finite point or at infinity, on the axis if the lines do intersect.

To draw a plane through a line and a point without the line, we join the given point to any point in the line and determine the plane through this and the given line.

To draw a plane through three points which are not in a line, we draw two of the lines which each join two of the given points and draw the plane through them. If the traces of all three lines AB, BC, CA be found, these must lie in two lines which meet on the axis.

§ 10. We have in the last example got more points, or can easily get more points, than are necessary for the determination of the figure required—in this case the traces of the plane. This will happen in a great many constructions and is of considerable importance. It may happen that some of the points or lines obtained are not convenient in the actual construction. The horizontal traces of the lines AB and AC may, for instance, fall very near together, in which case the line joining them is not well defined. Or, one or both of them may fall beyond the drawing paper, so that they are practically non-existent for the construction. In this case the traces of the line BC may be used. Or, if the vertical traces of AB and AC are both in convenient position, so that the vertical trace of the required plane is found and one of the horizontal traces is got, then we may join the latter to the point where the vertical trace cuts the axis.

The draughtsman must remember that the lines which he draws are not mathematical lines without thickness, and therefore every drawing is affected by some errors. It is therefore very desirable to be able constantly to check the latter. Such checks always present themselves when the same result can be obtained by different constructions, or when, as in the above case, some lines must meet on the axis, or if three points must lie in a line. A careful draughtsman will always avail himself of these checks.

§ 11. *To draw a plane through a given point parallel to a given plane α*, we draw through the point two lines which are parallel to the plane α, and determine the plane through them; or, as we know that the traces of the required plane are parallel to those of the given one (§ 8), we need only draw one line l through the point parallel to the plane and find one of its traces, say the vertical trace l″; a line through this parallel to the vertical trace of α will be the vertical trace β″ of the required plane β, and a line parallel to the horizontal trace of α meeting β″ on the axis will be the horizontal trace β′.

Let A_1 A_2 (fig. 42) be the given point, α′ α″ the given plane, a line l_1 through A_1, parallel to α′ and a horizontal line l_2 through A_2 will be the projections of a line l through A parallel to the plane, because the horizontal plane through this line will cut the plane α in a line c which has its horizontal projection c_1 parallel to α′.

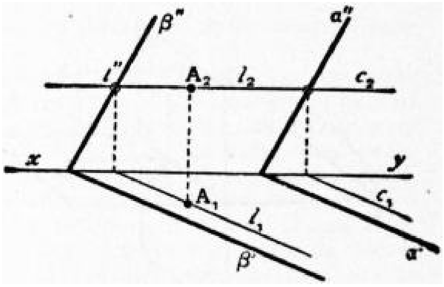


FIG. 42.

§ 12. We now come to the metrical properties of figures.

A line is perpendicular to a plane if the projections of the line are perpendicular to the traces of the plane. We prove it for the horizontal projection. If a line p is perpendicular to a plane α, every plane through p is perpendicular to α; hence also the vertical plane which projects the line p to p_1. As this plane is perpendicular both to the horizontal plane and to the plane α, it is also perpendicular to their intersection—that is, to the horizontal trace of α. It follows that every line in this projecting plane, therefore also p_1, the plan of p, is perpendicular to the horizontal trace of α.

To draw a plane through a given point A perpendicular to a given line p, we first draw through some point O in the axis lines γ′, γ″ perpendicular respectively to the projections p_1 and p_2 of the given line. These will be the traces of a plane γ which is perpendicular to the given line. We next draw through the given point A a plane parallel to the plane γ; this will be the plane required.

Other metrical properties depend on the determination of the real size or shape of a 709
figure.

In general the projection of a figure differs both in size and shape from the figure itself. But figures in a plane parallel to a plane of projection will be identical with their projections, and will thus be given in their true dimensions. In other cases there is the problem, constantly recurring, either to find the true shape and size of a plane figure when plan and elevation are given, or, conversely, to find the latter from the known true shape of the figure itself. To do this, the plane is turned about one of its traces till it is laid down into that plane of projection to which the trace belongs. This is technically called rabatting the plane respectively into the plane of the plan or the elevation. As there is no difference in the treatment of the two cases, we shall consider only the case of rabatting a plane α into the plane of the plan. The plan of the figure is a parallel (orthographic) projection of the figure itself. The results of parallel projection (see PROJECTION, §§ 17 and 18) may therefore now be used. The trace α′ will hereby take the place of what formerly was called the axis of projection. Hence we see that corresponding points in the plan and in the rabatted plane are joined by lines which are perpendicular to the trace α′ and that corresponding lines meet on this trace. We also see that the correspondence is completely determined if we know for one point or one line in the plan the corresponding point or line in the rabatted plane.

Before, however, we treat of this we consider some special cases.

§ 13. *To determine the distance between two points A, B given by their projections* A_1, B_1 *and* A_2, B_2, *or, in other words, to determine the true length of a line the plan and elevation of which are given.*

Solution.—The two points A, B in space lie vertically above their plans A_1, B_1 (fig. 43) and $A_1A = A_0A_2$, $B_1B = B_0B_2$. The four points A, B, A_1, B_1 therefore form a plane quadrilateral on the base A_1B_1 and having right angles at the base. This plane we rabatt about A_1B_1 by drawing A_1A and B_1B perpendicular to A_1B_1 and making $A_1A = A_0A_2$, B_1B

$= B_0B_2$. Then AB will give the length required.

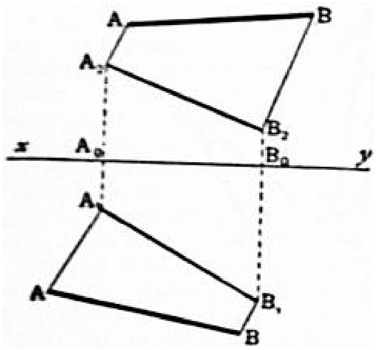


FIG. 43.

The construction might have been performed in the elevation by making $A_2A = A_0A_1$ and $B_2B = B_0B_1$ on lines perpendicular to A_2B_2. Of course AB must have the same length in both cases.

This figure may be turned into a model. Cut the paper along A_1A, AB and BB_1, and fold the piece A_1ABB_1 over along A_1B_1 till it stands upright at right angles to the horizontal plane. The points A, B will then be in their true position in space relative to π_1. Similarly if B_2BAA_2 be cut out and turned along A_2B_2 through a right angle we shall get AB in its true position relative to the plane π_2. Lastly we fold the whole plane of the paper along the axis x till the plane π_2 is at right angles to π_1. In this position the two sets of points AB will coincide if the drawing has been accurate.

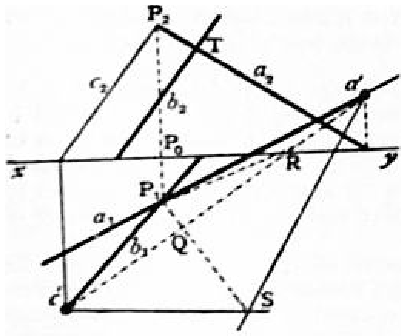


FIG. 44.

Models of this kind can be made in many cases and their construction cannot be too highly recommended in order to realize orthographic projection.

§ 14. *To find the angle between two given lines* a, b *of which the projections* a_1, b_1 *and* a_2, b_2 *are given.*

Solution.—Let a_1, b_1 (fig. 44) meet in P_1, a_2, b_2 in T, then if the line P_1T is not perpendicular to the axis the two lines will not meet. In this case we draw a line parallel to b to meet the line a. This is easiest done by drawing first the line P_1P_2 perpendicular to the axis to meet a_2 in P_2, and then drawing through P_2 a line c_2 parallel to b_2; then b_1, c_2 will be the projections of a line c which is parallel to b and meets a in P. The plane α which these two lines determine we rabatt to the plan. We determine the traces a′ and c′ of the lines a and c; then a′c′ is the trace α′ of their plane. On rabatting the point P comes to a point S on the line P_1Q perpendicular to a′c′, so that QS = QP. But QP is the hypotenuse of a triangle PP_1Q with a right angle P_1. This we construct by making $QR = P_0P_2$; then $P_1R = PQ$. The lines a′S and c′S will therefore include angles equal to those made by the given lines. It is to be remembered that two lines include two angles which are supplementary. Which of these is to be taken in any special case depends upon the circumstances.

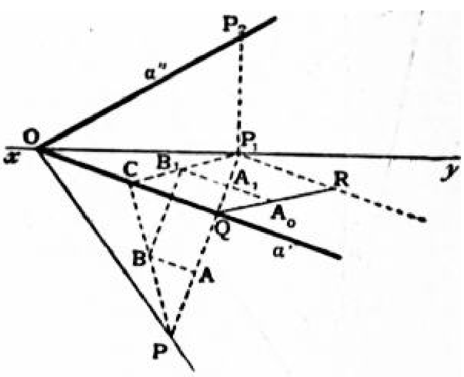
To determine the angle between a line and a plane, we draw through any point in the line a perpendicular to the plane (§ 12) and determine the angle between it and the given line. The complement of this angle is the required one.

To determine the angle between two planes, we draw through any point two lines perpendicular to the two planes and determine the angle between the latter as above.

In special cases it is simpler to determine at once the angle between the two planes by taking a plane section perpendicular to the intersection of the two planes and rabatt this. This is especially the case if one of the planes is the horizontal or vertical plane of projection.

Thus in fig. 45 the angle P_1QR is the angle which the plane α makes with the horizontal plane.

§ 15. We return to the general case of rabatting a plane α of which the traces α′ α″ are given.



Here it will be convenient to determine first the position which the trace α″—which is a line in α—assumes when rabatted. Points in this line coincide with their elevations. Hence it is given in its true dimension, and we can measure off along it the true distance between two points in it. If therefore (fig. 45) P is any point in α″ originally coincident with its elevation P_2, and if O is the point where α″ cuts the axis xy, so that O is also in α′, then the point P will after rabatting the plane assume such a position that $OP = OP_2$. At the same time the plan is an orthographic projection of

FIG. 45.

the plane α. Hence the line joining P to the plan P_1 will after rabatting be perpendicular to α′. But P_1 is known; it is the foot of the perpendicular from P_2 to the axis xy. We draw therefore, to find P, from P_1 a perpendicular P_1Q to α′ and find on it a point P such that OP = OP_2. Then the line OP will be the position of α″ when rabatted. This line corresponds therefore to the plan of α″—that is, to the axis xy, corresponding points on these lines being those which lie on a perpendicular to α′.

We have thus one pair of corresponding lines and can now find for any point B_1 in the plan the corresponding point B in the rabatted plane. We draw a line through B_1, say B_1P_1, cutting α′ in C. To it corresponds the line CP, and the point where this is cut by the projecting ray through B_1, perpendicular to α′, is the required point B.

Similarly any figure in the rabatted plane can be found when the plan is known; but this is usually found in a different manner without any reference to the general theory of parallel projection. As this method and the reasoning employed for it have their peculiar advantages, we give it also.

Supposing the planes π_1 and π_2 to be in their positions in space perpendicular to each other, we take a section of the whole figure by a plane perpendicular to the trace α′ about which we are going to rabatt the plane α. Let this section pass through the point Q in α′. Its traces will then be the lines QP_1 and P_1P_2 (fig. 9). These will be at right angles, and will therefore, together with the section QP_2 of the plane α, form a right-angled triangle QP_1P_2 with the right angle at P_1, and having the sides P_1Q and P_1P_2 which both are given in their true lengths. This triangle we rabatt about its base P_1Q, making $P_1R = P_1P_2$. The line QR will then give the true length of the line QP in space. If now the plane α be turned about α′ the point P will describe a circle about Q as centre with radius QP = QR, in a plane perpendicular to the trace α′. Hence when the plane α has been rabatted into the horizontal plane the point P will lie in the perpendicular P_1Q to α′, so that QP = QR.

If A_1 is the plan of a point A in the plane α, and if A_1 lies in QP_1, then the point A will lie vertically above A_1 in the line QP. On turning down the triangle QP_1P_2, the point A will come to A_0, the line A_1A_0 being perpendicular to QP_1. Hence A will be a point in QP such that QA = QA_0.

If B_1 is the plan of another point, but such that A_1B_1 is parallel to α′, then the corresponding line AB will also be parallel to α′. Hence, if through A a line AB be drawn parallel to α′, and B_1B perpendicular to α′, then their intersection gives the point B. Thus of any point given in plan the real position in the plane α, when rabatted, can be found by this second method. This is the one most generally given in books on geometrical drawing. The first method explained is, however, in most cases preferable as it gives the draughtsman a greater variety of constructions. It requires a somewhat greater amount of theoretical knowledge.

If instead of our knowing the plan of a figure the latter is itself given, then the process of finding the plan is the reverse of the above and needs little explanation. We give an example.

§ 16. *It is required to draw the plan and elevation of a polygon of which the real shape and position in a given plane α are known.*

We first rabatt the plane α (fig. 46) as before so that P_1 comes to P, hence OP_1 to OP. Let the given polygon in α be the figure ABCDE. We project, not the vertices, but the sides. To project the line AB, we produce it to cut α′ in F and OP in G, and draw GG_1 perpendicular to α′; then G_1 corresponds to G, therefore FG_1 to FG. In the same manner we might project all the other sides, at least those which 710
cut OF and OP in convenient points. It will be best, however, first to produce all the sides to cut OP and α′ and then to draw all the projecting rays through A, B, C ... perpendicular to α′, and in the same direction the lines G, G_1, &c. By drawing FG we get the points A_1, B_1 on the projecting ray through A and B. We then join B to the point M where

BC produced meets the trace α′. This gives C_1. So we go on till we have found E_1. The line A_1 E_1 must then meet AE in α′, and this gives a check. If one of the sides cuts α′ or OP beyond the drawing paper this method fails, but then we may easily find the projection of some other line, say of a diagonal, or directly the projection of a point, by the former methods. The diagonals may also serve to check the drawing, for two corresponding diagonals must meet in the trace α′.

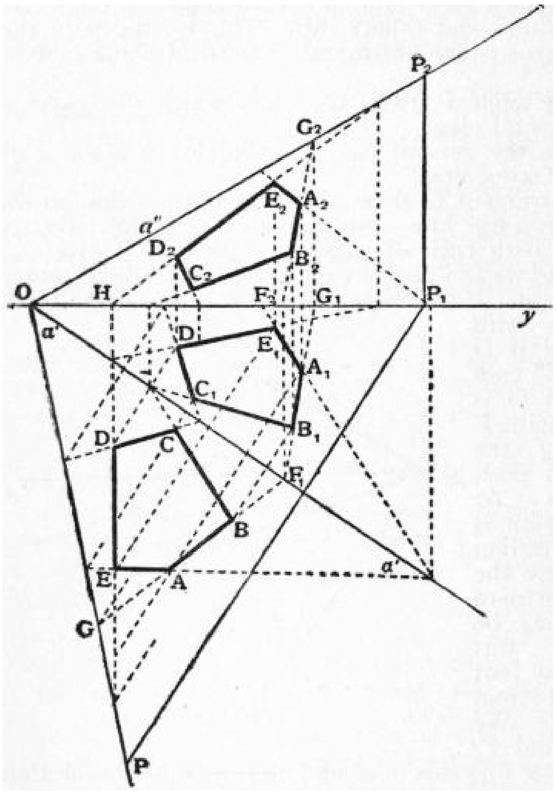


FIG. 46.

Having got the plan we easily find the elevation. The elevation of G is above G_1 in α″, and that of F is at F_2 in the axis. This gives the elevation F_2G_2 of FG and in it we get A_2B_2 in the verticals through A_1 and B_1. As a check we have OG = OG_2. Similarly the elevation of the other sides and vertices are found.

§ 17. We proceed to give some applications of the above principles to the representation of solids and of the solution of problems connected with them.

Of a pyramid are given its base, the length of the perpendicular from the vertex to the base, and the point where this perpendicular cuts the base; it is required first to develop the whole surface of the pyramid into one plane, and second to determine its section by a plane which cuts the plane of the base in a given line and makes a given angle with it.

1. As the planes of projection are not given we can take them as we like, and we select them in such a manner that the solution becomes as simple as possible. We take the plane of the base as the horizontal plane and the vertical plane perpendicular to the plane of the section. Let then (fig. 47) ABCD be the base of the pyramid, V_1 the plan of the vertex, then the elevations of A, B, C, D will be in the axis at A_2, B_2, C_2, D_2, and the vertex at some point V_2 above V_1 at a known distance from the axis. The lines V_1A, V_1B, &c., will be the plans and the lines V_2A_2, V_2B_2, &c., the elevations of the edges of the pyramid, of which thus plan and elevation are known.

We develop the surface into the plane of the base by turning each lateral face about its lower edge into the horizontal plane by the method used in § 14. If one face has been turned down, say ABV to ABP, then the point Q to which the vertex of the next face BCV comes can be got more simply by finding on the line V_1Q perpendicular to BC the point Q such that BQ = BP, for these lines represent the same edge BV of the pyramid. Next R is found by making CR = CQ, and so on till we have got the last vertex—in this case S. The fact that AS must equal AP gives a convenient check.

2. The plane α whose section we have to determine has its horizontal trace given perpendicular to the axis, and its vertical trace makes the given angle with the axis. This determines it. To find the section of the pyramid by this plane there are two methods applicable: we find the sections of the plane either with the faces or with the edges of the pyramid. We use the latter.

As the plane α is perpendicular to the vertical plane, the trace α″ contains the projection of every figure in it; the points E_2, F_2, G_2, H_2 where this trace cuts the elevations of the edges will therefore be the elevations of the points where the edges cut α. From these we find the plans E_1, F_1, G_1, H_1, and by joining them the plan of the section. If from E_1, F_1 lines be drawn perpendicular to AB, these will determine the points E, F on the developed face in which the plane α cuts it; hence also the line EF. Similarly on the other faces. Of course BF must be the same length on BP and on BQ. If the plane α be rabatted to the plan, we get the real shape of the section as shown in the figure in EFGH. This is done easily by making F_0F = OF_2, &c. If the figure representing the development of the pyramid, or better a copy of it, is cut out, and if the lateral faces be bent along the lines AB, BC, &c., we get a model of the pyramid with the section marked on its faces. This may be placed on its plan ABCD and the plane of elevation bent about the axis x. The pyramid stands then in front of its elevations. If next the plane α with a hole cut out representing the true section be bent along the trace α′

till its edge coincides with α″, the edges of the hole ought to coincide with the lines EF, FG, &c., on the faces.

§ 18. Polyhedra like the pyramid in § 17 are represented by the projections of their edges and vertices. But solids bounded by curved surfaces, or surfaces themselves, cannot be thus represented.

For a surface we may use, as in case of the plane, its traces—that is, the curves in which it cuts the planes of projection. We may also project points and curves on the surface. A ray cuts the surface generally in more than one point; hence it will happen that some of the rays touch the surface, if two of these points coincide. The points of contact of these rays will form some curve on the surface, and this will appear from the centre of projection as the boundary of the surface or of part of the surface. The outlines of all surfaces of solids which we see about us are formed by the points at which rays through our eye touch the surface. The projections of these contours are therefore best adapted to give an idea of the shape of a surface.

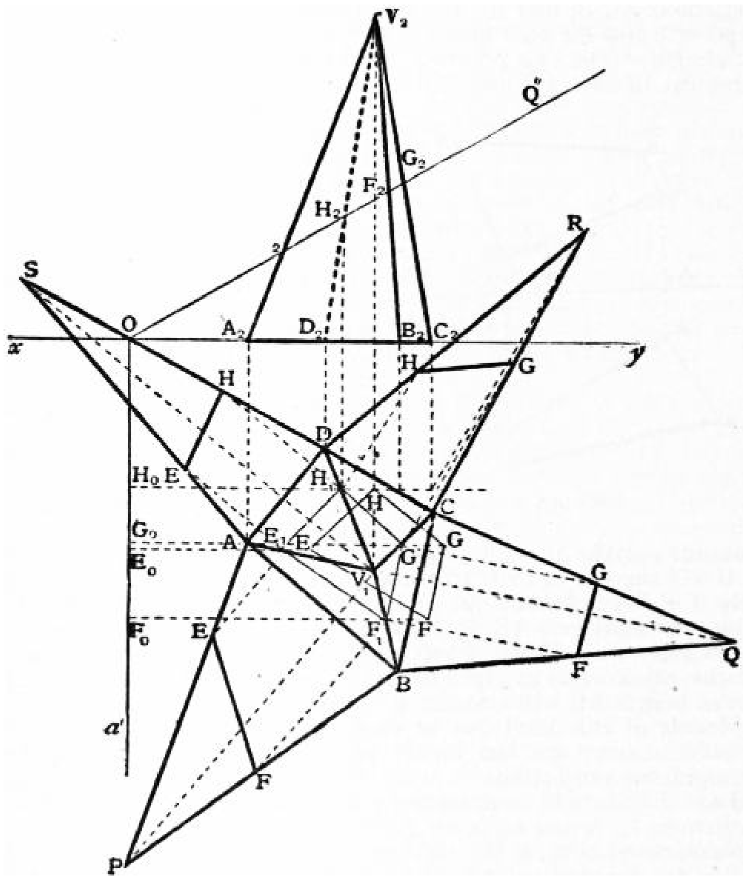


Fig. 47.

Thus the tangents drawn from any finite centre to a sphere form a right circular cone, and this will be cut by any plane in a conic. It is often called the projection of a sphere, but it is better called the contour-line of the sphere, as it is the boundary of the projections of all points on the sphere.

If the centre is at infinity the tangent cone becomes a right circular cylinder touching the sphere along a great circle, and if the projection is, as in our case, orthographic, then the section of this cone by a plane of projection will be a circle equal to the great circle of the sphere. We get such a circle in the plan and another in the elevation, their centres being plan and elevation of the centre of the sphere.

Similarly the rays touching a cone of the second order will lie in two planes which pass through the vertex of the cone, the contour-line of the projection of the cone consists therefore of two lines meeting in the projection of the vertex. These may, however, be invisible if no real tangent rays can be drawn from the centre of projection; and this happens when the ray projecting the centre of the vertex lies within the cone. In this case the traces of the cone are of importance. Thus in representing a cone of revolution with a vertical axis we get in the plan a circular trace of the surface whose centre is the plan of the vertex of the cone, and in the elevation the contour, consisting of a pair of lines intersecting in the elevation of the vertex of the cone. The circle in the plan and the pair of lines in the elevation do not determine the surface, for an infinite number of surfaces might be conceived which pass through the circular trace and touch two planes through the contour lines in the vertical plane. The surface becomes only completely defined if we write down to the figure that it shall represent a cone. The same holds for all surfaces. Even a plane is fully represented by its traces only under the silent understanding that the traces are those of a plane.

§ 19. Some of the simpler problems connected with the representation of surfaces are the determination of plane sections and of the curves of intersection of two such surfaces. The former is constantly used in nearly all problems concerning surfaces. Its solution depends of course on the nature of the surface.

To determine the curve of intersection of two surfaces, we take a plane and determine its section with each of the two surfaces, rabatting this plane if necessary. This gives two curves which lie in the same plane and whose intersections will give us points on both surfaces. It must here be remembered that two curves in space do not necessarily intersect, hence that the points in which their projections intersect are not necessarily the projections of points common to the two curves. This will, however, be the case if the two curves lie in a common plane. By taking then a number of plane sections of the surfaces we can get as many points on their curve of intersection as we like. These planes have, of course, to be selected in such a way that the sections are curves as simple as the case permits of, and such that they can be easily and accurately drawn. Thus when possible the sections should be straight lines or circles. This not only saves time in drawing but determines all points on the sections, and therefore also the points where the two curves meet, with equal accuracy.

§ 20. We give a few examples how these sections have to be selected. A cone is cut by every plane through the vertex in lines, and if it is a cone of revolution by planes perpendicular to the axis in circles.

A cylinder is cut by every plane parallel to the axis in lines, and if it is a cylinder of revolution by planes perpendicular to the axis in circles.

A sphere is cut by every plane in a circle.

Hence in case of two cones situated anywhere in space we take sections through both vertices. These will cut both cones in lines. Similarly in case of two cylinders we may take sections parallel to the axis of both. In case of a sphere and a cone of revolution with vertical axis, horizontal sections will cut both surfaces in circles whose plans are circles and whose elevations are lines, whilst vertical sections through the vertex of the cone cut the latter in lines and the sphere in circles. To avoid drawing the projections of these circles, which would in general be ellipses, we rabatt the plane and then draw the circles in their real shape. And so on in other cases.

Special attention should in all cases be paid to those points in which the tangents to the projection of the curve of intersection are parallel or perpendicular to the axis x, or where these projections touch the contour of one of the surfaces.

(O. H.)

IV. Analytical Geometry

1. In the name *geometry* there is a lasting record that the science had its origin in the knowledge that two distances may be compared by measurement, and in the idea that measurement must be effectual in the dissociation of different directions as well as in the comparison of distances in the same direction. The distance from an observer’s eye of an object seen would be specified as soon as it was ascertained that a rod, straight to the eye and of length taken as known, could be given the direction of the line of vision, and had to be moved along it a certain number of times through lengths equal to its own in order to reach the object from the eye. Moreover, if a field had for two of its boundaries lines straight to the eye, one running from south to north and the other from west to east, the position of a point in the field would be specified if the rod, when directed west, had to be shifted from the point one observed number of times westward to meet the former boundary, and also, when directed south, had to be shifted another observed number of times southward to meet the latter. Comparison by measurement, the beginning of geometry, involved counting, the basis of arithmetic; and the science of number was marked out from the first as of geometrical importance.

But the arithmetic of the ancients was inadequate as a science of number. Though a length might be recognized as known when measurement certified that it was so many times a standard length, it was not every length which could be thus specified in terms of the same standard length, even by an arithmetic enriched with the notion of fractional number. The idea of possible incommensurability of lengths was introduced into Europe by Pythagoras; and the corresponding idea of irrationality of number was absent from a crude arithmetic, while there were great practical difficulties in the way of its introduction. Hence perhaps it arose that, till comparatively modern times, appeal to arithmetical aid in geometrical reasoning was in all possible ways restrained. Geometry figured rather as the helper of the more difficult science of arithmetic.

2. It was reserved for algebra to remove the disabilities of arithmetic, and to restore the earliest ideas of the land-measurer to the position of controlling ideas in geometrical investigation. This unified science of pure number made comparatively little headway in the hands of the ancients, but began to receive due attention shortly after the revival of learning. It expresses whole classes of arithmetical facts in single statements, gives to arithmetical laws the form of equations involving symbols which may mean any known or sought numbers, and provides processes which enable us to analyse the information given by an equation and derive from that equation other equations, which express laws that are in effect consequences or causes of a law started from, but differ greatly from it in form. Above all, for present purposes, it deals not only with integral and fractional number, but with number regarded as capable of continuous growth, just as distance is capable of continuous growth. The difficulty of the arithmetical expression of irrational number, a difficulty considered by the modern school of analysts to have been at length surmounted (see FUNCTION), is not vital to it. It can call the ratio of the diagonal of a square to a side, for instance, or that of the circumference of a circle to a diameter, a number, and let a or x denote that number, just as properly as it may allow either letter to denote any rational number which may be greater or less than the ratio in question by a difference less than any minute one we choose to assign.

Counting only, and not the counting of objects, is of the essence of arithmetic, and of algebra. But it is lawful to count objects, and in particular to count equal lengths by measure. The widened idea is that even when a or x is an irrational number we may speak of a or x unit lengths by measure. We may give concrete interpretation to an algebraical equation by allowing its terms all to mean numbers of times the same unit length, or the same unit area, or &c. and in any equation lawfully derived from the first by algebraical processes we may do the same. Descartes in his *Géométrie* (1637) was the first to systematize the application of this principle to the inherent first notions of geometry; and the methods which he instituted have become the most potent methods of all in geometrical research. It is hardly too much to say that, when known facts as to a geometrical figure have once been expressed in algebraical terms, all strictly consequential facts as to the figure can be deduced by almost mechanical processes. Some may well be unexpected consequences; and in obtaining those of which there has been suggestion beforehand the often bewildering labour of constant attention to the figure is obviated. These are the methods of what is now called *analytical*, or sometimes *algebraical*, *geometry*.

3. The modern use of the term “analytical” in geometry has obscured, but not made obsolete, an earlier use, one as old as Plato. There is nothing algebraical in this analysis, as distinguished from synthesis, of the Greeks, and of the expositors of pure geometry. It has reference to an order of ideas in demonstration, or, more frequently, in discovering means to effect the geometrical construction of a figure with an assigned special property. We have to suppose hypothetically that the construction has been performed, drawing a rough figure which exhibits it as nearly as is practicable. We then analyse or critically examine the figure, treated as correct, and ascertain other properties which it can only possess in association with the one in question. Presently one of these properties will often be found which is of such a character that the construction of a figure possessing it is simple. The means of effecting synthetically a construction such as was desired is thus brought to light by what Plato called *analysis*. Or again, being asked to prove a theorem A, we ascertain that it must be true if another theorem B is, that B must be if C is, and so on, thus eventually finding that the theorem A is the consequence, through a chain of intermediaries, of a theorem Z of which the establishment is easy. This geometrical analysis is not the subject of the present article; but in the reasoning from form to form of an equation or system of equations, with 712
the object of basing the algebraical proof of a geometrical fact on other facts of a more obvious character, the same logic is utilized, and the name “analytical geometry” is thus in part explained.

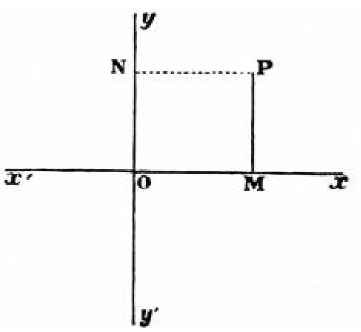
4. In algebra real positive number was alone at first dealt with, and in geometry actual signless distance. But in algebra it became of importance to say that every equation of the first degree has a root, and the notion of negative number was introduced. The negative unit had to be defined as what can be added to the positive unit and produce the sum zero. The corresponding notion was readily at hand in geometry, where it was clear that a unit distance can be measured to the left or down from the farther end of a unit distance already

measured to the right or up from a point O, with the result of reaching O again. Thus, to give full interpretation in geometry to the algebraically negative, it was only necessary to associate distinctness of sign with oppositeness of direction. Later it was discovered that algebraical reasoning would be much facilitated, and that conclusions as to the real would retain all their soundness, if a pair of imaginary units $\pm\sqrt{-1}$ of what might be called number were allowed to be contemplated, the pair being defined, though not separately, by the two properties of having the real sum 0 and the real product 1. Only in these two real combinations do they enter in conclusions as to the real. An advantage gained was that every quadratic equation, and not some quadratics only, could be spoken of as having two roots. These admissions of new units into algebra were final, as it admitted of proof that all equations of degrees higher than two have the full numbers of roots possible for their respective degrees in any case, and that every root has a value included in the form $a + b\sqrt{-1}$, with a, b, real. The corresponding enrichment could be given to geometry, with corresponding advantages and the same absence of danger, and this was done. On a line of measurement of distance we contemplate as existing, not only an infinite continuum of points at real distances from an origin of measurement O, but a doubly infinite continuum of points, all but the singly infinite continuum of real ones imaginary, and imaginary in conjugate pairs, a conjugate pair being at imaginary distances from O, which have a real arithmetic and a real geometric mean. To geometry enriched with this conception all algebra has its application.

5. Actual geometry is one, two or three-dimensional, *i.e.* lineal, plane or solid. In one-dimensional geometry positions and measurements in a single line only are admitted. Now descriptive constructions for points in a line are impossible without going out of the line. It has therefore been held that there is a sense in which no science of geometry strictly confined to one dimension exists. But an algebra of one variable can be applied to the study of distances along a line measured from a chosen point on it, so that the idea of construction as distinct from measurement is not essential to a one-dimensional geometry aided by algebra. In geometry of two dimensions, the flat of the land-measurer, the passage from one point O to any other point, can be effected by two successive marches, one east or west and one north or south, and, as will be seen, an algebra of two variables suffices for geometrical exploitation. In geometry of three dimensions, that of space, any point can be reached from a chosen one by three marches, one east or west, one north or south, and one up or down; and we shall see that an algebra of three variables is all that is necessary. With three dimensions actual geometry stops; but algebra can supply any number of variables. Four or more variables have been used in ways analogous to those in which one, two and three variables are used for the purposes of one, two and three-dimensional geometry, and the results have been expressed in quasi-geometrical language on the supposition that a higher space can be conceived of, though not realized, in which four independent directions exist, such that no succession of marches along three of them can effect the same displacement of a point as a march along the fourth; and similarly for higher numbers than four. Thus analytical, though not actual, geometries exist for four and more dimensions. They are in fact algebras furnished with nomenclature of a geometrical cast, suggested by convenient forms of expression which actual geometry has, in return for benefits received, conferred on algebras of one, two and three variables.

We will confine ourselves to the dimensions of actual geometry, and will devote no space to the one-dimensional, except incidentally as existing within the two-dimensional. The analytical method will now be explained for the cases of two and three dimensions in succession. The form of it originated by Descartes, and thence known as Cartesian, will alone be considered in much detail.

I. *Plane Analytical Geometry.*



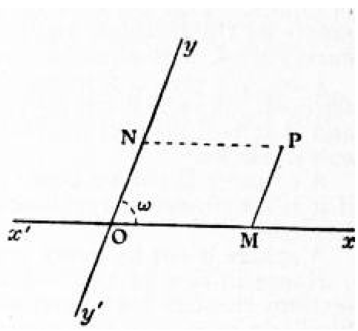


FIG. 48. FIG. 49.

6. *Coordinates.*—It is assumed that the points, lines and figures considered lie in one and the same plane, which plane therefore need not be in any way referred to. In the plane a point O, and two lines x′Ox, y′Oy, intersecting in O, are taken once for all, and regarded as fixed. O is called the origin, and x′Ox, y′Oy the axes of x and y respectively. Other positions in the plane are specified in relation to this fixed origin and these fixed axes. From any point P we suppose PM drawn parallel to the axis of y to meet the axis of x in M, and may also suppose PN drawn parallel to the axis of x to meet the axis of y in N, so that OMPN is a parallelogram. The position of P is determined when we know OM (= NP) and MP (= ON). If OM is x times the unit of a scale of measurement chosen at pleasure, and MP is y times the unit, so that x and y have numerical values, we call x and y the (Cartesian) coordinates of P. To distinguish them we often speak of y as the ordinate, and of x as the abscissa.

It is necessary to attend to signs; x has one sign or the other according as the point P is on one side or the other of the axis of y, and y one sign or the other according as P is on one side or the other of the axis of x. Using the letters N, E, S, W, as in a map, and considering the plane as divided into four quadrants by the axes, the signs are usually taken to be:

x	y	For quadrant
+	+	N E
+	−	S E
−	+	N W
−	−	S W

A point is referred to as the point (a, b), when its coordinates are x = a, y = b. A point may be fixed, or it may be variable, *i.e.* be regarded for the time being as free to move in the plane. The coordinates (x, y) of a variable point are algebraic variables, and are said to be "current coordinates."

The axes of x and y are usually (as in fig. 48) taken at right angles to one another, and we then speak of them as rectangular axes, and of x and y as "rectangular coordinates" of a point P; OMPN is then a rectangle. Sometimes, however, it is convenient to use axes which are oblique to one another, so that (as in fig. 49) the angle xOy between their positive directions is some known angle ω distinct from a right angle, and OMPN is always an oblique parallelogram with given angles; and we then speak of x and y as "oblique coordinates." The coordinates are as a rule taken to be rectangular in what follows.

7. *Equations and loci.* If (x, y) is the point P, and if we are given that x = 0, we are told that, in fig. 48 or fig. 49, the point M lies at O, whatever value y may have, *i.e.* we are told the one fact that P lies on the axis of y. Conversely, if P lies anywhere on the axis of y, we have always OM = 0, *i.e.* x = 0. Thus the equation x = 0 is one satisfied by the coordinates (x, y) of every point in the axis of y, and not by those of any other point. We say that x = 0 is the equation of the axis of y, and that the axis of y is the locus represented by the equation x = 0. Similarly y = 0 is the equation of the axis of x. An equation x = a, where a is a constant, expresses that P lies on a parallel to the axis of y through a point M on the axis of x such that OM = a. Every line parallel to the axis of y has an equation of this form. Similarly, every line parallel to the axis of x has an equation of the form y = b, where b is some definite constant.

These are simple cases of the fact that a single equation in the current coordinates of a variable point (x, y) imposes one limitation on the freedom of that point to vary. The coordinates of a point taken at random in the plane will, as a rule, not satisfy the equation, 713
but infinitely many points, and in most cases infinitely many real ones, have coordinates which do satisfy it, and these points are exactly those which lie upon some locus of one dimension, a straight line or more frequently a curve, which is said to be represented by the equation. Take, for instance, the equation y = mx, where m is a given constant. It is satisfied by the coordinates of every point P, which is such that, in fig. 48, the distance MP, with its proper sign, is m times the distance OM, with its proper sign, *i.e.* by the coordinates of every point in the straight line through O which we arrive at by making a line, originally coincident with x′Ox, revolve about O in the direction opposite to that of the hands of a watch through an angle of which m is the tangent, and by those of no other points. That line is the locus which it represents. Take, more generally, the equation y = φ(x), where φ(x) is any given non-ambiguous function of x. Choosing any point M on x′Ox in fig. 1, and giving to x the value of the numerical measure of OM, the equation determines a single corresponding y, and so determines a single point P on the line through M parallel to y′Oy. This is one point whose coordinates satisfy the equation. Now let M move from the extreme left to the extreme right of the line x′Ox, regarded as extended both ways as far as we like, *i.e.* let x take all real values from −∞ to ∞. With every value goes a point P, as above, on the parallel

to $y'Oy$ through the corresponding M; and we thus find that there is a path from the extreme left to the extreme right of the figure, all points P along which are distinguished from other points by the exceptional property of satisfying the equation by their coordinates. This path is a locus; and the equation $y = \varphi(x)$ represents it. More generally still, take an equation $f(x, y) = 0$ which involves both x and y under a functional form. Any particular value given to x in it produces from it an equation for the determination of a value or values of y, which go with that value of x in specifying a point or points (x, y), of which the coordinates satisfy the equation $f(x, y) = 0$. Here again, as x takes all values, the point or points describe a path or paths, which constitute a locus represented by the equation. Except when y enters to the first degree only in $f(x, y)$, it is not to be expected that all the values of y, determined as going with a chosen value of x, will be necessarily real; indeed it is not uncommon for all to be imaginary for some ranges of values of x. The locus may largely consist of continua of imaginary points; but the real parts of it constitute a real curve or real curves. Note that we have to allow x to admit of all imaginary, as well as of all real, values, in order to obtain all imaginary parts of the locus.

A locus or curve may be algebraically specified in another way; viz. we may be given two equations $x = f(\theta)$, $y = F(\theta)$, which express the coordinates of any point of it as two functions of the same variable parameter θ to which all values are open. As θ takes all values in turn, the point (x, y) traverses the curve.

It is a good exercise to trace a number of curves, taken as defined by the equations which represent them. This, in simple cases, can be done approximately by plotting the values of y given by the equation of a curve as going with a considerable number of values of x, and connecting the various points (x, y) thus obtained. But methods exist for diminishing the labour of this tentative process.

Another problem, which will be more attended to here, is that of determining the equations of curves of known interest, taken as defined by geometrical properties. It is not a matter for surprise that the curves which have been most and longest studied geometrically are among those represented by equations of the simplest character.

8. *The Straight Line.*—This is the simplest type of locus. Also the simplest type of equation in x and y is $Ax + By + C = 0$, one of the first degree. Here the coefficients A, B, C are constants. They are, like the current coordinates, x, y, numerical. But, in giving interpretation to such an equation, we must of course refer to numbers Ax, By, C of unit magnitudes of the same kind, of units of counting for instance, or unit lengths or unit squares. It will now be seen that every straight line has an equation of the first degree, and that every equation of the first degree represents a straight line.

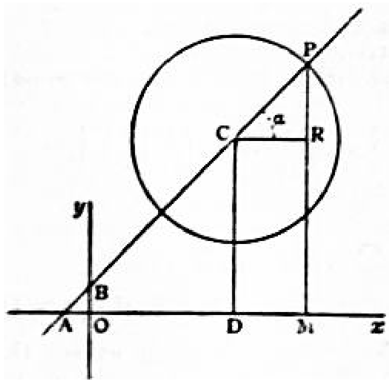


FIG. 50.

It has been seen (§ 7) that lines parallel to the axes have equations of the first degree, free from one of the variables. Take now a straight line ABC inclined to both axes. Let it make a given angle α with the positive direction of the axis of x, *i.e.* in fig. 50 let this be the angle through which Ax must be revolved counter-clockwise about A in order to be made coincident with the line. Let C, of coordinates (h, k), be a fixed point on the line, and P(x, y) any other point upon it. Draw the ordinates CD, PM of C and P, and let the parallel to the axis of x through C meet PM, produced if necessary, in R. The right-angled triangle CRP tells us that, with the signs appropriate to their directions attached to CR and RP,

$$RP = CR \tan \alpha, \textit{ i.e. } MP - DC = (OM - OD) \tan \alpha,$$

and this gives that

$$y - k = \tan \alpha \, (x - h),$$

an equation of the first degree satisfied by x and y. No point not on the line satisfies the same equation; for the line from C to any point off the line would make with CR some angle β different from α, and the point in question would satisfy an equation $y - k = \tan \beta(x - h)$, which is inconsistent with the above equation.

The equation of the line may also be written $y = mx + b$, where $m = \tan \alpha$, and $b = k - h \tan \alpha$. Here b is the value obtained for y from the equation when 0 is put for x, *i.e.* it is the numerical measure, with proper sign, of OB, the intercept made by the line on the axis of y, measured from the origin. For different straight lines, m and b may have any constant values we like.

Now the general equation of the first degree $Ax + By + C = 0$ may be written $y = -(A/B)x - C/B$, unless $B = 0$, in which case it represents a line parallel to the axis of y; and $-A/B$, $-C/B$ are values which can be given to m and b, so that every equation of the first degree represents a straight line. It is important to notice that the general equation, which in appearance contains three constants A, B, C, in effect depends on two only, the ratios of two of them to the third. In virtue of this last remark, we see that two distinct conditions suffice to determine a straight line. For instance, it is easy from the above to see that

$$\frac{x}{a} + \frac{y}{b} = 1$$

is the equation of a straight line determined by the two conditions that it makes intercepts OA, OB on the two axes, of which a and b are the numerical measures with proper signs: note that in fig. 50 a is negative. Again,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

i.e.

$$(y_1 - y_2)\,x - (x_1 - x_2)\,y + x_1y_2 - x_2y_1 = 0,$$

represents the line determined by the data that it passes through two given points (x_1, y_1) and (x_2, y_2). To prove this find m in the equation $y - y_1 = m(x - x_1)$ of a line through (x_1, y_1), from the condition that (x_2, y_2) lies on the line.

In this paragraph the coordinates have been assumed rectangular. Had they been oblique, the doctrine of similar triangles would have given the same results, except that in the forms of equation $y - k = m(x - h)$, $y = mx + b$, we should not have had $m = \tan\alpha$.

9. *The Circle.*—It is easy to write down the equation of a given circle. Let (h, k) be its given centre C, and ρ the numerical measure of its given radius. Take P (x, y) any point on its circumference, and construct the triangle CRP, in fig. 50 as above. The fact that this is right-angled tells us that

$$CR^2 + RP^2 = CP^2,$$

and this at once gives the equation

$$(x - h)^2 + (y - k)^2 = \rho^2.$$

A point not upon the circumference of the particular circle is at some distance from (h, k) different from ρ, and satisfies an equation inconsistent with this one; which accordingly represents the circumference, or, as we say, the circle.

The equation is of the form

$$x^2 + y^2 + 2Ax + 2By + C = 0.$$

Conversely every equation of this form represents a circle: we have only to take $-A$, $-B$, $A^2 + B^2 - C$ for h, k, ρ^2 respectively, to obtain its centre and radius. But this statement must appear too unrestricted. Ought we not to require $A^2 + B^2 - C$ to be positive? Certainly, if by circle we are only to mean the visible round circumference of the geometrical definition. Yet, analytically, we contemplate altogether imaginary circles, for which ρ^2 is negative, and circles, for which $\rho = 0$, with all their reality condensed into their centres. Even when ρ^2 is positive, so that a visible round circumference exists, we do not regard this as constituting the whole of the circle. Giving to x any value whatever in $(x - h)^2 + (y - k)^2 = \rho^2$, we obtain two values of y, real, coincident or imaginary, each of which goes with the abscissa x as the ordinate of a point, real or imaginary, on what is represented by the equation of the circle.

The doctrine of the imaginary on a circle, and in geometry generally, is of purely algebraical inception; but it has been in its entirety accepted by modern pure geometers, and signal success has attended the efforts of those who, like K.G.C. von Staudt, have striven to base its conclusions on principles not at all algebraical in form, though of course cognate to those adopted in introducing the imaginary into algebra.

A circle with its centre at the origin has an equation $x^2 + y^2 = \rho^2$.

In oblique coordinates the general equation of a circle is $x^2 + 2xy\cos\omega + y^2 + 2Ax + 2By + C = 0$.

10. The conic sections are the next simplest loci; and it will be seen later that they are the loci represented by equations of the second degree. Circles are particular cases of conic sections; and they have just been seen to have for their equations a particular class of equations of the second degree. Another particular class of such equations is that included in the form $(Ax + By + C)(A'x + B'y + C') = 0$, which represents two straight lines, because the

product on the left vanishes if, and only if, one of the two factors does, *i.e.* if, and only if, (x, y) lies on one or other of two straight lines. The condition that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, which is often written $(a, b, c, f, g, h)(x, y, I)^2 = 0$, takes this form is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. Note that the two lines may, in particular cases, be parallel or coincident.

Any equation like $F_1(x, y)\ F_2(x, y) \dots F_n(x, y) = 0$, of which the left-hand side breaks up into factors, represents all the loci separately represented by $F_1(x, y) = 0$, $F_2(x, y) = 0$, ... $F_n(x, y) = 0$. In particular an equation of degree n which is free from x represents n straight lines parallel to the axis of x, and one of degree n which is homogeneous in x and y, *i.e.* one which upon division by x^n, becomes an equation in the ratio y/x, represents n straight lines through the origin.

Curves represented by equations of the third degree are called cubic curves. The general equation of this degree will be written $(*)(x, y, I)^3 = 0$.

11. *Descriptive Geometry.*—A geometrical proposition is either descriptive or metrical: in the former case the statement of it is independent of the idea of magnitude (length, inclination, &c.), and in the latter it has reference to this idea. The method of coordinates seems to be by its inception essentially metrical. Yet in dealing by this method with descriptive propositions we are eminently free from metrical considerations, because of our power to use general equations, and to avoid all assumption that measurements implied are any particular measurements.

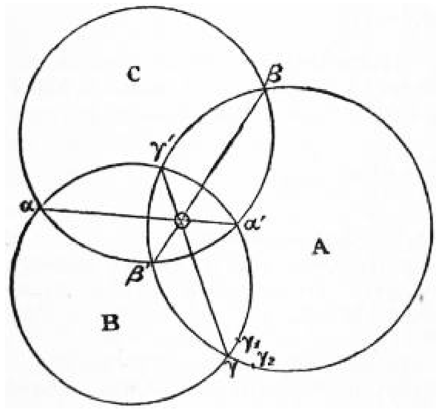


FIG. 51.

12. It is worth while to illustrate this by the instance of the well-known theorem of the radical centre of three circles. The theorem is that, given any three circles A, B, C (fig. 51), the common chords $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ of the three pairs of circles meet in a point.

The geometrical proof is metrical throughout:—

Take O the point of intersection of $\alpha\alpha'$, $\beta\beta'$, and joining this with γ', suppose that $\gamma'O$ does not pass through γ, but that it meets the circles A, B in two distinct points γ_2, γ_1 respectively. We have then the known metrical property of intersecting chords of a circle; viz. in circle C, where $\alpha\alpha'$, $\beta\beta'$, are chords meeting at a point O,

$$O\alpha \cdot O\alpha' = O\beta \cdot O\beta',$$

where, as well as in what immediately follows, $O\alpha$, &c. denote, of course, *lengths* or *distances*.

Similarly in circle A,

$$O\beta \cdot O\beta' = O\gamma_2 \cdot O\gamma',$$

and in circle B,

$$O\alpha \cdot O\alpha' = O\gamma_1 \cdot O\gamma'.$$

Consequently $O\gamma_1 \cdot O\gamma' = O\gamma_2 \cdot O\gamma'$, that is, $O\gamma_1 = O\gamma_2$, or the points γ_1 and γ_2 coincide; that is, they each coincide with γ.

We contrast this with the analytical method:—

Here it only requires to be known that an equation $Ax + By + C = 0$ represents a line, and an equation $x^2 + y^2 + Ax + By + C = 0$ represents a circle. A, B, C have, in the two cases respectively, metrical significations; but these we are not concerned with. Using S to denote the function $x^2 + y^2 + Ax + By + C$, the equation of a circle is $S = o$. Let the equation of any other circle be $S', = x^2 + y^2 + A'x + B'y + C' = 0$; the equation $S\text{-}S' = 0$ is a linear equation ($S - S'$ is in fact $= (A - A')x + (B - B')y + C\text{-}C)$, and it thus represents a line; this equation is satisfied by the coordinates of each of the points of intersection of the two circles (for at each of these points $S = 0$ and $S' = 0$, therefore also $S - S' = 0$); hence the equation $S - S' = 0$ is that of the line joining the two points of intersection of the two circles, or say it is the equation of the common chord of the two circles. Considering then a third circle $S'', = x^2 + y^2 + A''x + B''y + C'' = 0$, the equations of the common chords are $S - S' = 0$, $S - S'' = 0$, $S' - S'' = 0$ (each of these a linear equation); at the intersection of the first and second of these lines $S = S'$ and $S = S''$, therefore also $S' = S''$, or the equation of the third line is satisfied by the coordinates of the point in question; that is, the three chords intersect in a point O, the coordinates of which are determined by the equations $S = S' = S''$.

It further appears that if the two circles $S = 0$, $S' = 0$ do not intersect in any real points, they must be regarded as intersecting in two imaginary points, such that the line joining them is the real line represented by the equation $S - S' = 0$; or that two circles, whether their intersections be real or imaginary, have always a real common chord (or radical axis), and that for *any* three circles the common chords intersect in a point (of course real) which is the radical centre. And by this very theorem, given two circles with imaginary intersections, we can, by drawing circles which meet each of them in real points, construct the radical axis of the first-mentioned two circles.

13. The principle employed in showing that the equation of the common chord of two circles is $S - S' = 0$ is one of very extensive application, and some more illustrations of it may be given.

Suppose $S = 0$, $S' = 0$ are lines (that is, let S, S' now denote linear functions $Ax + By + C$, $A'x + B'y + C'$), then $S - kS' = 0$ (k an arbitrary constant) is the equation of any line passing through the point of intersection of the two given lines. Such a line may be made to pass through any given point, say the point (x_0, y_0); if S_0, S'_0 are what S, S' respectively become on writing for (x, y) the values (x_0, y_0), then the value of k is $k = S_0 \div S'_0$. The equation in fact is $SS'_0 - S_0S' = 0$; and starting from this equation we at once verify it *a posteriori*; the equation is a linear equation satisfied by the values of (x, y) which make $S = 0$, $S' = 0$; and satisfied also by the values (x_0, y_0); and it is thus the equation of the line in question.

If, as before, $S = 0$, $S' = 0$ represent circles, then (k being arbitrary) $S - kS' = 0$ is the equation of any circle passing through the two points of intersection of the two circles; and to make this pass through a given point (x_0, y_0) we have again $k = S_0 \div S'_0$. In the particular case $k = 1$, the circle becomes the common chord (more accurately it becomes the common chord together with the line infinity; see § 23 below).

If S denote the general quadric function,

$$S = ax^2 + 2hxy + by^2 + 2fy + 2gx + c,$$

then the equation $S = 0$ represents a conic; assuming this, then, if $S' = 0$ represents another conic, the equation $S - kS' = 0$ represents *any* conic through the four points of intersection of the two conics.

14. The object still being to illustrate the mode of working with coordinates for descriptive purposes, we consider the theorem of the polar of a point in regard to a circle. Given a circle and a point O (fig. 52), we draw through O any two lines meeting the circle in the points A, A′ and B, B′ respectively, and then taking Q as the intersection of the lines AB′ and A′B, the theorem is that the locus of the point Q is a right line depending only upon O and the circle, but independent of the particular lines OAA′ and OBB′.

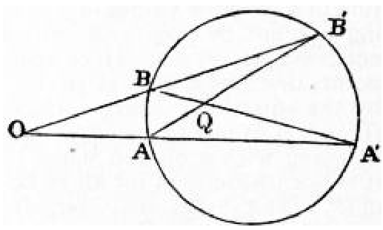


FIG. 52.

Taking O as the origin, and for the axes any two lines through O at right angles to each other, the equation of the circle will be

$$x^2 + y^2 + 2Ax + 2By + C = 0;$$

and if the equation of the line OAA′ is taken to be $y = mx$, then the points A, A′ are found as the intersections of the straight line with the circle; or to determine x we have

$$x^2 (1 + m^2) + 2x (A + Bm) + C = 0.$$

If(x_1, y_1) are the coordinates of A, and (x_2, y_2) of A′, then the roots of this equation are x_1, x_2, whence easily

$$\frac{1}{x_1} + \frac{1}{x_2} = -2\,\frac{A + Bm}{C}\,.$$

And similarly, if the equation of the line OBB′ is taken to be $y = m'x_1$ and the coordinates of B, B′ to be (x_3, y_3) and (x_4, y_4) respectively, then

$$\frac{1}{x_3} + \frac{1}{x_4} = -2\,\frac{A + Bm'}{C'}\,.$$

We have then by § 8

$$x (y_1 - y_4) - y (x_1 - x_4) + x_1y_4 - x_4y_1 = 0,$$
$$x (y_2 - y_3) - y (x_2 - x_3) + x_2y_3 - x_3y_2 = 0,$$

as the equations of the lines AB′ and A′B respectively. Reducing by means of the relations $y_1 - mx_1 = 0$, $y_2 - mx_2 = 0$, $y_3 - m'x_3 = 0$, $y_4 - m'x_4 = 0$, the two equations become

$$x\,(mx_1 - m'x_4) - y\,(x_1 - x_4) + (m' - m)\,x_1x_4 = 0,$$
$$x\,(mx_2 - m'x_3) - y\,(x_2 - x_3) + (m' - m)\,x_2x_3 = 0,$$

and if we divide the first of these equations by x_1x_4, and the second by x_2x_3 and then add, we obtain

$$x\left\{ m\left(\frac{1}{x_3}+\frac{1}{x_4}\right) - m'\left(\frac{1}{x_1}+\frac{1}{x_2}\right) \right\} - y\left\{ \frac{1}{x_3}+\frac{1}{x_4} - \left(\frac{1}{x_1}+\frac{1}{x_2}\right) \right\} + 2m' - 2m = 0,$$

or, what is the same thing,

$$\left(\frac{1}{x_1}+\frac{1}{x_2}\right)(y - m'x) - \left(\frac{1}{x_3}+\frac{1}{x_4}\right)(y - mx) + 2m' - 2m = 0,$$

which by what precedes is the equation of a line through the point Q. Substituting herein for $1/x_1 + 1/x_2$, $1/x_3 + 1/x_4$ their foregoing values, the equation becomes

$$-(A + Bm)\,(y - m'x) + (A + Bm')\,(y - mx) + C\,(m' - m) = 0;$$

that is,

$$(m - m')\,(Ax + By + C) = 0;$$

or finally it is $Ax + By + C = 0$, showing that the point Q lies in a line the position of which is 715
independent of the particular lines OAA′, OBB′ used in the construction. It is proper to notice that there is no correspondence to each other of the points A, A′ and B, B′; the grouping might as well have been A, A′ and B′, B; and it thence appears that the line $Ax + By + C = 0$ just obtained is in fact the line joining the point Q with the point R which is the intersection of AB and A′B′.

15. In § 8 it has been seen that two conditions determine the equation of a straight line, because in $Ax + By + C = 0$ one of the coefficients may be divided out, leaving only two parameters to be determined. Similarly five conditions instead of six determine an equation of the second degree $(a, b, c, f, g, h)(x, y, 1)^2 = 0$, and nine instead of ten determine a cubic $(*)(x, y, 1)^3 = 0$. It thus appears that a cubic can be made to pass through 9 given points, and that the cubic so passing through 9 given points is completely determined. There is, however, a remarkable exception. Considering two given cubic curves $S = 0$, $S' = 0$, these intersect in 9 points, and through these 9 points we have the whole series of cubics $S - kS' = 0$, where k is an arbitrary constant: k may be determined so that the cubic shall pass through a given tenth point ($k = S_0 \div S'_0$, if the coordinates are (x_0, y_0), and S_0, S'_0 denote the corresponding values of S, S′). The resulting curve $SS'_0 - S'S_0 = 0$ may be regarded as the cubic determined by the conditions of passing through 8 of the 9 points and through the given point (x_0, y_0); and from the equation it thence appears that the curve passes through the remaining one of the 9 points. In other words, we thus have the theorem, any cubic curve which passes through 8 of the 9 intersections of two given cubic curves passes through the 9th intersection.

The applications of this theorem are very numerous; for instance, we derive from it Pascal's theorem of the inscribed hexagon. Consider a hexagon inscribed in a conic. The three alternate sides constitute a cubic, and the other three alternate sides another cubic. The cubics intersect in 9 points, being the 6 vertices of the hexagon, and the 3 Pascalian points, or intersections of the pairs of opposite sides of the hexagon. Drawing a line through two of the Pascalian points, the conic and this line constitute a cubic passing through 8 of the 9 points of intersection, and it therefore passes through the remaining point of intersection—that is, the third Pascalian point; and since obviously this does not lie on the conic, it must lie on the line—that is, we have the theorem that the three Pascalian points (or points of intersection of the pairs of opposite sides) lie on a line.

16. *Metrical Theory resumed. Projections and Perpendiculars.*—It is a metrical fact of fundamental importance, already used in § 8, that, if a finite line PQ be projected on any other line OO′ by perpendiculars PP′, QQ′ to OO′, the length of the projection P′Q′ is equal to that of PQ multiplied by the cosine of the acute angle between the two lines. Also the algebraical sum of the projections of the sides of any closed polygon upon any line is zero, because as a point goes round the polygon, from any vertex A to A again, the point which is its projection on the line passes from A′ the projection of A to A′ again, *i.e.* traverses equal distances along the line in positive and negative senses. If we consider the polygon as consisting of two broken lines, each extending from the same initial to the same terminal point, the sum of the projections of the lines which compose the one is equal, in sign and

magnitude, to the sum of the projections of the lines composing the other. Observe that the projection on a line of a length perpendicular to the line is zero.

Let us hence find the equation of a straight line such that the perpendicular OD on it from the origin is of length ρ taken as positive, and is inclined to the axis of x at an angle xOD = α, measured counter-clockwise from Ox. Take any point P(x, y) on the line, and construct OM and MP as in fig. 48. The sum of the projections of OM and MP on OD is OD itself; and this gives the equation of the line

$$x \cos \alpha + y \sin \alpha = \rho.$$

Observe that cos α and sin α here are the sin α and $-\cos \alpha$, or the $-\sin \alpha$ and cos α of § 8 according to circumstances.

We can write down an expression for the perpendicular distance from this line of any point (x', y') which does not lie upon it. If the parallel through (x', y') to the line meet OD in E, we have $x' \cos \alpha + y' \sin \alpha = OE$, and the perpendicular distance required is OD − OE, *i.e.* $\rho - x' \cos \alpha - y' \sin \alpha$; it is the perpendicular distance taken positively or negatively according as (x', y') lies on the same side of the line as the origin or not.

The general equation $Ax + By + C = 0$ may be given the form $x \cos \alpha + y \sin \alpha - \rho = 0$ by dividing it by $\sqrt{(A^2 + B^3)}$. Thus $(Ax' + By' + C) \div \sqrt{(A^2 + B^2)}$ is in absolute value the perpendicular distance of (x', y') from the line $Ax + By + C = 0$. Remember, however, that there is an essential ambiguity of sign attached to a square root. The expression found gives the distance taken positively when (x', y') is on the origin side of the line, if the sign of C is given to $\sqrt{(A^2 + B^2)}$.

17. *Transformation of Coordinates.*—We often need to adopt new axes of reference in place of old ones; and the above principle of projections readily expresses the old coordinates of any point in terms of the new.

Suppose, for instance, that we want to take for new origin the point O′ of old coordinates OA = h, AO′ = k, and for new axes of X and Y lines through O′ obtained by rotating parallels to the old axes of x and y through an angle θ counter-clockwise. Construct (fig. 53) the old and new coordinates of any point P. Expressing that the projections, first on the old axis of x and secondly on the old axis of y, of OP are equal to the sums of the projections, on those axes respectively, of the parts of the broken line OO′M′P, we obtain:

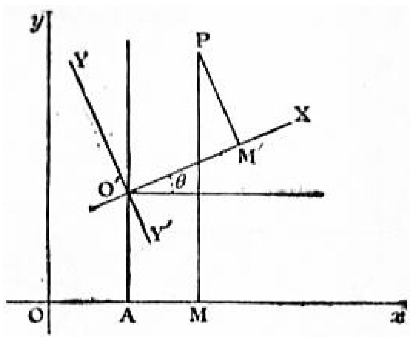


Fig. 53.

$$x = h + X \cos \theta + Y \cos (\theta + \tfrac{1}{2}\pi) = h + X \cos \theta - Y \sin \theta,$$

and

$$y = k + X \cos (\tfrac{1}{2}\pi - \theta) + Y \cos \theta = k + X \sin \theta + Y \cos \theta.$$

Be careful to observe that these formulae do not apply to every conceivable change of reference from one set of rectangular axes to another. It might have been required to take O′X, O′Y′ for the positive directions of the new axes, so that the change of directions of the axes could not be effected by rotation. We must then write −Y for Y in the above.

Were the new axes oblique, making angles α, β respectively with the old axis of x, and so inclined at the angle $\beta - \alpha$, the same method would give the formulae

$$x = h + X \cos \alpha + Y \cos \beta, \quad y = k + X \sin \alpha + Y \sin \beta.$$

18. *The Conic Sections.*—The conics, as they are now called, were at first defined as curves of intersection of planes and a cone; but Apollonius substituted a definition free from reference to space of three dimensions. This, in effect, is that a conic is the locus of a point the distance of which from a given point, called the focus, has a given ratio to its distance from a given line, called the directrix (see Conic Section). If e : 1 is the ratio, e is called the eccentricity. The distances are considered signless.

Take (h, k) for the focus, and $x \cos \alpha + y \sin \alpha - p = 0$ for the directrix. The absolute values of $\sqrt{\{(x - h)^2 + (y - k)^2\}}$ and $p - x \cos \alpha - y \sin \alpha$ are to have the ratio e : 1; and this gives

$$(x - h)^2 + (y - k)^2 = e^2 (p - x \cos \alpha - y \sin \alpha)^2$$

as the general equation, in rectangular coordinates, of a conic.

It is of the second degree, and is the general equation of that degree. If, in fact, we multiply it by an unknown λ, we can, by solving six simultaneous equations in the six unknowns λ, h, k, e, p, α, so choose values for these as to make the coefficients in the equation equal to those in any equation of the second degree which may be given. There is no failure of this statement in the special case when the given equation represents two straight lines, as in § 10, but there is speciality: if the two lines intersect, the intersection and either bisector of the angle between them are a focus and directrix; if they are united in one line, any point on the line and a perpendicular to it through the point are: if they are parallel, the case is a limiting one in which e and $h^2 + k^2$ have become infinite while $e^{-2}(h^2 + k^2)$ remains finite. In the case (§ 9) of an equation such as represents a circle there is another instance of proceeding to a limit: e has to become 0, while ep remains finite: moreover α is indeterminate. The centre of a circle is its focus, and its directrix has gone to infinity, having no special direction. This last fact illustrates the necessity, which is also forced on plane geometry by three-dimensional considerations, of treating all points at infinity in a plane as lying on a single straight line.

Sometimes, in reducing an equation to the above focus and directrix form, we find for h, k, e, p, tan α, or some of them, only imaginary values, as quadratic equations have to be solved; and we have in fact to contemplate the existence of entirely imaginary conics. For instance, no real values of x and y satisfy $x^2 + 2y^2 + 3 = 0$. Even when the locus represented is real, we obtain, as a rule, four sets of values of h, k, e, p, of which two sets are imaginary; a real conic has, besides two real foci and corresponding directrices, two others that are imaginary.

In oblique as well as rectangular coordinates equations of the second degree represent conics.

19. *The three Species of Conics.*—A real conic, which does not degenerate into straight lines, is called an ellipse, parabola or hyperbola according as e <, = , or > 1. To trace the three forms it is best so to choose the axes of reference as to simplify their equations.

In the case of a parabola, let 2c be the distance between the given focus and directrix, and take axes referred to which these are the point (c, 0) and the line x = − c. The equation becomes $(x - c)^2 + y^2 = (x + c)^2$, *i.e.* $y^2 = 4cx$.

In the other cases, take a such that $a(e \sim e^{-1})$ is the distance of focus from directrix, and so choose axes that these are (ae, 0) and $x = ae^{-1}$, thus getting the equation $(x - ae)^2 + y^2 = e^2(x - ae^{-1})^2$, *i.e.* $(1 - e^2)x^2 + y^2 = a^2(1 - e^2)$. When e < 1, *i.e.* in the case of an ellipse, this may be written $x^2/a^2 + y^2/b^2 = 1$, where $b^2 = a^2(1 - e^2)$; and when e > 1, *i.e.* in the case of an hyperbola, $x^2/a^2 - y^2/b^2 = 1$, where $b^2 = a^2(e^2 - 1)$. The axes thus chosen for the ellipse and 716
hyperbola are called the principal axes.

In figs. 54, 55, 56 in order, conics of the three species, thus referred, are depicted.

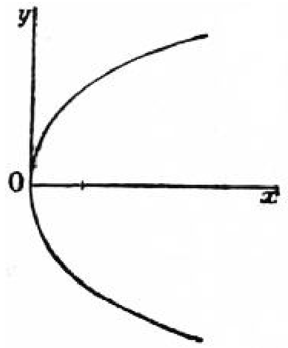


FIG. 54

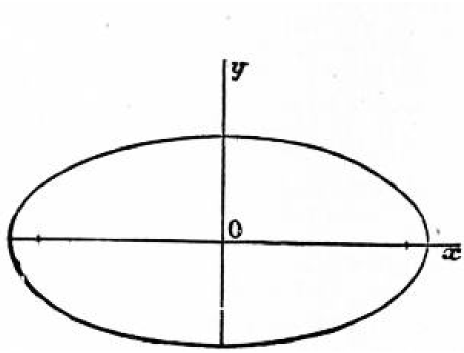


FIG. 55

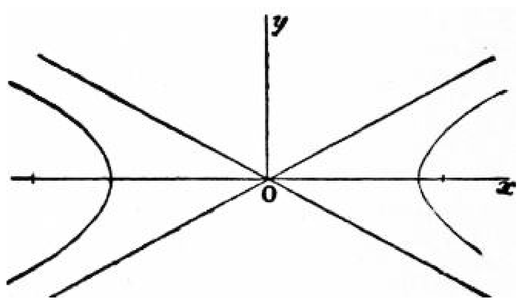


FIG. 56.

The oblique straight lines in fig. 56 are the *asymptotes* x/a = ±y/b of the hyperbola, lines to which the curve tends with unlimited closeness as it goes to infinity. The hyperbola would have an equation of the form $xy = c$ if referred to its asymptotes as axes, the coordinates being then oblique, unless $a = b$, in which case the hyperbola is called rectangular. An ellipse has two imaginary asymptotes. In particular a circle $x^2 + y^2 = a^2$, a particular ellipse, has for asymptotes the imaginary lines $x = \pm y \sqrt{-1}$. These run from the centre to the so-called circular points at infinity.

20. *Tangents and Curvature.*—Let (x', y') and $(x' + h, y' + k)$ be two neighbouring points P, P′ on a curve. The equation of the line on which both lie is $h(y - y') = k(x - x')$. Now keep P fixed, and let P′ move towards coincidence with it along the curve. The connecting line will tend towards a limiting position, to which it can never attain as long as P and P′ are distinct. The line which occupies this limiting position is the tangent at P. Now if we subtract the equation of the curve, with (x', y') for the coordinates in it, from the like equation in $(x' + h, y' + k)$, we obtain a relation in h and k, which will, as a rule, be of the form $0 = Ah + Bk +$ terms of higher degrees in h and k, where A, B and the other coefficients involve x' and y'. This gives $k/h = -A/B +$ terms which tend to vanish as h and k do, so that $-A : B$ is the limiting value tended to by $k : h$. Hence the equation of the tangent is $B(y - y') + A(x - x') = 0$.

The *normal* at (x', y') is the line through it at right angles to the tangent, and its equation is $A(y - y') - B(x - x') = 0$.

In the case of the conic $(a, b, c, f, g, h)\ (x, y, 1)^2 = 0$ we find that $A/B = (ax' + hy' + g)/(hx' + by' + f)$.

We can obtain the coordinates of Q, the intersection of the normals QP, QP′ at (x', y') and $(x' + h, y' + k)$, and then, using the limiting value of $k : h$, deduce those of its limiting position as P′ moves up to P. This is the *centre of curvature* of the curve at P (x', y'), and is so called because it is the centre of the circle of closest contact with the curve at that point. That it is so follows from the facts that the closest circle is the limit tended to by the circle which touches the curve at P and passes through P′, and that the arc from P to P′ of this circle lies between the circles of centre Q and radii QP, QP′, which circles tend, not to different limits as P′ moves up to P, but to one. The distance from P to the centre of curvature is the *radius of curvature*.

21. *Differential Plane Geometry.*—The language and notation of the differential calculus are very useful in the study of tangents and curvature. Denoting by (ξ, η) the current coordinates, we find, as above, that the tangent at a point (x, y) of a curve is $\eta - y = (\xi - x)dy/dx$, where dy/dx is found from the equation of the curve. If this be $f(x, y) = 0$ the tangent is $(\xi - x)\ (\partial f/\partial x) + (\eta - y)\ (\partial f/\partial y) = 0$. If ρ and (α, β) are the radius and centre of curvature at (x, y), we find that $q(\alpha - x) = -p(1 + p^2)$, $q(\beta - y) = 1 + p^2$, $q^2\rho^2 = (1 + p^2)^3$, where p, q denote dy/dx, d^2y/dx^2 respectively. (See Infinitesimal Calculus.)

In any given case we can, at all events in theory, eliminate x, y between the above equations for $\alpha - x$ and $\beta - y$, and the equation of the curve. The resulting equation in (α, β) represents the locus of the centre of curvature. This is the *evolute* of the curve.

22. *Polar Coordinates.*—In plane geometry the distance of any point P from a fixed origin (or pole) O, and the inclination xOP of OP to a fixed line Ox, determine the point: r, the numerical measure of OP, the *radius vector*, and θ, the circular measure of xOP, the *inclination*, are called polar coordinates of P. The formulae $x = r \cos \theta$, $y = r \sin \theta$ connect Cartesian and polar coordinates, and make transition from either system to the other easy. In polar coordinates the equations of a circle through O, and of a conic with O as focus, take the simple forms $r = 2a \cos (\theta - \alpha)$, $r\{1 - e \cos (\theta - \alpha)\} = l$. The use of polar coordinates is very convenient in discussing curves which have properties of symmetry akin to that of a regular polygon, such curves for instance as $r = a \cos m\theta$, with m integral, and also the curves called spirals, which have equations giving r as functions of θ itself, and not merely of $\sin \theta$ and $\cos \theta$. In the geometry of motion under central forces the advantage of working with polar coordinates is great.

23. *Trilinear and Areal Coordinates.*—Consider a fixed triangle ABC, and regard its sides as produced without limit. Denote, as in trigonometry, by a, b, c the positive numbers of units of a chosen scale contained in the lengths BC, CA, AB, by A, B, C the angles, and by Δ the area, of the triangle. We might, as in § 6, take CA, CB as axes of x and y, inclined at an angle C. Any point P (x, y) in the plane is at perpendicular distances $y \sin C$ and $x \sin C$ from CA and CB. Call these β and α respectively. The signs of β and α are those of y and x, *i.e.* β is positive or negative according as P lies on the same side of CA as B does or the opposite, and similarly for α. An equation in (x, y) of any degree may, upon replacing in it x and y by α cosec C and β cosec C, be written as one of the same degree in (α, β). Now let γ be the perpendicular distance of P from the third side AB, taken as positive or negative as P is on the C side of AB or not. The geometry of the figure tells us that $a\alpha + b\beta + c\gamma = 2\Delta$. By means

of this relation in α, β, γ we can give an equation considered countless other forms, involving two or all of α, β, γ. In particular we may make it *homogeneous* in α, β, γ: to do this we have only to multiply the terms of every degree less than the highest present in the equation by a power of $(a\alpha + b\beta + c\gamma)/2\Delta$ just sufficient to raise them, in each case, to the highest degree.

We call (α, β, γ) *trilinear coordinates*, and an equation in them the trilinear equation of the locus represented. Trilinear equations are, as a rule, dealt with in their homogeneous forms. An advantage thus gained is that we need not mean by (α, β, γ) the actual measures of the perpendicular distances, but any properly signed numbers which have the same ratio two and two as these distances.

In place of α, β, γ it is lawful to use, as coordinates specifying the position of a point in the plane of a triangle of reference ABC, any given multiples of these. For instance, we may use $x = a\alpha/2\Delta$, $y = b\beta/2\Delta$, $z = c\gamma/2\Delta$, the properly signed ratios of the triangular areas PBC, PCA, PAB to the triangular area ABC. These are called the *areal* coordinates of P. In areal coordinates the relation which enables us to make any equation homogeneous takes the simple form $x + y + z = 1$; and, as before, we need mean by x, y, z, in a homogeneous equation, only signed numbers in the right ratios.

Straight lines and conics are represented in trilinear and in areal, because in Cartesian, coordinates by equations of the first and second degrees respectively, and these degrees are preserved when the equations are made homogeneous. What must be said about points infinitely far off in order to make universal the statement, to which there is no exception as long as finite distances alone are considered, that *every* homogeneous equation of the first degree represents a straight line? Let the point of areal coordinates (x', y', z') move infinitely far off, and mean by x, y, z finite quantities in the ratios which x', y', z' tend to assume as they become infinite. The relation $x' + y' + z' = 1$ gives that the limiting state of things tended to is expressed by $x + y + z = 0$. This particular equation of the first degree is satisfied by no point at a finite distance; but we see the propriety of saying that it has to be taken as satisfied by all the points conceived of as actually at infinity. Accordingly the special property of these points is expressed by saying that they lie on a special straight line, of which the areal equation is $x + y + z = 0$. In trilinear coordinates this *line at infinity* has for equation $a\alpha + b\beta + c\gamma = 0$.

On the one special line at infinity parallel lines are treated as meeting. There are on it two special (imaginary) points, the circular points at infinity of § 19, through which all circles pass in the same sense. In fact if $S = O$ be one circle, in areal coordinates, $S + (x + y + z)(lx + my + nz) = 0$ may, by proper choice of l, m, n, be made any other; since the added terms are once $lx + my + nz$, and have the generality of any expression like $a'x + b'y + c'$ in Cartesian coordinates. Now these two circles intersect in the two points where either meets $x + y + z = 0$ as well as in two points on the radical axis $lx + my + nz = 0$.

24. Let us consider the perpendicular distance of a point $(\alpha', \beta', \gamma')$ from a line $l\alpha + m\beta + n\gamma$. We can take rectangular axes of Cartesian coordinates (for clearness as to equalities of angle it is best to choose an origin inside ABC), and refer to them, by putting expressions $p - x \cos\theta - y \sin\theta$, &c., for α &c.; we can then apply § 16 to get the perpendicular distance; and finally revert to the trilinear notation. The result is to find that the required distance is

$$(l\alpha' + m\beta' + n\gamma') / \{l, m, n\},$$

where $\{l, m, n\}^2 = l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C$.

In areal coordinates the perpendicular distance from (x', y', z') to $lx + my + nz = 0$ is $2\Delta(lx' + my' + nz')/\{al, bm, cn\}$. 717
In both cases the coordinates are of course actual values.

Now let ξ, η, ζ be the perpendiculars on the line from the vertices A, B, C, *i.e.* the points (1, 0, 0), (0, 1, 0), (0, 0, 1), with signs in accord with a convention that oppositeness of sign implies distinction between one side of the line and the other. Three applications of the result above give

$$\xi/l = 2\Delta / \{al, bm, cn\} = \eta/m = \zeta/n;$$

and we thus have the important fact that $\xi x' + \eta y' + \zeta z'$ is the perpendicular distance between a point of areal coordinates $(x'y'z')$ and a line on which the perpendiculars from A, B, C are ξ, η, ζ respectively. We have also that $\xi x + \eta y + \zeta z = 0$ is the areal equation of the line on which the perpendiculars are ξ, η, ζ; and, by equating the two expressions for the perpendiculars from (x', y', z') on the line, that in all cases $\{a\xi, b\eta, c\zeta\}^2 = 4\Delta^2$.

25. *Line-coordinates. Duality.*—A quite different order of ideas may be followed in applying analysis to geometry. The notion of a straight line specified may precede that of a point, and points may be dealt with as the intersections of lines. The specification of a line may be by means of coordinates, and that of a point by an equation, satisfied by the coordinates of lines which pass through it. Systems of *line-coordinates* will here be only briefly considered. Every

such system is allied to some system of point-coordinates; and space will be saved by giving prominence to this fact, and not recommencing *ab initio*.

Suppose that any particular system of point-coordinates, in which $lx + my + nz = 0$ may represent any straight line, is before us: notice that not only are trilinear and areal coordinates such systems, but Cartesian coordinates also, since we may write x/z, y/z for the Cartesian x, y, and multiply through by z. The line is exactly assigned if l, m, n, or their mutual ratios, are known. Call (l, m, n) the *coordinates* of the line. Now keep x, y, z constant, and let the coordinates of the line vary, but always so as to satisfy the equation. This equation, which we now write $xl + ym + zn = 0$, is satisfied by the coordinates of every line through a certain fixed point, and by those of no other line; it is the equation of that point in the line-coordinates l, m, n.

Line-coordinates are also called *tangential* coordinates. A curve is the envelope of lines which touch it, as well as the locus of points which lie on it. A homogeneous equation of degree above the first in l, m, n is a relation connecting the coordinates of every line which touches some curve, and represents that curve, regarded as an envelope. For instance, the condition that the line of coordinates (l, m, n), *i.e.* the line of which the allied point-coordinate equation is $lx + my + nz = 0$, may touch a conic $(a, b, c, f, g, h) (x, y, z)^2 = 0$, is readily found to be of the form $(A, B, C, F, G, H) (l, m, n)^2 = 0$, *i.e.* to be of the second degree in the line-coordinates. It is not hard to show that the *general* equation of the second degree in l, m, n thus represents a conic; but the degenerate conics of line-coordinates are not line-pairs, as in point-coordinates, but point-pairs.

The degree of the point-coordinate equation of a curve is the *order* of the curve, the number of points in which it cuts a straight line. That of the line-coordinate equation is its *class*, the number of tangents to it from a point. The order and class of a curve are generally different when either exceeds two.

26. The system of line-coordinates allied to the areal system of point-coordinates has special interest.

The l, m, n of this system are the perpendiculars ξ, η, ζ of § 24; and $x'\xi + y'\eta + z'\zeta = 0$ is the equation of the point of areal coordinates (x′, y′, z′), *i.e.* is a relation which the perpendiculars from the vertices of the triangle of reference on every line through the point, but no other line, satisfy. Notice that a non-homogeneous equation of the first degree in ξ, η, ζ does not, as a homogeneous one does, represent a point, but a circle. In fact $x'\xi + y'\eta + z'\zeta = R$ expresses the constancy of the perpendicular distance of the fixed point $x'\xi + y'\eta + z'\zeta = 0$ from the variable line (ξ, η, ζ), *i.e.* the fact that (ξ, η, ζ) touches a circle with the fixed point for centre. The relation in any ξ, η, ζ which enables us to make an equation homogeneous is not linear, as in point-coordinates, but quadratic, viz. it is the relation $\{a\xi, b\eta, c\zeta\}^2 = 4\Delta^2$ of § 24. Accordingly the homogeneous equation of the above circle is

$$4\Delta^2 (x'\xi + y'\eta + z'\zeta)^2 = R^2 \{a\xi, b\eta, c\zeta\}^2.$$

Every circle has an equation of this form in the present system of line-coordinates. Notice that the equation of any circle is satisfied by those coordinates of lines which satisfy both $x'\xi + y'\eta + z'\zeta = 0$, the equation of its centre, and $\{a\xi, b\eta, c\zeta\}^2 = 0$. This last equation, of which the left-hand side satisfies the condition for breaking up into two factors, represents the two imaginary circular points at infinity, through which all circles and their asymptotes pass.

There is strict duality in descriptive geometry between point-line-locus and line-point-envelope theorems. But in metrical geometry duality is encumbered by the fact that there is in a plane one special line only, associated with distance, while of special points, associated with direction, there are two: moreover the line is real, and the points both imaginary.

II. *Solid Analytical Geometry.*

27. Any point in space may be specified by three coordinates. We consider three fixed planes of reference, and generally, as in all that follows, three which are at right angles two and two. They intersect, two and two, in lines x′Ox, y′Oy, z′Oz, called the axes of x, y, z respectively, and divide all space into eight parts called octants. If from any point P in space we draw PN parallel to zOz′ to meet the plane xOy in N, and then from N draw NM parallel to yOy′ to meet x′Ox in M, the coordinates (x, y, z) of P are the numerical measures of OM, MN, NP; in the case of rectangular coordinates these are the perpendicular distances of P from the three planes of reference. The sign of each coordinate is positive or negative as P lies on one side or the other of the corresponding plane. In the octant delineated the signs are taken all positive.

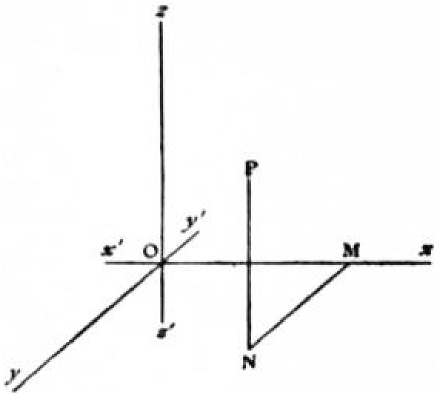


Fig. 57.

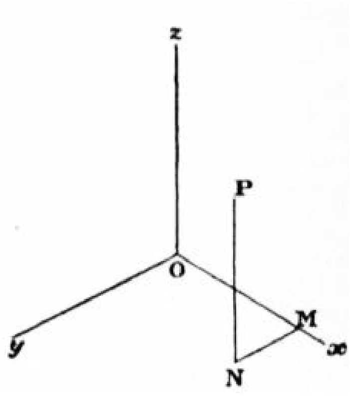


Fig. 58.

In fig. 57 the delineation is on a plane of the paper taken parallel to the plane zOx, the points of a solid figure being projected on that plane by parallels to some chosen line through O in the positive octant. Sometimes it is clearer to delineate, as in fig. 58, by projection parallel to that line in the octant which is equally inclined to Ox, Oy, Oz upon a plane of the paper perpendicular to it. It is possible by parallel projection to delineate equal scales along Ox, Oy, Oz by scales having any ratios we like along lines in a plane having any mutual inclinations we like.

For the delineation of a surface of simple form it frequently suffices to delineate the sections by the coordinate planes; and, in particular, when the surface has symmetry about each coordinate plane, to delineate the quarter-sections belonging to a single octant. Thus fig. 59 conveniently represents an octant of the wave surface, which cuts each coordinate plane in a circle and an ellipse. Or we may delineate a series of contour lines, *i.e.* sections by planes parallel to xOy, or some other chosen plane; of course other sections may be indicated too for greater clearness. For the delineation of a curve a good method is to represent, as above, a series of points P thereof, each accompanied by its ordinate PN, which serves to refer it to the plane of xy. The employment of stereographic projection is also interesting.

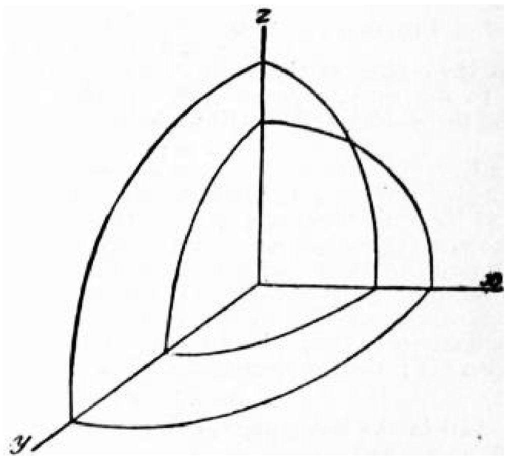


Fig. 59.

28. In plane geometry, reckoning the line as a curve of the first order, we have only the point and the curve. In solid geometry, reckoning a line as a curve of the first order, and the plane as a surface of the first order, we have the point, the curve and the surface; but the increase of complexity is far greater than would hence at first sight appear. In plane geometry a curve is considered in connexion with lines (its tangents); but in solid geometry the curve is considered in connexion with lines and planes (its tangents and osculating planes), and the surface also in connexion with lines and planes (its tangent lines and tangent planes); there are surfaces arising out of the line—cones, skew surfaces, developables, doubly and triply infinite systems of lines, and whole classes of theories which have nothing analogous to them in plane geometry: it is thus a very small part indeed of the subject which can be even referred to in the present article.

In the case of a surface we have between the coordinates (x, y, z) a single, or say a onefold relation, which can be represented by a single relation $f(x, y, z) = 0$; or we may consider the coordinates expressed each of them as a given function of two variable parameters p, q; the form $z = f(x, y)$ is a particular case of each of these modes of representation; in other words, we have in the first mode $f(x, y, z) = z - f(x, y)$, and in the second mode $x = p$, $y = q$ for the expression of two of the coordinates in terms of the parameters.

In the case of a curve we have between the coordinates (x, y, z) a twofold relation: two 718
equations $f(x, y, z) = 0$, $\varphi(x, y, z) = 0$ give such a relation; *i.e.* the curve is here considered as the intersection of two surfaces (but the curve is not always the complete intersection of two surfaces, and there are hence difficulties); or, again, the coordinates may be given each of them as a function of a single variable parameter. The form $y = \varphi(x)$, $z = \psi(x)$, where two of the coordinates are given in terms of the third, is a particular case of each of these modes of representation.

29. The remarks under plane geometry as to descriptive and metrical propositions, and as to the non-metrical character of the method of coordinates when used for the proof of a descriptive proposition, apply also to solid geometry; and they might be illustrated in like manner by the instance of the theorem of the radical centre of four spheres. The proof is obtained from the consideration that S and S′ being each of them a function of the form $x^2 + y^2 + z^2 + ax + by + cz + d$, the difference S-S′ is a mere linear function of the coordinates, and consequently that S-S′ = 0 is the equation of the plane containing the circle of intersection of the two spheres S = 0 and S′ = 0.

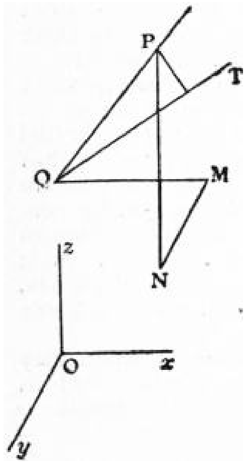


FIG. 60.

30. *Metrical Theory.*—The foundation in solid geometry of the metrical theory is in fact the before-mentioned theorem that if a finite right line PQ be projected upon any other line OO′ by lines perpendicular to OO′, then the length of the projection P′Q′ is equal to the length of PQ into the cosine of its inclination to P′Q′—or (in the form in which it is now convenient to state the theorem) the perpendicular distance P′Q′ of two parallel planes is equal to the inclined distance PQ into the cosine of the inclination. The principle of § 16, that the algebraical sum of the projections of the sides of any closed polygon on any line is zero, or that the two sets of sides of the polygon which connect a vertex A and a vertex B have the same sum of projections on the line, in sign and magnitude, as we pass from A to B, is applicable when the sides do not all lie in one plane.

31. Consider the skew quadrilateral QMNP, the sides QM, MN, NP being respectively parallel to the three rectangular axes Ox, Oy, Oz; let the lengths of these sides be ξ, η, ζ, and that of the side QP be $= \rho$; and let the cosines of the inclinations (or say the cosine-inclinations) of ρ to the three axes be α, β, γ; then projecting successively on the three sides and on QP we have

$$\xi, \eta, \zeta = \rho\alpha, \rho\beta, \rho\gamma,$$

and

$$\rho = \alpha\xi + \beta\eta + \gamma\zeta,$$

whence $\rho^2 = \xi^2 + \eta^2 + \zeta^2$, which is the relation between a distance ρ and its projections ξ, η, ζ upon three rectangular axes. And from the same equations we obtain $\alpha^2 + \beta^2 + \gamma^2 = 1$, which is a relation connecting the cosine-inclinations of a line to three rectangular axes.

Suppose we have through Q any other line QT, and let the cosine-inclinations of this to the axes be α', β', γ', and δ be its cosine-inclination to QP; also let ρ be the length of the projection of QP upon QT; then projecting on QT we have

$$\rho = \alpha'\xi + \beta'\eta + \gamma'\zeta = \rho\delta.$$

And in the last equation substituting for ξ, η, ζ their values $\rho\alpha$, $\rho\beta$, $\rho\gamma$ we find

$$\delta = \alpha\alpha' + \beta\beta' + \gamma\gamma',$$

which is an expression for the mutual cosine-inclination of two lines, the cosine-inclinations of which to the axes are α, β, γ and α', β', γ' respectively. We have of course $\alpha^2 + \beta^2 + \gamma^2 = 1$ and $\alpha'^2 + \beta'^2 + \gamma'^2 = 1$; and hence also

$$\begin{aligned} 1 - \delta^2 &= (\alpha^2 + \beta^2 + \gamma^2)(\alpha'^2 + \beta'^2 + \gamma'^2) - (\alpha\alpha' + \beta\beta' + \gamma\gamma')^2, \\ &= (\beta\gamma' - \beta'\gamma)^2 + (\gamma\alpha' - \gamma'\alpha)^2 + (\alpha\beta' - \alpha'\beta)^2; \end{aligned}$$

so that the sine of the inclination can only be expressed as a square root. These formulae are the foundation of spherical trigonometry.

32. *Straight Lines, Planes and Spheres.*—The foregoing formulae give at once the equations of these loci.

For first, taking Q to be a fixed point, coordinates (a, b, c), and the cosine-inclinations (α, β, γ) to be constant, then P will be a point in the line through Q in the direction thus determined; or, taking (x, y, z) for its coordinates, these will be the current coordinates of a point in the line. The values of ξ, η, ζ then are $x - a$, $y - b$, $z - c$, and we thus have

$$\frac{x - a}{\alpha} = \frac{y - b}{\beta} = \frac{z - c}{\gamma} \ (= \rho),$$

which (omitting the last equation, $= \rho$) are the equations of the line through the point (a, b, c), the cosine-inclinations to the axes being α, β, γ, and these quantities being connected by the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$. This equation may be omitted, and then α, β, γ, instead of being equal, will only be proportional, to the cosine-inclinations.

Using the last equation, and writing

$$x, y, z = a + \alpha\rho, b + \beta\rho, c + \gamma\rho,$$

these are expressions for the current coordinates in terms of a parameter ρ, which is in fact the distance from the fixed point (a, b, c).

It is easy to see that, if the coordinates (x, y, z) are connected by any two linear equations, these equations can always be brought into the foregoing form, and hence that the two linear equations represent a line.

Secondly, taking for greater simplicity the point Q to be coincident with the origin, and α', β', γ', p to be constant, then p is the perpendicular distance of a plane from the origin, and α', β', γ' are the cosine-inclinations of this distance to the axes ($\alpha'^2 + \beta'^2 + \gamma'^2 = 1$). P is any point in this plane, and taking its coordinates to be (x, y, z) then (ξ, η, ζ) are = (x, y, z), and the foregoing equation $p = \alpha'\xi + \beta'\eta + \gamma'\zeta$ becomes

$$\alpha'x + \beta'y + \gamma'z = p,$$

which is the equation of the plane in question.

If, more generally, Q is not coincident with the origin, then, taking its coordinates to be (a, b, c), and writing p_1 instead of p, the equation is

$$\alpha' (x - a) + \beta' (y - b) + \gamma' (z - c) = p_1;$$

and we thence have $p_1 = p - (a\alpha' + b\beta' + c\gamma')$, which is an expression for the perpendicular distance of the point (a, b, c) from the plane in question.

It is obvious that any linear equation $Ax + By + Cz + D = O$ between the coordinates can always be brought into the foregoing form, and hence that such an equation represents a plane.

Thirdly, supposing Q to be a fixed point, coordinates (a, b, c), and the distance $QP = \rho$, to be constant, say this is = d, then, as before, the values of ξ, η, ζ are $x - a$, $y - b$, $z - c$, and the equation $\xi^2 + \eta^2 + \zeta^2 = \rho^2$ becomes

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = d^2,$$

which is the equation of the sphere, coordinates of the centre = (a, b, c), and radius = d.

A quadric equation wherein the terms of the second order are $x^2 + y^2 + z^2$, viz. an equation

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0,$$

can always, it is clear, be brought into the foregoing form; and it thus appears that this is the equation of a sphere, coordinates of the centre $-\frac{1}{2}A$, $-\frac{1}{2}B$, $-\frac{1}{2}C$, and squared radius $= \frac{1}{4}(A^2 + B^2 + C^2) - D$.

33. *Cylinders, Cones, ruled Surfaces.*—If the two equations of a straight line involve a parameter to which any value may be given, we have a singly infinite system of lines. They cover a surface, and the equation of the surface is obtained by eliminating the parameter between the two equations.

If the lines all pass through a given point, then the surface is a cone; and, in particular, if the lines are all parallel to a given line, then the surface is a cylinder.

Beginning with this last case, suppose the lines are parallel to the line $x = mz$, $y = nz$, the equations of a line of the system are $x = mz + a$, $y = nz + b$,—where a, b are supposed to be functions of the variable parameter, or, what is the same thing, there is between them a relation $f(a, b) = 0$: we have $a = x - mz$, $b = y - nz$, and the result of the elimination of the parameter therefore is $f(x - mz, y - nz) = 0$, which is thus the general equation of the cylinder the generating lines whereof are parallel to the line $x = mz$, $y = nz$. The equation of the section by the plane $z = 0$ is $f(x, y) = 0$, and conversely if the cylinder be determined by means of its curve of intersection with the plane $z = 0$, then, taking the equation of this curve to be $f(x, y) = 0$, the equation of the cylinder is $f(x - mz, y - nz) = 0$. Thus, if the curve of intersection be the circle $(x - \alpha)^2 + (y - \beta)^2 = \gamma^2$, we have $(x - mz - \alpha)^2 + (y - nz - \beta)^2 = \gamma^2$ as the equation of an oblique cylinder on this base, and thus also $(x - \alpha)^2 + (y - \beta)^2 = \gamma^2$ as the equation of the right cylinder.

If the lines all pass through a given point (a, b, c), then the equations of a line are $x - a = \alpha(z - c)$, $y - b = \beta(z - c)$, where α, β are functions of the variable parameter, or, what is the same thing, there exists between them an equation $f(\alpha, \beta) = 0$; the elimination of the parameter gives, therefore, $f[(x - a)/(x - c'), (y - b)/(z - c)] = 0$; and this equation, or, what is the same thing, any homogeneous equation $f(x - a, y - b, z - c) = 0$, or, taking f to be a rational and integral function of the order n, say $(*)(x - a, y - b, z - c)^n = 0$, is the general equation of the cone having the point (a, b, c) for its vertex. Taking the vertex to be at the origin, the equation is $(*)(x, y, z)^n = 0$; and, in particular, $(*)(x, y, z)^2 = 0$ is the equation of a cone of the second order, or quadricone, having the origin for its vertex.

34. In the general case of a singly infinite system of lines, the locus is a ruled surface (or *regulus*). Now, when a line is changing its position in space, it may be looked upon as in a state of turning about some point in itself, while that point is, as a rule, in a state of moving out of the plane in which the turning takes place. If instantaneously it is only in a state of turning, it is usual, though not strictly accurate, to say that it intersects its consecutive position. A regulus such that consecutive lines on it do not intersect, in this sense, is called a skew surface, or *scroll*; one on which they do is called a developable surface or *torse*.

Suppose, for instance, that the equations of a line (depending on the variable parameter θ) 719
are $x/a + y/c = \theta (1 + y/b)$, $x/a - z/c = (1/\theta)(1 - y/b)$; then, eliminating θ we have $x^2/a^2 - z^2/c^2 = 1 - y^2/b^2$, or say, $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, the equation of a quadric surface, afterwards called the hyperboloid of one sheet; this surface is consequently a scroll. It is to be remarked that we have upon the surface a second singly infinite series of lines; the equations of a line of this second system (depending on the variable parameter φ) are

$$\frac{x}{a} + \frac{z}{c} = \varphi\left(1 - \frac{y}{b}\right), \quad \frac{x}{a} - \frac{z}{c} = \frac{1}{\varphi}\left(1 + \frac{y}{b}\right).$$

It is easily shown that any line of the one system intersects every line of the other system.

Considering any curve (of double curvature) whatever, the tangent lines of the curve form a singly infinite system of lines, each line intersecting the consecutive line of the system,—that is, they form a developable, or torse; the curve and torse are thus inseparably connected together, forming a single geometrical figure. An osculating plane of the curve (see § 38 below) is a tangent plane of the torse all along a generating line.

35. *Transformation of Coordinates.*—There is no difficulty in changing the origin, and it is for brevity assumed that the origin remains unaltered. We have, then, two sets of rectangular axes, Ox, Oy, Oz, and Ox_1, Oy_1, Ozx_1, the mutual cosine-inclinations being shown by the diagram—

	x	y	z
x_1	α	β	γ
y_1	α	β'	γ'
z_1	α''	β''	γ''

that is, α, β, γ are the cosine-inclinations of Ox_1 to Ox, Oy, Oz; α′, β′, γ′ those of Oy_1, &c.

And this diagram gives also the linear expressions of the coordinates (x_1, y_1, z_1) or (x, y, z) of either set in terms of those of the other set; we thus have

$$\begin{aligned} x_1 &= \alpha x + \beta y + \gamma z, & x &= \alpha x_1 + \alpha' y_1 + \alpha'' z_1, \\ y_1 &= \alpha' x + \beta' y + \gamma' z, & y &= \beta x_1 + \beta' y_1 + \beta'' z_1, \\ z_1 &= \alpha'' x + \beta'' y + \gamma'' z, & z &= \gamma x_1 + \gamma' y_1 + \gamma'' z_1, \end{aligned}$$

which are obtained by projection, as above explained. Each of these equations is, in fact, nothing else than the before-mentioned equation $p = \alpha'\xi + \beta'\eta + \gamma'\zeta$, adapted to the problem in hand.

But we have to consider the relations between the nine coefficients. By what precedes, or by the consideration that we must have identically $x^2 + y^2 + z^2 = x_1^2 + y_1^2 + z_1^2$, it appears that these satisfy the relations—

$$\begin{array}{llll} \alpha^2 + \beta^2 + \gamma^2 = 1, & & \alpha^2 + \alpha'^2 + \alpha''^2 = 1, \\ \alpha'^2 + \beta'^2 + \gamma'^2 = 1, & & \beta^2 + \beta'^2 + \beta''^2 = 1, \\ \alpha''^2 + \beta''^2 + \gamma''^2 = 1, & & \gamma^2 + \gamma'^2 + \gamma''^2 = 1, \\ \alpha' a'' + \beta'\beta'' + \gamma'\gamma'' = 0, & & \beta\gamma + \beta'\gamma' + \beta''\gamma'' = 0, \\ \alpha''\alpha + \beta''\beta + \gamma''\gamma = 0, & & \gamma\alpha + \gamma'\alpha' + \gamma''\alpha'' = 0, \\ \alpha\alpha' + \beta\beta' + \gamma\gamma' = 0, & & \alpha\beta + \alpha'\beta' + \alpha''\beta'' = 0, \end{array}$$

either set of six equations being implied in the other set.

It follows that the square of the determinant

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}$$

is = 1; and hence that the determinant itself is = ±1. The distinction of the two cases is an important one: if the determinant is = + 1, then the axes Ox_1, Oy_1, Oz_1 are such that they can by a rotation about O be brought to coincide with Ox, Oy, Oz respectively; if it is = −1, then they cannot. But in the latter case, by measuring x_1, y_1, z_1 in the opposite directions we change the signs of all the coefficients and so make the determinant to be = + 1; hence the former case need alone be considered, and it is accordingly assumed that the determinant is = +1. This being so, it is found that we have the equality $\alpha = \beta'\gamma'' - \beta''\gamma'$, and eight like ones, obtained from this by cyclical interchanges of the letters α, β, γ, and of unaccented, singly and doubly accented letters.

36. The nine cosine-inclinations above are, as has been seen, connected by six equations. It ought then to be possible to express them all in terms of three parameters. An elegant means of doing this has been given by Rodrigues, who has shown that the tabular expression of the formulae of transformation may be written

	x	y	z
x_1	$1 + \lambda^2 - \mu^2 - \nu^2$	$2(\lambda\mu - \nu)$	$2(\nu\lambda + \mu)$
y_1	$2(\lambda\mu + \nu)$	$1 - \lambda^2 + \mu^2 - \nu^2$	$2(\mu\nu + \lambda)$
z_1	$2(\nu\lambda - \mu)$	$2(\mu\nu + \lambda)$	$1 - \lambda^2 - \mu^2 + \nu^2$

$$\div (1 + \lambda^2 + \mu^2 + \nu^2),$$

the meaning being that the coefficients in the transformation are fractions, with numerators expressed as in the table, and the common denominator.

37. *The Species of Quadric Surfaces.*—Surfaces represented by equations of the second degree are called *quadric* surfaces. Quadric surfaces are either *proper* or *special*. The special ones arise when the coefficients in the general equation are limited to satisfy certain special equations; they comprise (1) plane-pairs, including in particular one plane twice repeated, and (2) cones, including in particular cylinders; there is but one form of cone, but cylinders may be elliptic, parabolic or hyperbolic.

A discussion of the general equation of the second degree shows that the *proper* quadric surfaces are of five kinds, represented respectively, when referred to the most convenient axes of reference, by equations of the five types (a and b positive):

(1) $z = x^2/2a + y^2/2b$, elliptic paraboloid.
(2) $z = x^2/2a - y^2/2b$, hyperbolic paraboloid.
(3) $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, ellipsoid.
(4) $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, hyperboloid of one sheet.
(5) $x^2/a^2 + y^2/b^2 - z^2/c^2 = -1$, hyperboloid of two sheets.

It is at once seen that these are distinct surfaces; and the equations also show very readily the general form and mode of generation of the several surfaces.

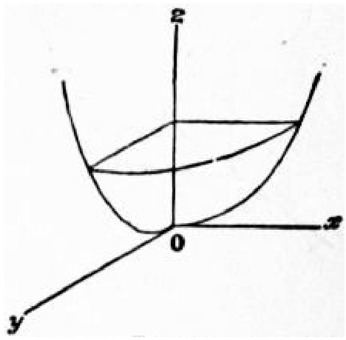


FIG. 61.

In the elliptic paraboloid (fig. 61) the sections by the planes of zx and zy are the parabolas

$$z = \frac{x^2}{2a}, \quad z = \frac{y^2}{2b},$$

having the common axes Oz; and the section by any plane z = γ parallel to that of xy is the ellipse

$$\gamma = \frac{x^2}{2a} + \frac{y^2}{2b};$$

so that the surface is generated by a variable ellipse moving parallel to itself along the parabolas as directrices.

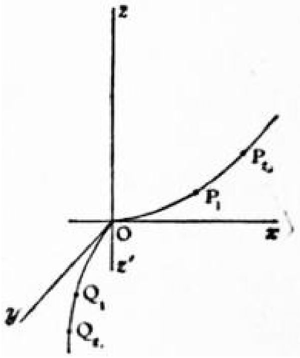


FIG. 62.

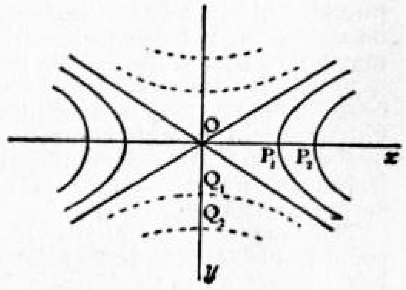


FIG. 63.

In the hyperbolic paraboloid (figs. 62 and 63) the sections by the planes of zx, zy are the parabolas $z = x^2/2a$, $z = -y^2/2b$, having the opposite axes Oz, Oz′, and the section by a plane $z = \gamma$ parallel to that of xy is the hyperbola $\gamma = x^2/2a - y^2/2b$, which has its transverse axis parallel to Ox or Oy according as γ is positive or negative. The surface is thus generated by a variable hyperbola moving parallel to itself along the parabolas as directrices. The form is best seen from fig. 63, which represents the sections by planes parallel to the plane of xy, or say the contour lines; the continuous lines are the sections above the plane of xy, and the dotted lines the sections below this plane. The form is, in fact, that of a saddle.

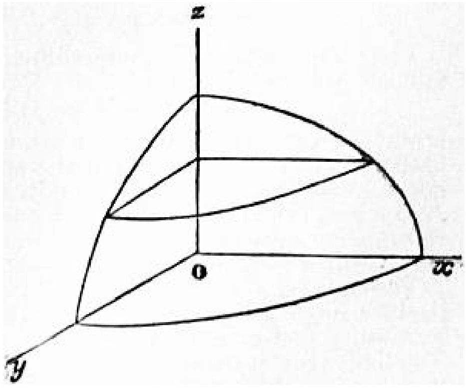


FIG. 64.

In the ellipsoid (fig. 64) the sections by the planes of zx, zy, and xy are each of them an ellipse, and the section by any parallel plane is also an ellipse. The surface may be considered as generated by an ellipse moving parallel to itself along two ellipses as directrices.

In the hyperboloid of one sheet (fig. 65), the sections by the planes of zx, zy are the hyperbolas 720

$$\frac{x^2}{c^2} - \frac{z^2}{c^2} = 1, \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

having a common conjugate axis zOz′; the section by the plane of x, y, and that by any parallel plane, is an ellipse; and the surface may be considered as generated by a variable ellipse moving parallel to itself along the two hyperbolas as directrices. If we imagine two equal and parallel circular disks, their points connected by strings of equal lengths, so that these are the generators of a right circular cylinder, and if we turn one of the disks about its centre through an angle in its plane, the strings in their new positions will be one system of generators of a hyperboloid of one sheet, for which $a = b$; and if we turn it through the same angle in the opposite direction, we get in like manner the generators of the other system; there will be the same general configuration when $a \neq b$. The hyperbolic paraboloid is also covered by two systems of rectilinear generators as a method like that used in § 34 establishes without difficulty. The figures should be studied to see how they can lie.

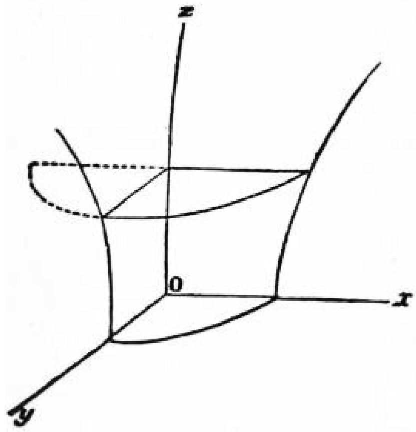


FIG. 65.

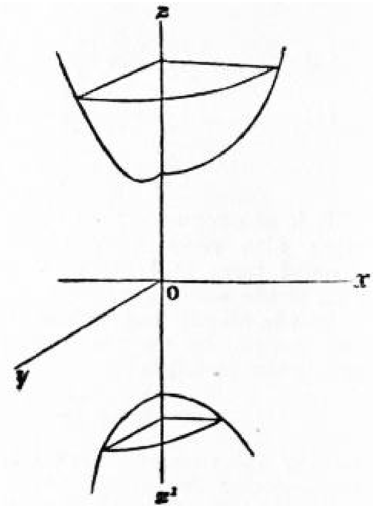


FIG. 66.

In the hyperboloid of two sheets (fig. 66) the sections by the planes of zx and zy are the hyperbolas

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1, \quad \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1,$$

having a common transverse axis along z′Oz; the section by any plane $z = \pm\gamma$ parallel to that of xy is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\gamma^2}{c^2} - 1,$$

provided $\gamma^2 > c^2$, and the surface, consisting of two distinct portions or sheets, may be considered as generated by a variable ellipse moving parallel to itself along the hyperbolas as directrices.

38. *Differential Geometry of Curves.*—For convenience consider the coordinates (x, y, z) of a point on a curve in space to be given as functions of a variable parameter θ, which may in particular be one of themselves. Use the notation x′, x″ for $dx/d\theta$, $d^2x/d\theta^2$, and similarly as to y and z. Only a few formulae will be given. Call the current coordinates (ξ, η, ζ).

The *tangent* at (x, y, z) is the line tended to as a limit by the connector of (x, y, z) and a neighbouring point of the curve when the latter moves up to the former: its equations are

$$(\xi - x)/x' = (\eta - y)/y' = (\zeta - z)/z'.$$

The *osculating plane* at (x, y, z) is the plane tended to as a limit by that through (x, y, z) and two neighbouring points of the curve as these, remaining distinct, both move up to (x, y, z): its one equation is

$$(\xi - x)(y'z'' - y''z') + (\eta - y)(z'x'' - z''x') + (\zeta - z)(x'y'' - x''y') = 0.$$

The *normal plane* is the plane through (x, y, z) at right angles to the tangent line, *i.e.* the plane

$$x'(\xi - x) + y'(\eta - y) + z'(\zeta - z) = 0.$$

It cuts the osculating plane in a line called the *principal normal*. Every line through (x, y, z) in the normal plane is a normal. The normal perpendicular to the osculating plane is called the *binormal*. A tangent, principal normal, and binormal are a convenient set of rectangular axes to use as those of reference, when the nature of a curve near a point on it is to be discussed.

Through (x, y, z) and three neighbouring points, all on the curve, passes a single sphere; and as the three points all move up to (x, y, z) continuing distinct, the sphere tends to a limiting size and position. The limit tended to is the sphere of closest contact with the curve at (x, y, z); its centre and radius are called the centre and radius of *spherical curvature*. It cuts the osculating plane in a circle, called the *circle of absolute curvature*; and the centre and radius of this circle are the centre and radius of absolute curvature. The centre of absolute curvature is the limiting position of the point where the principal normal at (x, y, z) is cut by the normal plane at a neighbouring point, as that point moves up to (x, y, z).

39. *Differential Geometry of Surfaces.*—Let (x, y, z) be any chosen point on a surface ƒ(x, y, z) = 0. As a second point of the surface moves up to (x, y, z), its connector with (x, y, z) tends to a limiting position, a tangent line to the surface at (x, y, z). All these tangent lines at (x, y, z), obtained by approaching (x, y, z) from different directions on a surface, lie in one plane

$$\frac{\partial f}{\partial x}(\xi - x) + \frac{\partial f}{\partial y}(\eta - y) + \frac{\partial f}{\partial z}(\zeta - z) = 0.$$

This plane is called the *tangent plane* at (x, y, z). One line through (x, y, z) is at right angles to the tangent plane. This is the normal

$$(\xi - x) \Big/ \frac{\partial f}{\partial x} = (\eta - y) \Big/ \frac{\partial f}{\partial y} = (\zeta - z) \Big/ \frac{\partial f}{\partial z}.$$

The tangent plane is cut by the surface in a curve, real or imaginary, with a node or double point at (x, y, z). Two of the tangent lines touch this curve at the node. They are called the “chief tangents” (*Haupt-tangenten*) at (x, y, z); they have closer contact with the surface than any other tangents.

In the case of a quadric surface the curve of intersection of a tangent and the surface is of the second order and has a node, it must therefore consist of two straight lines. Consequently a quadric surface is covered by two sets of straight lines, a pair through every point on it; these are imaginary for the ellipsoid, hyperboloid of two sheets, and elliptic paraboloid.

A surface of any order is covered by two singly infinite systems of curves, a pair through every point, the tangents to which are all chief tangents at their respective points of contact. These are called *chief-tangent curves*; on a quadric surface they are the above straight lines.

40. The tangents at a point of a surface which bisect the angles between the chief tangents are called the *principal tangents* at the point. They are at right angles, and together with the normal constitute a convenient set of rectangular axes to which to refer the surface when its properties near the point are under discussion. At a special point which is such that the chief tangents there run to the circular points at infinity in the tangent plane, the principal tangents are indeterminate; such a special point is called an umbilic of the surface.

There are two singly infinite systems of curves on a surface, a pair cutting one another at right angles through every point upon it, all tangents to which are principal tangents of the surface at their respective points of contact. These are called *lines of curvature*, because of a property next to be mentioned.

As a point Q moves in an arbitrary direction on a surface from coincidence with a chosen point P, the normal at it, as a rule, at once fails to meet the normal at P; but, if it takes the direction of a line of curvature through P, this is instantaneously not the case. We have thus on the normal two centres of curvature, and the distances of these from the point on the surface are the two *principal radii of curvature* of the surface at that point; these are also the radii of curvature of the sections of the surface by planes through the normal and the two principal tangents respectively; or say they are the radii of curvature of the normal sections through the two principal tangents respectively. Take at the point the axis of z in the direction of the normal, and those of x and y in the directions of the principal tangents respectively, then, if the radii of curvature be a, b (the signs being such that the coordinates of the two centres of curvature are $z = a$ and $z = b$ respectively), the surface has in the neighbourhood of the point the form of the paraboloid

$$z = \frac{x^2}{2a} + \frac{y^2}{2b},$$

and the chief-tangents are determined by the equation $0 = x^2/2a + y^2/2b$. The two centres of curvature may be on the same side of the point or on opposite sides; in the former case a and b have the same sign, the paraboloid is elliptic, and the chief-tangents are imaginary; in the latter case a and b have opposite signs, the paraboloid is hyperbolic, and the chief-tangents are real.

The normal sections of the surface and the paraboloid by the same plane have the same radius of curvature; and it thence readily follows that the radius of curvature of a normal section of the surface by a plane inclined at an angle θ to that of zx is given by the equation

$$\frac{1}{\rho} = \frac{\cos^2\theta}{a} + \frac{\sin^2\theta}{b}.$$

The section in question is that by a plane through the normal and a line in the tangent plane inclined at an angle θ to the principal tangent along the axis of x. To complete the theory, consider the section by a plane having the same trace upon the tangent plane, but inclined to the normal at an angle φ; then it is shown without difficulty (Meunier’s theorem) that the radius of curvature of this inclined section of the surface is $= \rho \cos\varphi$.

AUTHORITIES.—The above article is largely based on that by Arthur Cayley in the 9th edition of this work. Of early and important recent publications on analytical geometry, special mention is to be made of R. Descartes, *Géométrie* (Leyden, 1637); John Wallis, *Tractatus de

sectionibus conicis nova methodo expositis (1655, *Opera mathematica*, i., Oxford, 1695); de l'Hospital, *Traité analytique des sections coniques* (Paris, 1720); Leonhard Euler, *Introductio in analysin infinitorum*, ii. (Lausanne, 1748); Gaspard Monge, "Application d'algèbre à la géométrie" (*Journ. École Polytech.*, 1801); Julius Plücker, *Analytisch-geometrische Entwickelungen*, 3 Bde. (Essen, 1828-1831); *System der analytischen Geometrie* (Berlin, 1835); G. Salmon, *A Treatise on Conic Sections* (Dublin, 1848; 6th ed., London, 1879); Ch. Briot and J. Bouquet, *Leçons de géométrie analytique* (Paris, 1851; 16th ed., 1897); M. Chasles, *Traité de géométrie supérieure* (Paris, 1852); Wilhelm Fiedler, *Analytische Geometrie der Kegelschnitte* nach G. Salmon frei bearbeitet (Leipzig, 5te Aufl., 1887-1888); N.M. Ferrers, *An Elementary Treatise on Trilinear Coordinates* (London, 1861); Otto Hesse, *Vorlesungen aus der analytischen Geometrie* (Leipzig, 1865, 1881); W.A. Whitworth, *Trilinear Coordinates and other Methods of Modern Analytical Geometry* (Cambridge, 1866); J. Booth, *A Treatise on Some New Geometrical Methods* (London, i., 1873; ii., 1877); A. Clebsch-F. Lindemann, *Vorlesungen über Geometrie*, Bd. i. (Leipzig, 1876, 2te Aufl., 1891); R. Baltser, *Analytische Geometrie* (Leipzig, 1882); Charlotte A. Scott, *Modern Methods of Analytical Geometry* (London, 1894); G. Salmon, *A Treatise on the Analytical Geometry of three Dimensions* (Dublin, 1862; 4th ed., 1882); Salmon-Fiedler, *Analytische Geometrie des Raumes* (Leipzig, 1863; 4te Aufl., 1898); P. Frost, *Solid Geometry* (London, 3rd ed., 1886; 1st ed., Frost and J. Wolstenholme). See also E. Pascal, *Repertorio di matematiche superiori, II. Geometria* (Milan, 1900), and articles now appearing in the *Encyklopädie der mathematischen Wissenschaften*, Bd. iii. 1, 2.

(E. B. El.)

V. Line Geometry

Line geometry is the name applied to those geometrical investigations in which the straight line replaces the point as element. Just as ordinary geometry deals primarily with points and systems of points, this theory deals in the first instance with straight lines and systems of straight lines. In two dimensions there is no necessity for a special line geometry, inasmuch as the straight line and the point are interchangeable by the principle of duality; but in three dimensions the straight line is its own reciprocal, and for the better discussion of systems of lines we require some new apparatus, *e.g.*, a system of coordinates applicable to straight lines rather than to points. The essential features of the subject are most easily elucidated by analytical methods: we shall therefore begin with the notion of line coordinates, and in order to emphasize the merits of the system of coordinates ultimately adopted, we first notice a system without these advantages, but often useful in special investigations.

In ordinary Cartesian coordinates the two equations of a straight line may be reduced to the form $y = rx + s$, $z = tx + u$, and r, s, t, u may be regarded as the four coordinates of the line. These coordinates lack symmetry: moreover, in changing from one base of reference to another the transformation is not linear, so that the degree of an equation is deprived of real significance. For purposes of the general theory we employ homogeneous coordinates; if $x_1y_1z_1w_1$ and $x_2y_2z_2w_2$ are two points on the line, it is easily verified that the six determinants of the array

$$\left|\begin{matrix} x_1y_1z_1w_1 \\ x_2y_2z_2w_2 \end{matrix}\right|$$

are in the same ratios for all point-pairs on the line, and further, that when the point coordinates undergo a linear transformation so also do these six determinants. We therefore adopt these six determinants for the coordinates of the line, and express them by the symbols l, λ, m, μ, n, ν where $l = x_1w_2 - x_2w_1$, $\lambda = y_1z_2 - y_2z_1$, &c. There is the further advantage that if $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$ be two planes through the line, the six determinants

$$\left|\begin{matrix} a_1b_1c_1d_1 \\ a_2b_2c_2d_2 \end{matrix}\right|$$

are in the same ratios as the foregoing, so that except as regards a factor of proportionality we have $\lambda = b_1c_2 - b_2c_1$, $l = c_1d_2 - c_2d_1$, &c. The identical relation $l\lambda + m\mu + n\nu = o$ reduces the number of independent constants in the six coordinates to four, for we are only concerned with their mutual ratios; and the quadratic character of this relation marks an essential difference between point geometry and line geometry. The condition of intersection of two lines is

$$l\lambda' + l'\lambda + m\mu' + m'\mu + n\nu' + n'\nu = 0$$

where the accented letters refer to the second line. If the coordinates are Cartesian and l, m, n are direction cosines, the quantity on the left is the mutual moment of the two lines.

Since a line depends on four constants, there are three distinct types of configurations arising in line geometry—those containing a triply-infinite, a doubly-infinite and a singly-infinite number of lines; they are called Complexes, Congruences, and Ruled Surfaces or Skews respectively. A *Complex* is thus a system of lines satisfying one condition—that is, the coordinates are connected by a single relation; and the degree of the complex is the degree of this equation supposing it to be algebraic. The lines of a complex of the nth degree which pass through any point lie on a cone of the nth degree, those which lie in any plane envelop a curve of the nth class and there are n lines of the complex in any plane pencil; the last statement combines the former two, for it shows that the cone is of the nth degree and the curve is of the nth class. To find the lines common to four complexes of degrees n_1, n_2, n_3, n_4, we have to solve five equations, viz. the four complex equations together with the quadratic equation connecting the line coordinates, therefore the number of common lines is $2n_1n_2n_3n_4$. As an example of complexes we have the lines meeting a twisted curve of the nth degree, which form a complex of the nth degree.

A *Congruence* is the set of lines satisfying two conditions: thus a finite number m of the lines pass through any point, and a finite number n lie in any plane; these numbers are called the degree and class respectively, and the congruence is symbolically written (m, n).

The simplest example of a congruence is the system of lines constituted by all those that pass through m points and those that lie in n planes; through any other point there pass m of these lines, and in any other plane there lie n, therefore the congruence is of degree m and class n. It has been shown by G.H. Halphen that the number of lines common to two congruences is mm′ + nn′, which may be verified by taking one of them to be of this simple type. The lines meeting two fixed lines form the general (1, 1) congruence; and the chords of a twisted cubic form the general type of a (1, 3) congruence; Halphen’s result shows that two twisted cubics have in general ten common chords. As regards the analytical treatment, the difficulty is of the same nature as that arising in the theory of curves in space, for a congruence is not in general the complete intersection of two complexes.

A *Ruled Surface*, *Regulus* or *Skew* is a configuration of lines which satisfy three conditions, and therefore depend on only one parameter. Such lines all lie on a surface, for we cannot draw one through an arbitrary point; only one line passes through a point of the surface; the simplest example, that of a quadric surface, is really two skews on the same surface.

The degree of a ruled surface *qua* line geometry is the number of its generating lines contained in a linear complex. Now the number which meets a given line is the degree of the surface *qua* point geometry, and as the lines meeting a given line form a particular case of linear complex, it follows that the degree is the same from whichever point of view we regard it. The lines common to three complexes of degrees, $n_1n_2n_3$, form a ruled surface of degree $2n_1n_2n_3$; but not every ruled surface is the complete intersection of three complexes.

Linear complex.

In the case of a complex of the first degree (or linear complex) the lines through a fixed point lie in a plane called the polar plane or nul-plane of that point, and those lying in a fixed plane pass through a point called the nul-point or pole of the plane. If the nul-plane of A pass through B, then the nul-plane of B will pass through A; the nul-planes of all points on one line l_1 pass through another line l_2. The relation between l_1 and l_2 is reciprocal; any line of the complex that meets one will also meet the other, and every line meeting both belongs to the complex. They are called conjugate or polar lines with respect to the complex. On these principles can be founded a theory of reciprocation with respect to a linear complex.

This may be aptly illustrated by an elegant example due to A. Voss. Since a twisted cubic can be made to satisfy twelve conditions, it might be supposed that a finite number could be drawn to touch four given lines, but this is not the case. For, suppose one such can be drawn, then its reciprocal with respect to any linear complex containing the four lines is a curve of the third class, *i.e.* another twisted cubic, touching the same four lines, which are unaltered in the process of reciprocation; as there is an infinite number of complexes containing the four lines, there is an infinite number of cubics touching the four lines, and the problem is poristic.

The following are some geometrical constructions relating to the unique linear complex that can be drawn to contain five arbitrary lines:

To construct the nul-plane of any point O, we observe that the two lines which meet any four of the given five are conjugate lines of the complex, and the line drawn through O to meet them is therefore a ray of the complex; similarly, by choosing another four we can find another ray through O: these rays lie in the nul-plane, and there is clearly a result involved that the five lines so obtained all lie in one plane. A reciprocal construction will enable us to find the nul-point of any plane. Proceeding now to the metrical properties and the statical and dynamical applications, we remark that there is just one line such that the nul-plane of any point on it is perpendicular to it. This is called the central axis; if d be the shortest distance, θ the angle between it and a ray of the complex, then d tan θ = p, where p is a constant called the pitch or parameter. Any system of forces can be reduced to a force R along a certain line, and a couple G perpendicular to that line; the lines of nul-moment for 722 the system form a linear complex of which the given line is the central axis and the quotient G/R is the pitch. Any motion of a rigid body can be reduced to a screw motion about a certain line, *i.e.* to an angular velocity ω about that line combined with a linear velocity u along the line. The plane drawn through any point perpendicular to the direction of its motion is its nul-plane with respect to a linear complex having this line for central axis, and the quotient u/ω for pitch (cf. Sir R.S. Ball, *Theory of Screws*).

The following are some properties of a configuration of two linear complexes:

The lines common to the two-complexes also belong to an infinite number of linear complexes, of which two reduce to single straight lines. These two lines are conjugate lines with respect to each of the complexes, but they may coincide, and then some simple modifications are required. The locus of the central axis of this system of complexes is a surface of the third degree called the cylindroid, which plays a leading part in the theory of screws as developed synthetically by Ball. Since a linear complex has an invariant of the second degree in its coefficients, it follows that two linear complexes have a lineo-linear invariant. This invariant is fundamental: if the complexes be both straight lines, its vanishing is the condition of their intersection as given above; if only one of them be a straight line, its vanishing is the condition that this line should belong to the other complex. When it vanishes for any two complexes they are said to be in *involution* or *apolar*; the nul-points P, Q of any plane then divide harmonically the points in which the plane meets the common conjugate lines, and each complex is its own reciprocal with respect to the other. As regards a configuration of these linear complexes, the common lines from one system of generators of a quadric, and the doubly infinite system of complexes containing the common lines, include an infinite number of straight lines which form the other system of generators of the same quadric.

General line coordinates.

If the equation of a linear complex is Al + Bm + Cn + Dλ + Eμ + Fν = 0, then for a line not belonging to the complex we may regard the expression on the left-hand side as a multiple of the moment of the line with respect to the complex, the word moment being used in the statical sense; and we infer that when the coordinates are replaced by linear functions of themselves the new coordinates are multiples of the moments of the line with respect to six fixed complexes. The essential features of this coordinate system are the same as those of the original one, viz. there are six coordinates connected by a quadratic equation, but this relation has in general a different form. By suitable choice of the six fundamental complexes, as they may be called, this connecting relation may be brought into other simple forms of which we mention two: (i.) When the six are mutually in involution it can be reduced to $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0$; (ii.) When the first four are in involution and the other two are the lines common to the first four it is $x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_5x_6 = 0$. These generalized coordinates might be explained without reference to actual magnitude, just as homogeneous point coordinates can be; the essential remark is that the equation of any coordinate to zero represents a linear complex, a point of view which includes our original system, for the equation of a coordinate to zero represents all the lines meeting an edge of the fundamental tetrahedron.

The system of coordinates referred to six complexes mutually in involution was introduced by Felix Klein, and in many cases is more useful than that derived directly from point coordinates; *e.g.* in the discussion of quadratic complexes: by means of it Klein has developed an analogy between line geometry and the geometry of spheres as treated by G. Darboux and others. In fact, in that geometry a point is represented by *five* coordinates, connected by a relation of the same type as the one just mentioned when the five fundamental spheres are mutually at right angles and the equation of a sphere is of the first degree. Extending this to four dimensions of space, we obtain an exact analogue of line geometry, in which (i.) a point corresponds to a line; (ii.) a linear complex to a hypersphere; (iii.) two linear complexes in involution to two orthogonal hyperspheres; (iv.) a linear complex and two conjugate lines to a hypersphere and two inverse points. Many results may be obtained by this principle, and more still are suggested by trying to extend the properties of circles to spheres in three and four dimensions. Thus the elementary theorem, that, given four lines, the circles circumscribed to the four triangles formed by them are concurrent, may be extended to six hyperplanes in four dimensions; and then we can derive a result in line geometry by translating the inverse of this theorem. Again, just as there is an infinite number of spheres touching a surface at a given point, two of them having contact of a closer nature, so there is an infinite number of linear complexes touching a non-linear complex at a given line, and *three* of these have contact of a closer nature (cf. Klein, *Math. Ann.* v.).

Sophus Lie has pointed out a different analogy with sphere geometry. Suppose, in fact, that the equation of a sphere of radius r is

$$x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0,$$

so that $r^2 = a^2 + b^2 + c^2 - d$; then introducing the quantity e to make this equation homogeneous, we may regard the sphere as given by the six coordinates a, b, c, d, e, r connected by the equation $a^2 + b^2 + c^2 - r^2 - de = 0$, and it is easy to see that two spheres touch, if the polar form $2aa_1 + 2bb_1 + 2cc_1 - 2rr_1 - de_1 - d_1e$ vanishes. Comparing this with the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_5x_6 = 0$ given above, it appears that this sphere geometry and line geometry are identical, for we may write $a = x_1$, $b = x_2$, $c = x_3$, $r = x_4\overline{\delta - 1}$, $d = x_5$, $e = ½x_6$; but it is to be noticed that a sphere is really replaced by two lines whose coordinates only differ in the sign of x_4, so that they are polar lines with respect to the complex $x_4 = 0$. Two spheres which touch correspond to two lines which intersect, or more accurately to two pairs of lines (p, p′) and (q, q′), of which the pairs (p, q) and (p′, q′) both intersect. By this means the problem of describing a sphere to touch four given spheres is reduced to that of drawing a pair of lines (t, t′) (of which t intersects one line of the four pairs (pp′), (qq′), (rr′), (ss′), and t′ intersects the remaining four). We may, however, ignore the accented letters in translating theorems, for a configuration of lines and its polar with respect to a linear complex have the same projective properties. In Lie’s transformation a linear complex corresponds to the totality of spheres cutting a given sphere at a given angle. A most remarkable result is that lines of curvature in the sphere geometry become asymptotic lines in the line geometry.

Some of the principles of line geometry may be brought into clearer light by admitting the ideas of space of four and five dimensions.

Thus, regarding the coordinates of a line as homogeneous coordinates in five dimensions, we may say that line geometry is equivalent to geometry on a quadric surface in five dimensions. A linear complex is represented by a hyperplane section; and if two such complexes are in involution, the corresponding hyperplanes are conjugate with respect to the fundamental quadric. By projecting this quadric stereographically into space of four dimensions we obtain Klein’s analogy. In the same way geometry in a linear complex is equivalent to geometry on a quadric in four dimensions; when two lines intersect the representative points are on the same generator of this quadric. Stereographic projection, therefore, converts a curve in a linear complex, *i.e.* one whose tangents all belong to the complex, into one whose tangents intersect a fixed conic: when this conic is the imaginary circle at infinity the curve is what Lie calls a minimal curve. Curves in a linear complex have been extensively studied. The osculating plane at any point of such a curve is the nul-plane of the point with respect to the complex, and points of superosculation always coincide in pairs at the points of contact of stationary tangents. When a point of such a curve is given, the osculating plane is determined, hence all the curves through a given point with the same tangent have the same torsion.

Non-linear complexes.

The lines through a given point that belong to a complex of the nth degree lie on a cone of the nth degree: if this cone has a double line the point is said to be a singular point. Similarly, a plane is said to be singular when the envelope of the lines in it has a double tangent. It is very remarkable that the same surface is the locus of the singular points and the envelope of the singular planes: this surface is called the singular surface, and both its degree and class are in general $2n(n - 1)^2$, which is equal to four for the quadratic complex.

The singular lines of a complex F = 0 are the lines common to F and the complex

$$\frac{\delta F}{\delta l}\,\frac{\delta F}{\delta \lambda} + \frac{\delta F}{\delta m}\,\frac{\delta F}{\delta \mu} + \frac{\delta F}{\delta n}\,\frac{\delta F}{\delta \nu} = 0.$$

As already mentioned, at each line l of a complex there is an infinite number of tangent linear complexes, and they all contain the lines adjacent to l. If now l be a singular line, these complexes all reduce to straight lines which form a plane pencil containing the line l. Suppose the vertex of the pencil is A, its plane a, and one of its lines ξ, then l′ being a complex line near l, meets ξ, or more accurately the mutual moment of l′, and is of the second order of small quantities. If P be a point on l, a line through P quite near l in the plane a will meet ξ and is therefore a line of the complex; hence the complex-cones of all points on l touch a and the complex-curves of all planes through l touch l at A. It follows that

l is a double line of the complex-cone of A, and a double tangent of the complex-curve of a. Conversely, a double line of a cone or curve is a singular line, and a singular line clearly touches the curves of all planes through it in the same point. Suppose now that the consecutive line l' is also a singular line, A' being the allied singular point, a' the singular plane and ξ' any line of the pencil (A', a') so that ξ' is a tangent line at l' to the complex: the mutual moments of the pairs l', ξ and l, ξ are each of the second order; hence the plane a' meets the lines l and ξ' in two points very near A. This being true for all singular planes, near a the point of contact of a with its envelope is in A, *i.e.* the locus of singular points is the same as the envelope of singular planes. Further, when a line touches a complex it touches the singular surface, for it belongs to a plane pencil like (Aa), and thus in Klein's analogy the analogue of a focus of a hyper-surface being a bitangent line of the complex is also a bitangent line of the singular surface. The theory of cosingular complexes is thus brought into line with that of confocal surfaces in four dimensions, and guided by these principles the existence of cosingular quadratic complexes can easily be established, the analysis required being almost the same as that invented for confocal cyclides by Darboux and others. Of 723
cosingular complexes of higher degree nothing is known.

Following J. Plücker, we give an account of the lines of a quadratic complex that meet a given line.

The cones whose vertices are on the given line all pass through eight fixed points and envelop a surface of the fourth degree; the conics whose planes contain the given line all lie on a surface of the fourth class and touch eight fixed planes. It is easy to see by elementary geometry that these two surfaces are identical. Further, the given line contains four singular points A_1, A_2, A_3, A_4, and the planes into which their cones degenerate are the eight common tangent planes mentioned above; similarly, there are four singular planes, a_1, a_2, a_3, a_4, through the line, and the eight points into which their conics degenerate are the eight common points above. The locus of the pole of the line with respect to all the conics in planes through it is a straight line called the *polar line* of the given one; and through this line passes the polar plane of the given line with respect to each of the cones. The name polar is applied in the ordinary analytical sense; any line has an infinite number of polar complexes with respect to the given complex, for the equation of the latter can be written in an infinite number of ways; one of these polars is a straight line, and is the polar line already introduced. The surface on which lie all the conics through a line l is called the Plücker surface of that line: from the known properties of (2, 2) correspondences it can be shown that the Plücker surface of l cuts l_1 in a range of the same cross ratio as that of the range in which the Plücker surface of l_1 cuts l. Applying this to the case in which l_1 is the polar of l, we find that the cross ratios of (A_1, A_2, A_3, A_4) and (a_1, a_2, a_3, a_4) are equal. The identity of the locus of the A′s with the envelope of the a′s follows at once; moreover, a line meets the singular surface in four points having the same cross ratio as that of the four tangent planes drawn through the line to touch the surface. The Plücker surface has eight nodes, eight singular tangent planes, and is a double line. The relation between a line and its polar line is not a reciprocal one with respect to the complex; but W. Stahl has pointed out that the relation is reciprocal as far as the singular surface is concerned.

Quadratic complexes.

To facilitate the discussion of the general quadratic complex we introduce Klein's canonical form. We have, in fact, to deal with two quadratic equations in six variables; and by suitable linear transformations these can be reduced to the form

$$\begin{aligned} a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + a_5x_5^2 + a_6x_6^2 &= 0 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 &= 0 \end{aligned}$$

subject to certain exceptions, which will be mentioned later.

Taking the first equation to be that of the complex, we remark that both equations are unaltered by changing the sign of any coordinate; the geometrical meaning of this is, that the quadratic complex is its own reciprocal with respect to each of the six fundamental complexes, for changing the sign of a coordinate is equivalent to taking the polar of a line with respect to the corresponding fundamental complex. It is easy to establish the existence of six systems of bitangent linear complexes, for the complex $l_1x_1 + l_2x_2 + l_3x_3 + l_4x_4 + l_5x_5 + l_6x_6 = 0$ is a bitangent when

$$l_1 = 0, \text{ and } \frac{l_2^2}{a_2 - a_1} + \frac{l_3^2}{a_3 - a_1} + \frac{l_4^2}{a_4 - a_1} + \frac{l_5^2}{a_5 - a_1} + \frac{l_6^2}{a_6 - a_1} = 0,$$

and its lines of contact are conjugate lines with respect to the first fundamental complex. We therefore infer the existence of six systems of bitangent lines of the complex, of which the first is given by

$$x_1 = 0, \frac{x_2^2}{a_2 - a_1} + \frac{x_3^2}{a_3 - a_1} + \frac{x_4^2}{a_4 - a_1} + \frac{x_5^2}{a_5 - a_1} + \frac{x_6^2}{a_6 - a_1} = 0,$$

Each of these lines is a bitangent of the singular surface, which is therefore completely determined as being the focal surface of the (2, 2) congruence above. It is thence easy to verify that the two complexes $\Sigma ax^2 = 0$ and $\Sigma bx^2 = 0$ are cosingular if $b_r = a_r\lambda + \mu/a_r\nu + \rho$.

The singular surface of the general quadratic complex is the famous quartic, with sixteen nodes and sixteen singular tangent planes, first discovered by E.E. Kümmer.

We cannot give a full account of its properties here, but we deduce at once from the above that its bitangents break up into six (2, 2) congruences, and the six linear complexes containing these are mutually in involution. The nodes of the singular surface are points whose complex cones are coincident planes, and the complex conic in a singular tangent plane consists of two coincident points. This configuration of sixteen points and planes has many interesting properties; thus each plane contains six points which lie on a conic, while through each point there pass six planes which touch a quadric cone. In many respects the Kümmer quartic plays a part in three dimensions analogous to the general quartic curve in two; it further gives a natural representation of certain relations between hyperelliptic functions (cf. R.W.H.T. Hudson, *Kümmer's Quartic*, 1905).

As might be expected from the magnitude of a form in six variables, the number of projectively distinct varieties of quadratic complexes is very great; and in fact Adolf Weiler, by whom the question was first systematically studied on lines indicated by Klein, enumerated no fewer than forty-nine different types. But the principle of the classification is so important, and withal so simple, that we give a brief sketch which indicates its essential features.

Classification of quadratic complexes.

We have practically to study the intersection of two quadrics F and F′ in six variables, and to classify the different cases arising we make use of the results of Karl Weierstrass on the equivalence conditions of two pairs of quadratics. As far as at present required, they are as follows: Suppose that the factorized form of the determinantal equation Disct (F + λF′) = 0 is

$$(\lambda - \alpha)^{s_1 + s_2 + s_3 \dots} (\lambda - \beta)^{t_1 + t_2 + t_3 + \dots} \dots$$

where the root α occurs $s_1 + s_2 + s_3$ ... times in the determinant, $s_2 + s_3$ ... times in every first minor, s_3 + ... times in every second minor, and so on; the meaning of each exponent is then perfectly definite. Every factor of the type $(\lambda - \alpha)^s$ is called an *elementartheil* (elementary divisor) of the determinant, and the condition of equivalence of two pairs of quadratics is simply that their determinants have the same elementary divisors. We write the pair of forms symbolically thus $[(s_1 s_2 \dots), (t_1 t_2 \dots), \dots]$, letters in the inner brackets referring to the same factor. Returning now to the two quadratics representing the complex, the sum of the exponents will be six, and two complexes are put in the same class if they have the same symbolical expression; *i.e.* the actual values of the roots of the determinantal equation need not be the same for both, but their manner of occurrence, as far as here indicated, must be identical in the two. The enumeration of all possible cases is thus reduced to a simple question in combinatorial analysis, and the actual study of any particular case is much facilitated by a useful rule of Klein's for writing down in a simple form two quadratics belonging to a given class—one of which, of course, represents the equation connecting line coordinates, and the other the equation of the complex. The general complex is naturally [111111]; the complex of tangents to a quadric is [(111), (111)] and that of lines meeting a conic is [(222)]. Full information will be found in Weiler's memoir, *Math. Ann.* vol. vii.

The detailed study of each variety of complex opens up a vast subject; we only mention two special cases, the harmonic complex and the tetrahedral complex.

The harmonic complex, first studied by Battaglini, is generated in an infinite number of ways by the lines cutting two quadrics harmonically. Taking the most general case, and referring the quadrics to their common self-conjugate tetrahedron, we can find its equation in a simple form, and verify that this complex really depends only on seventeen constants, so that it is not the most general quadratic complex. It belongs to the general type in so far as it is discussed above, but the roots of the determinant are in involution. The singular surface is the "tetrahedroid" discussed by Cayley. As a particular case, from a metrical point of view, we have L.F. Painvin's complex generated by the lines of intersection of perpendicular tangent planes of a quadric, the singular surface now being Fresnel's wave surface. The tetrahedral or Reye complex is the simplest and best known of proper quadratic complexes. It is generated by the lines which cut the faces of a tetrahedron in a constant cross ratio, and therefore by those subtending the same cross ratio at the four vertices. The singular surface is made up of the faces or the vertices of the fundamental tetrahedron, and each edge of this tetrahedron is a double line of the complex. The complex was first discussed by K.T. Reye as the assemblage of lines joining corresponding points in a homographic transformation of space, and this point of view leads to many important and elegant properties. A (metrically) particular case of great interest is the complex generated by the normals to a family of confocal quadrics, and for many investigations it is convenient to deal with this complex referred to the principal axes. For example, Lie has developed the theory of curves in a Reye complex (*i.e.* curves whose tangents belong to the complex) as solutions of a differential equation of the form $(b - c)xdydz + (c - a)ydzdx + (a - b)zdxdy = 0$, and we can simplify this equation by a logarithmic transformation. Many theorems connecting complexes with differential equations have been given by Lie and his school. A line complex, in fact, corresponds to a Mongian equation having ∞^3 line integrals.

Congruences.

As the coordinates of a line belonging to a congruence are functions of two independent parameters, the theory of congruences is analogous to that of surfaces, and we may regard it as a fundamental inquiry to find the simplest form of surface into which a given congruence can be transformed. Most of those whose properties have been extensively discussed can be represented on a plane by a birational transformation. But in addition to the difficulties of the theory of algebraic surfaces, a subject still in its infancy, the theory of congruences has other difficulties in that a congruence is seldom completely represented, even by two equations.

A fundamental theorem is that the lines of a congruence are in general bitangents of a surface; in fact, since the condition of intersection of two consecutive straight lines is $ld\lambda + dmd\mu + dnd\nu = 0$, a line l of the congruence meets two adjacent lines, say l_1 and l_2. Suppose l, l_1 lie in the plane pencil (A_1a_1) and l, l_2 in the plane pencil (A_2a_2), then the locus of the A′s is the same as the envelope of the a′s, but a_2 is the tangent plane at A_1 and a_1 at A_2. This surface is called the focal surface of the congruence, and to it all the lines l are bitangent. The distinctive property of the points A is that two of the congruence lines through them coincide, and in like manner the planes a each contain two coincident lines. The focal surface consists of two sheets, but one or both may degenerate into curves; thus, for 724
example, the normals to a surface are bitangents of the surface of centres, and in the case of Dupin’s cyclide this surface degenerates into two conics.

In the discussion of congruences it soon becomes necessary to introduce another number r, called the rank, which expresses the number of plane pencils each of which contains an arbitrary line and two lines of the congruence. The order of the focal surface is $2m(n - 1) - 2r$, and its class is $m(m - 1) - 2r$. Our knowledge of congruences is almost exclusively confined to those in which either m or n does not exceed two. We give a brief account of those of the second order without singular lines, those of order unity not being especially interesting. A congruence generally has singular points through which an infinite number of lines pass; a singular point is said to be of order r when the lines through it lie on a cone of the rth degree. By means of formulae connecting the number of singular points and their orders with the class m of quadratic congruence Kümmer proved that the class cannot exceed seven. The focal surface is of degree four and class 2m; this kind of quartic surface has been extensively studied by Kümmer, Cayley, Rohn and others. The varieties (2, 2), (2, 3), (2, 4), (2, 5) all belong to at least one Reye complex; and so also does the most important class of (2, 6) congruences which includes all the above as special cases. The congruence (2, 2) belongs to a linear complex and forty different Reye complexes; as above remarked, the singular surface is Kümmer’s sixteen-nodal quartic, and the same surface is focal for six different congruences of this variety. The theory of (2, 2) congruences is completely analogous to that of the surfaces called cyclides in three dimensions. Further particulars regarding quadratic congruences will be found in Kümmer’s memoir of 1866, and the second volume of Sturm’s treatise. The properties of quadratic congruences having singular lines, *i.e.* degenerate focal surfaces, are not so interesting as those of the above class; they have been discussed by Kümmer, Sturm and others.

Ruled surfaces.

Since a ruled surface contains only ∞^1 elements, this theory is practically the same as that of curves. If a linear complex contains more than n generators of a ruled surface of the nth degree, it contains all the generators, hence for $n = 2$ there are three linearly independent complexes, containing all the generators, and this is a well-known property of quadric surfaces. In ruled cubics the generators all meet two lines which may or may not coincide; these two cases correspond to the two main classes of cubics discussed by Cayley and Cremona. As regards ruled quartics, the generators must lie in one and may lie in two linear complexes. The first class is equivalent to a quartic in four dimensions and is always rational, but the latter class has to be subdivided into the elliptic and the rational, just like twisted quartic curves. A quintic skew may not lie in a linear complex, and then it is unicursal, while of sextics we have two classes not in a linear complex, viz. the elliptic variety, having thirty-six places where a linear complex contains six consecutive generators, and the rational, having six such places.

The general theory of skews in two linear complexes is identical with that of curves on a quadric in three dimensions and is known. But for skews lying in only one linear complex there are difficulties; the curve now lies in four dimensions, and we represent it in three by stereographic projection as a curve meeting a given plane in n points on a conic. To find the maximum deficiency for a given degree would probably be difficult, but as far as degree eight the space-curve theory of Halphen and Nöther can be translated into line geometry at once. When the skew does not lie in a linear complex at all the theory is more difficult still, and the general theory clearly cannot advance until further progress is made in the study of twisted curves.

References.—The earliest works of a general nature are Plücker, *Neue Geometrie des Raumes* (Leipzig, 1868); and Kümmer, “Über die algebraischen Strahlensysteme,” *Berlin Academy* (1866). Systematic development on purely synthetic lines will be found in the three volumes of Sturm, *Liniengeometrie* (Leipzig, 1892, 1893, 1896); vol. i. deals with the linear and Reye complexes, vols. ii. and iii. with quadratic congruences and complexes respectively. For a highly suggestive review by Gino Loria see *Bulletin des sciences mathématiques* (1893, 1897). A shorter treatise, giving a very interesting account of Klein’s coordinates, is the work of Koenigs, *La Géométrie réglée et ses applications* (Paris, 1898). English treatises are C.M. Jessop, *Treatise on the Line Complex* (1903); R.W.H.T. Hudson, *Kümmer’s Quartic* (1905). Many references to memoirs on line geometry will be found in Hagen, *Synopsis der höheren Mathematik*, ii. (Berlin, 1894); Loria, *Il passato ed il presente delle principali teorie geometriche* (Milan, 1897); a clear résumé of the principal results is contained in the very elegant volume of Pascal, *Repertorio di mathematiche superiori*, ii. (Milan, 1900). Another treatise dealing extensively with line geometry is Lie, *Geometrie der Berührungstransformationen* (Leipzig, 1896). Many memoirs on the subject have appeared in the *Mathematische Annalen*; a full list of these will be found in the index to the first fifty volumes, p. 115. Perhaps the two memoirs which have left most impression on the subsequent development of the subject are Klein, “Zur Theorie der Liniencomplexe des ersten und zweiten Grades,” *Math. Ann.* ii.; and Lie, “Über Complexe, insbesondere Linien- und Kugelcomplexe,” *Math. Ann.* v.

(J. H. Gr.)

VI. Non-Euclidean Geometry

The various metrical geometries are concerned with the properties of the various types of congruence-groups, which are defined in the study of the *axioms* of *geometry* and of their immediate consequences. But this point of view of the subject is the outcome of recent research, and historically the subject has a different origin. Non-Euclidean geometry arose from the discussion, extending from the Greek period to the present day, of the various assumptions which are implicit in the traditional Euclidean system of geometry. In the course of these investigations it became evident that metrical geometries, each internally consistent but inconsistent in many respects with each other and with the Euclidean system, could be developed. A short historical sketch will explain this origin of the subject, and describe the famous and interesting progress of thought on the subject. But previously a description of the chief characteristic properties of elliptic and of hyperbolic geometries will be given, assuming the standpoint arrived at below under VII. *Axioms of Geometry*.

First assume the equation to the absolute (cf. *loc. cit.*) to be $w^2 - x^2 - y^2 - z^2 = 0$. The absolute is then real, and the geometry is hyberbolic.

The distance (d_{12}) between the two points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) is given by

$$\cosh (d_{12}/\gamma) = (w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2) / \{(w_1^2 - x_1^2 - y_1^2 - z_1^2) (w_2^2 - x_2^2 - y_2^2 - z_2^2)\}^{1/2} \quad (1)$$

The only points to which the metrical geometry applies are those within the region enclosed by the quadric; the other points are “improper ideal points.” The angle (θ_{12}) between two planes, $l_1x + m_1y + n_1z + r_1w = 0$ and $l_2x + m_2y + n_2z + r_2w = 0$, is given by

$$\cos \theta_{12} = (l_1l_2 + m_1m_2 + n_1n_2 - r_1r_2) / \{(l_1^2 + m_1^2 + n_1^2 - r_1^2) (l_2^2 + m_2^2 + n_2^2 - r_2^2)\}^{1/2} \quad (2)$$

These planes only have a real angle of inclination if they possess a line of intersection within the actual space, *i.e.* if they intersect. Planes which do not intersect possess a shortest distance along a line which is perpendicular to both of them. If this shortest distance is δ_{12}, we have

$$\cosh (\delta_{12}/\gamma) = (l_1l_2 + m_1m_2 + n_1n_2 - r_1r_2) / \{(l_1^2 + m_1^2 + n_1^2 - r_1^2) (l_2^2 + m_2^2 + n_2^2 - r_2^2)\}^{1/2} \quad (3)$$

Thus in the case of the two planes one and only one of the two, θ12 and δ_{12}, is real. The same considerations hold for coplanar straight lines (see VII. *Axioms of Geometry*). Let O (fig. 67) be the point (0, 0, 0, 1), OX the line $y = 0$, $z = 0$, OY the line $z = 0$, $x = 0$, and OZ the line $x = 0$, $y = 0$. These are the coordinate axes and are at right angles to each other. Let P be any point, and let ρ be the distance OP, θ the angle POZ, and φ the angle between the planes ZOX and ZOP. Then the coordinates of P can be taken to be

$$\sinh(\rho/\gamma)\sin\theta\cos\varphi,\ \sinh(\rho/\gamma)\sin\theta\sin\varphi,\ \sinh(\rho/\gamma)\cos\theta,\ \cosh(\rho/\gamma).$$

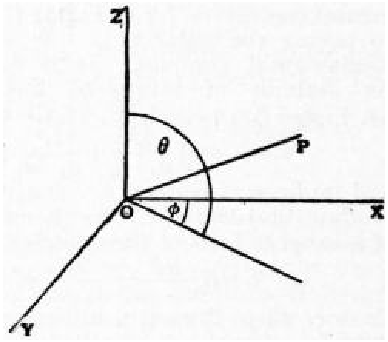


FIG. 67.

If ABC is a triangle, and the sides and angles are named according to the usual convention, we have

$$\sinh(a/\gamma) / \sin A = \sinh(b/\gamma) / \sin B = \sinh(c/\gamma) / \sin C, \tag{4}$$

and also

$$\cosh(a/\gamma) = \cosh(b/\gamma)\cosh(c/\gamma) - \sinh(b/\gamma)\sinh(c/\gamma)\cos A, \tag{5}$$

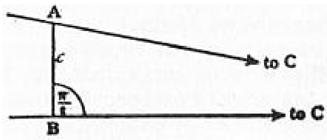


FIG. 68.

with two similar equations. The sum of the three angles of a triangle is always less than two right angles. The area of the triangle ABC is $\lambda^2(\pi - A - B - C)$. If the base BC of a triangle is kept fixed and the vertex A moves in the fixed plane ABC so that the area ABC is constant, then the locus of A is a line of equal distance from BC. This locus is not a straight line. The whole theory of similarity is inapplicable; two triangles are either congruent, or their angles are not equal two by two. Thus the elements of a triangle are determined when its three angles are given. By keeping A and B and the line BC fixed, but by making C move off to infinity along BC, the lines BC and AC become parallel, and the sides a and b become infinite. Hence from equation (5) above, it follows that two parallel lines (cf. Section VII. *Axioms of Geometry*) must be considered as making a zero angle with each other. Also if B be a right angle, from the equation (5), remembering that, in the limit,

$$\cosh(a/\gamma) / \cosh(b/\gamma) = \cosh(a/\gamma) / \sinh(b/\gamma) = 1,$$

we have 725

$$\cos A = \tanh(c/2\gamma) \tag{6}.$$

The angle A is called by N.I. Lobatchewsky the “angle of parallelism.”

The whole theory of lines and planes at right angles to each other is simply the theory of conjugate elements with respect to the absolute, where ideal lines and planes are introduced.

Thus if l and l′ be any two conjugate lines with respect to the absolute (of which one of the two must be improper, say l′), then any plane through l′ and containing proper points is perpendicular to l. Also if p is any plane containing proper points, and P is its pole, which is necessarily improper, then the lines through P are the normals to P. The equation of the sphere, centre (x_1, y_1, z_1, w_1) and radius ρ, is

$$(w_1^2 - x_1^2 - y_1^2 - z_1^2)(w^2 - x^2 - y^2 - z^2)\cosh^2(\rho/\gamma) = (w_1w - x_1x - y_1y - z_1z)^2 \tag{7}.$$

The equation of the surface of equal distance (σ) from the plane $lx + my + nz + rw = 0$ is

$$(l^2 + m^2 + n^2 - r^2)(w^2 - x^2 - y^2 - z^2)\sinh^2(\sigma/\gamma) = (rw + lx + my + nz)^2 \tag{8}.$$

A surface of equal distance is a sphere whose centre is improper; and both types of surface are included in the family

$$k^2(w^2 - x^2 - y^2 - z^2) = (ax + by + cz + dw)^2 \tag{9}.$$

But this family also includes a third type of surfaces, which can be looked on either as the limits of spheres whose centres have approached the absolute, or as the limits of surfaces of equal distance whose central planes have approached a position tangential to the absolute. These surfaces are called limit-surfaces. Thus (9) denotes a limit-surface, if $d^2 - a^2 - b^2 - c^2 = 0$. Two limit-surfaces only differ in position. Thus the two limit-surfaces which touch the plane YOZ at O, but have their concavities turned in opposite directions, have as their equations

$$w^2 - x^2 - y^2 - z^2 = (w \pm x)^2.$$

The geodesic geometry of a sphere is elliptic, that of a surface of equal distance is hyperbolic, and that of a limit-surface is parabolic (*i.e. Euclidean*). The equation of the surface (cylinder) of equal distance (δ) from the line OX is

$$(w^2 - x^2) \tanh^2 (\delta/\gamma) - y^2 - z^2 = 0.$$

This is not a ruled surface. Hence in this geometry it is not possible for two straight lines to be at a constant distance from each other.

Secondly, let the equation of the absolute be $x^2 + y^2 + z^2 + w^2 = 0$. The absolute is now imaginary and the geometry is elliptic.

The distance (d_{12}) between the two points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) is given by

$$\cos (d_{12}/\gamma) = \pm (x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2) / \{(x_1^2 + y_1^2 + z_1^2 + w_1^2) (x_2^2 + y_2^2 + z_2^2 + w_2^2)\}^{1/2} \quad (10).$$

Thus there are two distances between the points, and if one is d_{12}, the other is $\pi\gamma$-d_{12}. Every straight line returns into itself, forming a closed series. Thus there are two segments between any two points, together forming the whole line which contains them; one distance is associated with one segment, and the other distance with the other segment. The complete length of every straight line is $\pi\gamma$.

The angle between the two planes $l_1x + m_1y + n_1z + r + {}_1w = 0$ and $l_2x + m_2y + n_2z + r_2w = 0$ is

$$\cos \theta_{12} = (l_1l_2 + m_1m_2 + n_1n_2 + r_1r_2) / \{(l_1^2 + m_1^2 + n_1^2 + r_1^2) (l_2^2 + m_2^2 + n_2^2 + r_2^2)\}^{1/2} \quad (11).$$

The polar plane with respect to the absolute of the point (x_1, y_1, z_1, w_1) is the real plane $x_1x + y_1y + z_1z + w_1w = 0$, and the pole of the plane $l_1x + m_1y + n_1z + r_1w = 0$ is the point (l_1, m_1, n_1, r_1). Thus (from equations 10 and 11) it follows that the angle between the polar planes of the points (x_1, ...) and (x_2, ...) is d_{12}/γ, and that the distance between the poles of the planes (l_1, ...) and (l_2, ...) is $\gamma\theta_{12}$. Thus there is complete reciprocity between points and planes in respect to all properties. This complete reign of the principle of duality is one of the great beauties of this geometry. The theory of lines and planes at right angles is simply the theory of conjugate elements with respect to the absolute. A tetrahedron self-conjugate with respect to the absolute has all its intersecting elements (edges and planes) at right angles. If l and l′ are two conjugate lines, the planes through one are the planes perpendicular to the other. If P is the pole of the plane p, the lines through P are the normals to the plane p. The distance from P to p is ½$\pi\gamma$. Thus every sphere is also a surface of equal distance from the polar of its centre, and conversely. A plane does not divide space; for the line joining any two points P and Q only cuts the plane once, in L say, then it is always possible to go from P to Q by the segment of the line PQ which does not contain L. But P and Q may be said to be separated by a plane p, if the point in which PQ cuts p lies on the shortest segment between P and Q. With this sense of “separation,” it is possible[2] to find three points P, Q, R such that P and Q are separated by the plane p, but P and R are not separated by p, nor are Q and R.

Let A, B, C be any three non-collinear points, then four triangles are defined by these points. Thus if a, b, c and A, B, C are the elements of any one triangle, then the four triangles have as their elements:

(1)	a,	b,	c,	A,	B,	C.
(2)	a,	$\pi\gamma - b$,	$\pi\gamma - c$,	A,	$\pi - B$,	$\pi - C$.
(3)	$\pi\gamma - a$,	b,	$\pi\gamma - c$,	$\pi - A$,	B,	$\pi - C$.
(4)	$\pi\gamma - a$,	$\pi\gamma - b$,	c,	$\pi - A$,	$\pi - B$,	C.

The formulae connecting the elements are

$$\sin A/\sin (a/\gamma) = \sin B/\sin (b/\gamma) = \sin C/\sin (c/\gamma), \quad (12)$$

and

$$\cos (a/\gamma) = \cos (b/\gamma) \cos (c/\gamma) + \sin (b/\gamma) \sin (c/\gamma) \cos A, \quad (13)$$

with two similar equations.

Two cases arise, namely (I.) according as one of the four triangles has as its sides the shortest segments between the angular points, or (II.) according as this is not the case. When case I. holds there is said to be a “principal triangle.”[3] If all the figures considered lie within a sphere of radius $\frac{1}{4}\pi\gamma$ only case I. can hold, and the principal triangle is the triangle wholly within this sphere, also the peculiarities in respect to the separation of points by a plane cannot then arise. The sum of the three angles of a triangle ABC is always greater than two right angles, and the area of the triangle is $\gamma^2(A + B + C - \pi)$. Thus as in hyperbolic geometry the theory of similarity does not hold, and the elements of a triangle are determined when its three angles are given. The coordinates of a point can be written in the form

$$\sin(\rho/\gamma) \sin\Phi \cos\varphi,\ \sin(\rho/\gamma) \sin\Phi \sin\varphi,\ \sin(\rho/\gamma) \cos\Phi,\ \cos(\rho/\gamma),$$

where ρ, Φ and φ have the same meanings as in the corresponding formulae in hyperbolic geometry. Again, suppose a watch is laid on the plane OXY, face upwards with its centre at O, and the line 12 to 6 (as marked on dial) along the line YOY. Let the watch be continually pushed along the plane along the line OX, that is, in the direction 9 to 3. Then the line XOX being of finite length, the watch will return to O, but at its first return it will be found to be face downwards on the other side of the plane, with the line 12 to 6 reversed in direction along the line YOY. This peculiarity was first pointed out by Felix Klein. The theory of parallels as it exists in hyperbolic space has no application in elliptic geometry. But another property of Euclidean parallel lines holds in elliptic geometry, and by the use of it parallel lines are defined. For the equation of the surface (cylinder) of equal distance (δ) from the line XOX is

$$(x^2 + w^2) \tan^2(\delta/\gamma) - (y^2 + z^2) = 0.$$

This is also the surface of equal distance, $\frac{1}{2}\pi\gamma$-δ, from the line conjugate to XOX. Now from the form of the above equation this is a ruled surface, and through every point of it two generators pass. But these generators are lines of equal distance from XOX. Thus throughout every point of space two lines can be drawn which are lines of equal distance from a given line l. This property was discovered by W.K. Clifford. The two lines are called Clifford’s right and left parallels to l through the point. This property of parallelism is reciprocal, so that if m is a left parallel to l, then l is a left parallel to m. Note also that two parallel lines l and m are not coplanar. Many of those properties of Euclidean parallels, which do not hold for Lobatchewsky’s parallels in hyperbolic geometry, do hold for Clifford’s parallels in elliptic geometry. The geodesic geometry of spheres is elliptic, the geodesic geometry of surfaces of equal distance from lines (cylinders) is Euclidean, and surfaces of revolution can be found[4] of which the geodesic geometry is hyperbolic. But it is to be noticed that the connectivity of these surfaces is different to that of a Euclidean plane. For instance there are only ∞^2 congruence transformations of the cylindrical surfaces of equal distance into themselves, instead of the ∞^3 for the ordinary plane. It would obviously be possible to state “axioms” which these geodesics satisfy, and thus to define independently, and not as loci, quasi-spaces of these peculiar types. The existence of such Euclidean quasi-geometries was first pointed out by Clifford.[5]

In both elliptic and hyperbolic geometry the spherical geometry, *i.e.* the relations between the angles formed by lines and planes passing through the same point, is the same as the “spherical trigonometry” in Euclidean geometry. The constant γ, which appears in the formulae both of hyperbolic and elliptic geometry, does not by its variation produce different types of geometry. There is only one type of elliptic geometry and one type of hyperbolic geometry; and the magnitude of the constant γ in each case simply depends upon the magnitude of the arbitrary unit of length in comparison with the natural unit of length which 726
each particular instance of either geometry presents. The existence of a natural unit of length is a peculiarity common both to hyperbolic and elliptic geometries, and differentiates them from Euclidean geometry. It is the reason for the failure of the theory of similarity in them. If γ is very large, that is, if the natural unit is very large compared to the arbitrary unit, and if the lengths involved in the figures considered are not large compared to the arbitrary unit, then both the elliptic and hyperbolic geometries approximate to the Euclidean. For from formulae (4) and (5) and also from (12) and (13) we find, after retaining only the lowest powers of small quantities, as the formulae for any triangle ABC,

$$a / \sin A = b / \sin B = c / \sin C,$$

and

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

with two similar equations. Thus the geometries of small figures are in both types Euclidean.

History.—"In pulcherrimo Geometriae corpore," wrote Sir Henry Savile in 1621, "duo sunt naevi, duae labes nec quod sciam plures, in quibus eluendis et emaculendis cum veterum tum recentiorum ... vigilavit industria." These two blemishes are the theory of parallels and the theory of proportion. The "industry of the moderns," in both respects, has given rise to important branches of mathematics, while at the same time showing that Euclid is in these respects more free from blemish than had been previously credible. It was from endeavours to improve the theory of parallels that non-Euclidean geometry arose; and though it has now acquired a far wider scope, its historical origin remains instructive and interesting. Euclid's "axiom of parallels" appears as Postulate V. to the first book of his *Elements*, and is stated thus, "And that, if a straight line falling on two straight lines make the angles, internal and on the same side, less than two right angles, the two straight lines, being produced indefinitely, meet on the side on which are the angles less than two right angles." The original Greek is καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Theory of parallels before Gauss.

To Euclid's successors this axiom had signally failed to appear self-evident, and had failed equally to appear indemonstrable. Without the use of the postulate its converse is proved in Euclid's 28th proposition, and it was hoped that by further efforts the postulate itself could be also proved. The first step consisted in the discovery of equivalent axioms. Christoph Clavius in 1574 deduced the axiom from the assumption that a line whose points are all equidistant from a straight line is itself straight. John Wallis in 1663 showed that the postulate follows from the possibility of similar triangles on different scales. Girolamo Saccheri (1733) showed that it is sufficient to have a single triangle, the sum of whose angles is two right angles. Other equivalent forms may be obtained, but none shows any essential superiority to Euclid's. Indeed plausibility, which is chiefly aimed at, becomes a positive demerit where it conceals a real assumption.

A new method, which, though it failed to lead to the desired goal, proved in the end immensely fruitful, was invented by Saccheri, in a work entitled *Euclides ab omni naevo vindicatus* (Milan, 1733). If the postulate of parallels is involved in Euclid's other assumptions, contradictions must emerge when it is denied while the others are maintained. This led Saccheri to attempt a *reductio ad absurdum*, in which he mistakenly believed himself to have succeeded. What is interesting, however, is not his fallacious conclusion, but the non-Euclidean results which he obtains in the process. Saccheri distinguishes three hypotheses (corresponding to what are now known as Euclidean or parabolic, elliptic and hyperbolic geometry), and proves that some one of the three must be universally true. His three hypotheses are thus obtained: equal perpendiculars AC, BD are drawn from a straight line AB, and CD are joined. It is shown that the angles ACD, BDC are equal. The first hypothesis is that these are both right angles; the second, that they are both obtuse; and the third, that they are both acute. Many of the results afterwards obtained by Lobatchewsky and Bolyai are here developed. Saccheri fails to be the founder of non-Euclidean geometry only because he does not perceive the possible truth of his non-Euclidean hypotheses.

Saccheri.

Some advance is made by Johann Heinrich Lambert in his *Theorie der Parallellinien* (written 1766; posthumously published 1786). Though he still believed in the necessary truth of Euclidean geometry, he confessed that, in all his attempted proofs, something remained undemonstrated. He deals with the same three hypotheses as Saccheri, showing that the second holds on a sphere, while the third would hold on a sphere of purely imaginary radius. The second hypothesis he succeeds in condemning, since, like all who preceded Bernhard Riemann, he is unable to conceive of the straight line as finite and closed. But the third hypothesis, which is the same as Lobatchewsky's, is not even professedly refuted.[6]

Lambert.

Non-Euclidean geometry proper begins with Karl Friedrich Gauss. The advance which he made was rather philosophical than mathematical: it was he (probably) who first recognized that the postulate of parallels is possibly false, and should be empirically tested by measuring the angles of large triangles. The history of non-Euclidean geometry has been aptly divided by Felix Klein into three very distinct periods. The first—which contains only Gauss, Lobatchewsky and Bolyai—is characterized by its synthetic method and by its close relation to Euclid. The attempt at indirect proof of the disputed postulate would seem to have been the source of these three men's discoveries; but when the postulate had been denied, they found that the results, so far from showing contradictions,

Three periods of non-Euclidean geometry.

were just as self-consistent as Euclid. They inferred that the postulate, if true at all, can only be proved by observations and measurements. Only one kind of non-Euclidean space is known to them, namely, that which is now called hyperbolic. The second period is analytical, and is characterized by a close relation to the theory of surfaces. It begins with Riemann's inaugural dissertation, which regards space as a particular case of a *manifold*; but the characteristic standpoint of the period is chiefly emphasized by Eugenio Beltrami. The conception of measure of curvature is extended by Riemann from surfaces to spaces, and a new kind of space, finite but unbounded (corresponding to the second hypothesis of Saccheri and Lambert), is shown to be possible. As opposed to the second period, which is purely metrical, the third period is essentially projective in its method. It begins with Arthur Cayley, who showed that metrical properties are projective properties relative to a certain fundamental quadric, and that different geometries arise according as this quadric is real, imaginary or degenerate. Klein, to whom the development of Cayley's work is due, showed further that there are two forms of Riemann's space, called by him the elliptic and the spherical. Finally, it has been shown by Sophus Lie, that if figures are to be freely movable throughout all space in ∞^6 ways, no other three-dimensional spaces than the above four are possible.

Gauss.

Gauss published nothing on the theory of parallels, and it was not generally known until after his death that he had interested himself in that theory from a very early date. In 1799 he announces that Euclidean geometry would follow from the assumption that a triangle can be drawn greater than any given triangle. Though unwilling to assume this, we find him in 1804 still hoping to prove the postulate of parallels. In 1830 he announces his conviction that geometry is not an a priori science; in the following year he explains that non-Euclidean geometry is free from contradictions, and that, in this system, the angles of a triangle diminish without limit when all the sides are increased. He also gives for the circumference of a circle of radius r the 727
formula $\pi k(e^{r/k} - e^{-r/k})$, where k is a constant depending upon the nature of the space. In 1832, in reply to the receipt of Bolyai's *Appendix*, he gives an elegant proof that the amount by which the sum of the angles of a triangle falls short of two right angles is proportional to the area of the triangle. From these and a few other remarks it appears that Gauss possessed the foundations of hyperbolic geometry, which he was probably the first to regard as perhaps true. It is not known with certainty whether he influenced Lobatchewsky and Bolyai, but the evidence we possess is against such a view.[7]

Lobatchewsky.

The first to publish a non-Euclidean geometry was Nicholas Lobatchewsky, professor of mathematics in the new university of Kazañ.[8] In the place of the disputed postulate he puts the following: "All straight lines which, in a plane, radiate from a given point, can, with respect to any other straight line in the same plane, be divided into two classes, the *intersecting* and the *non-intersecting*. The *boundary line* of the one and the other class is called *parallel to the given line*." It follows that there are two parallels to the given line through any point, each meeting the line at infinity, like a Euclidean parallel. (Hence a line has two distinct points at infinity, and not one only as in ordinary geometry.) The two parallels to a line through a point make equal acute angles with the perpendicular to the line through the point. If p be the length of the perpendicular, either of these angles is denoted by $\Pi(p)$. The determination of $\Pi(p)$ is the chief problem (cf. equation (6) above); it appears finally that, with a suitable choice of the unit of length,

$$\tan \tfrac{1}{2}\,\Pi(p) = e^{-p}.$$

Before obtaining this result it is shown that spherical trigonometry is unchanged, and that the normals to a circle or a sphere still pass through its centre. When the radius of the circle or sphere becomes infinite all these normals become parallel, but the circle or sphere does not become a straight line or plane. It becomes what Lobatchewsky calls a limit-line or limit-surface. The geometry on such a surface is shown to be Euclidean, limit-lines replacing Euclidean straight lines. (It is, in fact, a surface of zero measure of curvature.) By the help of these propositions Lobatchewsky obtains the above value of $\Pi(p)$, and thence the solution of triangles. He points out that his formulae result from those of spherical trigonometry by substituting ia, ib, ic, for the sides a, b, c.

Bolyai.

John Bolyai, a Hungarian, obtained results closely corresponding to those of Lobatchewsky. These he published in an appendix to a work by his father, entitled *Appendix Scientiam spatii absolute veram exhibens: a veritate aut falsitate Axiomatis XI. Euclidei* (*a priori haud unquam decidenda*) *independentem: adjecta ad casum falsitatis, quadratura circuli geometrica*.[9] This work was published in

1831, but its conception dates from 1823. It reveals a profounder appreciation of the importance of the new ideas, but otherwise differs little from Lobatchewsky’s. Both men point out that Euclidean geometry as a limiting case of their own more general system, that the geometry of very small spaces is always approximately Euclidean, that no a priori grounds exist for a decision, and that observation can only give an approximate answer. Bolyai gives also, as his title indicates, a geometrical construction, in hyperbolic space, for the quadrature of the circle, and shows that the area of the greatest possible triangle, which has all its sides parallel and all its angles zero, is $\pi\iota^2$, where i is what we should now call the space-constant.

The works of Lobatchewsky and Bolyai, though known and valued by Gauss, remained obscure and ineffective until, in 1866, they were translated into French by J. Hoüel. But at this time Riemann’s dissertation, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*,[10] was already about to be published. In this work Riemann, without any knowledge of his predecessors in the same field, inaugurated a far more profound discussion, based on a far more general standpoint; and by its publication in 1867 the attention of mathematicians and philosophers was at last secured. (The dissertation dates from 1854, but owing to changes which Riemann wished to make in it, it remained unpublished until after his death.)

Riemann.

Riemann’s work contains two fundamental conceptions, that of a manifold and that of the *measure of curvature* of a continuous manifold possessed of what he calls flatness in the smallest parts. By means of these conceptions space is made to appear at the end of a gradual series of more and more specialized conceptions. Conceptions of magnitude, he explains, are only possible where we have a general conception capable of determination in various ways. The manifold consists of all these various determinations, each of which is an element of the manifold. The passage from one element to another may be discrete or continuous; the manifold is called discrete or continuous accordingly. Where it is discrete two portions of it can be compared, as to magnitude, by counting; where continuous, by measurement. But measurement demands superposition, and consequently some magnitude independent of its place in the manifold. In passing, in a continuous manifold, from one element to another in a determinate way, we pass through a series of intermediate terms, which form a one-dimensional manifold. If this whole manifold be similarly caused to pass over into another, each of its elements passes through a one-dimensional manifold, and thus on the whole a two-dimensional manifold is generated. In this way we can proceed to n dimensions. Conversely, a manifold of n dimensions can be analysed into one of one dimension and one of $(n - 1)$ dimensions. By repetitions of this process the position of an element may be at last determined by n magnitudes. We may here stop to observe that the above conception of a manifold is akin to that due to Hermann Grassmann in the first edition (1847) of his *Ausdehnungslehre*.[11]

Definition of a manifold.

Both concepts have been elaborated and superseded by the modern procedure in respect to the axioms of geometry, and by the conception of abstract geometry involved therein. Riemann proceeds to specialize the manifold by considerations as to measurement. If measurement is to be possible, some magnitude, we saw, must be independent of position; let us consider manifolds in which lengths of lines are such magnitudes, so that every line is measurable by every other. The coordinates of a point being x_1, x_2, ... x_n, let us confine ourselves to lines along which the ratios $dx_1 : dx_2 : ... : dx_n$ alter continuously. Let us also assume that the element of length, ds, is unchanged (to the first order) when all its points undergo the same infinitesimal motion. Then if all the increments dx be altered in the same ratio, ds is also altered in this ratio. Hence ds is a homogeneous function of the first degree of the increments dx. Moreover, ds must be unchanged when all the dx change sign. The simplest possible case is, therefore, that in which ds is the square root of a quadratic function of the dx. This case includes space, and is alone considered in what follows. It is called the case of flatness in the smallest parts. Its further discussion depends upon the measure of curvature, the second of Riemann’s fundamental conceptions. This conception, derived from the theory of surfaces, is applied as follows. Any one of the shortest lines which issue from a given point (say the origin) is completely determined by the initial ratios of the dx. Two such lines, defined by dx and δx say, determine a pencil, or one-dimensional series, of shortest lines, any one of which is defined by $\lambda dx + \mu\delta x$, where the parameter $\lambda : \mu$ may have any value. 728 This pencil generates a two-dimensional series of points, which may be regarded as a surface, and for which we may apply Gauss’s formula for the measure of curvature at any point. Thus at every point of our manifold there is a measure of curvature corresponding to every such pencil; but all these can be found when $n\cdot\overline{n - 1}/2$ of them are known. If figures

Measure of curvature.

are to be freely movable, it is necessary and sufficient that the measure of curvature should be the same for all points and all directions at each point. Where this is the case, if α be the measure of curvature, the linear element can be put into the form

$$ds = \sqrt{(\Sigma dx^2)} / (1 + \tfrac{1}{4}\alpha\Sigma x^2).$$

If α be positive, space is finite, though still unbounded, and every straight line is closed—a possibility first recognized by Riemann. It is pointed out that, since the possible values of a form a continuous series, observations cannot prove that our space is strictly Euclidean. It is also regarded as possible that, in the infinitesimal, the measure of curvature of our space should be variable.

There are four points in which this profound and epoch-making work is open to criticism or development—(1) the idea of a manifold requires more precise determination; (2) the introduction of coordinates is entirely unexplained and the requisite presuppositions are unanalysed; (3) the assumption that ds is the square root of a quadratic function of dx_1, dx_2, ... is arbitrary; (4) the idea of superposition, or congruence, is not adequately analysed. The modern solution of these difficulties is properly considered in connexion with the general subject of the axioms of geometry.

The publication of Riemann's dissertation was closely followed by two works of Hermann von Helmholtz,[12] again undertaken in ignorance of the work of predecessors. In these a proof is attempted that ds must be a rational integral quadratic function of the increments of the coordinates. This proof has since been shown by Lie to stand in need of correction (see VII. *Axioms of Geometry*). Helmholtz's remaining works on the subject[13] are of almost exclusively philosophical interest. We shall return to them later.

Helmholtz.

The only other writer of importance in the second period is Eugenio Beltrami, by whom Riemann's work was brought into connexion with that of Lobatchewsky and Bolyai. As he gave, by an elegant method, a convenient Euclidean interpretation of hyperbolic plane geometry, his results will be stated at some length[14]. The *Saggio* shows that Lobatchewsky's plane geometry holds in Euclidean geometry on surfaces of constant negative curvature, straight lines being replaced by geodesics. Such surfaces are capable of a conformal representation on a plane, by which geodesics are represented by straight lines. Hence if we take, as coordinates on the surface, the Cartesian coordinates of corresponding points on the plane, the geodesics must have linear equations.

Beltrami.

Hence it follows that

$$ds^2 = R^2w^{-4} \{(\alpha^2 - v^2)\, du^2 + 2uvdudv + (\alpha^2 - u^2)dv^2\}$$

where $w^2 = \alpha^2 - u^2 - v^2$, and $-1/R^2$ is the measure of curvature of our surface (note that k = γ as used above). The angle between two geodesics u = const., v = const. is θ, where

$$\cos\theta = uv / \sqrt{\{(\alpha^2 - u^2)(\alpha^2 - v^2)\}},\ \sin\theta = aw / \sqrt{\{(a^2 - u^2)(a^2 - v^2)\}}.$$

Thus u = 0 is orthogonal to all geodesies v = const., and vice versa. In order that sin θ may be real, w^2 must be positive; thus geodesics have no real intersection when the corresponding straight lines intersect outside the circle $u^2 + v^2 = \alpha^2$. When they intersect on this circle, θ = 0. Thus Lobatchewsky's parallels are represented by straight lines intersecting on the circle. Again, transforming to polar coordinates u = r cos μ, v = r sin μ, and calling ρ the geodesic distance of u, v from the origin, we have, for a geodesic through the origin,

$$d\rho = Radr / (a^2 - r^2),\ \rho = \tfrac{1}{2}R \log \frac{a + r}{a - r},\ r = a \tan h\, (\rho / R).$$

Thus points on the surface corresponding to points in the plane on the limiting circle r = a, are all at an infinite distance from the origin. Again, considering r constant, the arc of a geodesic circle subtending an angle μ at the origin is

$$\sigma = Rr\mu / \sqrt{(a^2 - r^2)} = \mu R \sin h\, (\rho/R),$$

whence the circumference of a circle of radius ρ is 2πR sin h (ρ/R). Again, if α be the angle between any two geodesics

$$V - v = m\,(U - u),\ V - v = n\,(U - u),$$

then

$$\tan\alpha = a\,(n - m)w / \{(1 + mn)a^2 - (v - mu)\,(v - nu)\}.$$

Thus α is imaginary when u, v is outside the limiting circle, and is zero when, and only when, u, v is on the limiting circle. All these results agree with those of Lobatchewsky and Bolyai. The maximum triangle, whose angles are all zero, is represented in the auxiliary plane by a triangle inscribed in the limiting circle. The angle of parallelism is also easily obtained. The perpendicular to $v = 0$ at a distance δ from the origin is $u = a \tan h\,(\delta/R)$, and the parallel to this through the origin is $u = v \sin h\,(\delta/R)$. Hence $\Pi(\delta)$, the angle which this parallel makes with $v = 0$, is given by

$$\tan \Pi(\delta)\,.\,\sin h\,(\delta/R) = 1, \text{ or } \tan \tfrac{1}{2}\Pi(\delta) = e^{-\delta/R}$$

which is Lobatchewsky's formula. We also obtain easily for the area of a triangle the formula $R^2(\pi - A - B - C)$.

Beltrami's treatment connects two curves which, in the earlier treatment, had no connexion. These are limit-lines and curves of constant distance from a straight line. Both may be regarded as circles, the first having an infinite, the second an imaginary radius. The equation to a circle of radius ρ and centre u_0v_0 is

$$(a^2 - uu_0 - vv_0)^2 = \cos h^2\,(\rho/R)\; w_0^2w^2 = C^2w^2 \qquad \text{(say).}$$

This equation remains real when ρ is a pure imaginary, and remains finite when $w_0 = 0$, provided ρ becomes infinite in such a way that $w_0 \cos h\,(\rho/R)$ remains finite. In the latter case the equation represents a limit-line. In the former case, by giving different values to C, we obtain concentric circles with the imaginary centre u_0v_0. One of these, obtained by putting $C = 0$, is the straight line $a^2 - uu_0 - vv_0 = 0$. Hence the others are each throughout at a constant distance from this line. (It may be shown that all motions in a hyperbolic plane consist, in a general sense, of rotations; but three types must be distinguished according as the centre is real, imaginary or at infinity. All points describe, accordingly, one of the three types of circles.)

The above Euclidean interpretation fails for three or more dimensions. In the *Teoria fondamentale*, accordingly, where n dimensions are considered, Beltrami treats hyperbolic space in a purely analytical spirit. The paper shows that Lobatchewsky's space of any number of dimensions has, in Riemann's sense, a constant negative measure of curvature. Beltrami starts with the formula (analogous to that of the *Saggio*)

$$ds^2 = R^2x^{-2}\,(dx^2 + dx_1^2 + dx_2^2 + \dots + dx_n^2)$$

where

$$x^2 + x_1^2 + x_2^2 + \dots + x_n^2 = a^2.$$

He shows that geodesics are represented by linear equations between $x_1, x_2, \dots, x_n$, and that the geodesic distance ρ between two points x and x′ is given by

$$\cos h\,\frac{\rho}{R} = \frac{a^2 - x_1x'_1 - x_2x'_2 - \dots - x_nx'_n}{\{(a^2 - x_1^2 - x_2^2 - \dots - x_n^2)\,(a^2 - x'^2_1 - x'^2_2 - \dots - x'^2_n)\}^{1/2}}$$

(a formula practically identical with Cayley's, though obtained by a very different method). In order to show that the measure of curvature is constant, we make the substitutions

$$x_1 = r\lambda_1,\ x_2 = r\lambda_2 \dots x_n = r\lambda_n, \text{ where } \Sigma\lambda^2 = 1.$$

Hence

$$ds^2 = (Radr / \overline{a^2 - r^2})^2 + R^2r^2d\Delta^2 / (a^2 - r^2).$$

where

$$d\Delta^2 = \Sigma d\lambda^2.$$

Also calling ρ the geodesic distance from the origin, we have

$$\cos h\,(\rho/R) = \frac{a}{\sqrt{(a^2 - r^2)}},\ \sin h\,(\rho/R) = \frac{r}{\sqrt{(a^2 - r^2)}}\,.$$

Hence

$$ds^2 = d\rho^2 + (R \sin h\,(\rho/R))^2\, d\Delta^2.$$

Putting

$$z_1 = \rho\lambda_1,\ z_2 = \rho\lambda_2, \dots z_n = \rho\lambda_n,$$

we obtain

1 R ρ 2

$ds^2 = \Sigma dz^2 +$ sinh

$$\overline{\rho^2} \left\{ \left(\overline{\rho} \qquad \overline{R} \right) - 1 \right\} \Sigma (z_i dz_k - z_k dz_i)^2.$$

Hence when ρ is small, we have approximately

$$ds^2 = \Sigma dz^2 + \frac{1}{3R^2} \Sigma (z_i dz_k - z_k dz_i)^2 \qquad (1).$$

Considering a surface element through the origin, we may choose our axes so that, for this element,

$$z_3 = z_4 = \dots = z_n = 0.$$

Thus

$$dz_1^2 + dz_2^2 + \frac{1}{3R^2} (z_1 dz_2 - z_2 dz_1)^2 \qquad (2).$$

Now the area of the triangle whose vertices are (0, 0), (z_1, z_2), (dz_1, dz_2) is $\frac{1}{2}(z_1, dz_2 - z_2 dz_1)$. Hence the quotient when the terms of the fourth order in (2) are divided by the square of this triangle is $4/3R^2$; hence, returning to general axes, the same is the quotient 729 when the terms of the fourth order in (1) are divided by the square of the triangle whose vertices are (0, 0, … 0), $(z_1, z_2, z_3, \dots z_n)$, $(dz_1, dz_2, dz_3 \dots dz_n)$. But $-\frac{3}{4}$ of this quotient is defined by Riemann as the measure of curvature.[15] Hence the measure of curvature is $-1/R^2$, *i.e.* is constant and negative. The properties of parallels, triangles, &c., are as in the *Saggio*. It is also shown that the analogues of limit surfaces have zero curvature; and that spheres of radius ρ have constant positive curvature $1/R^2 \sinh^2 (\rho/R)$, so that spherical geometry may be regarded as contained in the pseudo-spherical (as Beltrami calls Lobatchewsky’s system).

Transition to the projective method.

The *Saggio*, as we saw, gives a Euclidean interpretation confined to two dimensions. But a consideration of the auxiliary plane suggests a different interpretation, which may be extended to any number of dimensions. If, instead of referring to the pseudosphere, we merely *define* distance and angle, in the Euclidean plane, as those functions of the coordinates which gave us distance and angle on the pseudosphere, we find that the geometry of our plane has become Lobatchewsky’s. All the points of the limiting circle are now at infinity, and points beyond it are imaginary. If we give our circle an imaginary radius the geometry on the plane becomes elliptic. Replacing the circle by a sphere, we obtain an analogous representation for three dimensions. Instead of a circle or sphere we may take any conic or quadric. With this definition, if the fundamental quadric be $\Sigma_{xx} = 0$, and if $\Sigma_{xx'}$ be the polar form of Σ_{xx}, the distance ρ between x and x′ is given by the projective formula

$$\cos(\rho/k) = \Sigma_{xx'} / \{\Sigma_{xx} \cdot \Sigma_{x'x'}\}^{1/2}.$$

That this formula is projective is rendered evident by observing that $e^{-2i\rho/k}$ is the anharmonic ratio of the range consisting of the two points and the intersections of the line joining them with the fundamental quadric. With this we are brought to the third or projective period. The method of this period is due to Cayley; its application to previous non-Euclidean geometry is due to Klein. The projective method contains a generalization of discoveries already made by Laguerre[16] in 1853 as regards Euclidean geometry. The arbitrariness of this procedure of deriving metrical geometry from the properties of conics is removed by Lie’s theory of congruence. We then arrive at the stage of thought which finds its expression in the modern treatment of the axioms of geometry.

The two kinds of elliptic space.

The projective method leads to a discrimination, first made by Klein,[17] of two varieties of Riemann’s space; Klein calls these elliptic and spherical. They are also called the polar and antipodal forms of elliptic space. The latter names will here be used. The difference is strictly analogous to that between the diameters and the points of a sphere. In the polar form two straight lines in a plane always intersect in one and only one point; in the antipodal form they intersect always in two points, which are antipodes. According to the definition of geometry adopted in section VII. (*Axioms of Geometry*), the antipodal form is not to be termed “geometry,” since any pair of coplanar straight lines intersect each other in two points. It may be called a “quasi-geometry.” Similarly in the antipodal form two diameters always determine a plane, but two points on a sphere do not determine a great circle when they are antipodes, and two great circles always intersect in two points. Again, a plane does not form a boundary among lines through a point: we can pass from any one such line to any other without passing through the plane. But a great circle does divide the surface of a sphere. So, in the polar form, a complete straight line does not divide a plane, and a plane does not divide space, and does not, like a Euclidean plane, have two sides.[18] But, in the antipodal form, a plane is, in these respects, like a Euclidean plane.

It is explained in section VII. in what sense the metrical geometry of the material world can be considered to be determinate and not a matter of arbitrary choice. The scientific question as to the best available evidence concerning the nature of this geometry is one beset with difficulties of a peculiar kind. We are obstructed by the fact that all existing physical science assumes the Euclidean hypothesis. This hypothesis has been involved in all actual measurements of large distances, and in all the laws of astronomy and physics. The principle of simplicity would therefore lead us, in general, where an observation conflicted with one or more of those laws, to ascribe this anomaly, not to the falsity of Euclidean geometry, but to the falsity of the laws in question. This applies especially to astronomy. On the earth our means of measurement are many and direct, and so long as no great accuracy is sought they involve few scientific laws. Thus we acquire, from such direct measurements, a very high degree of probability that the space-constant, if not infinite, is yet large as compared with terrestrial distances. But astronomical distances and triangles can only be measured by means of the received laws of astronomy and optics, all of which have been established by assuming the truth of the Euclidean hypothesis. It therefore remains possible (until a detailed proof of the contrary is forthcoming) that a large but finite space-constant, with different laws of astronomy and optics, would have equally explained the phenomena. We cannot, therefore, accept the measurements of stellar parallaxes, &c., as conclusive evidence that the space-constant is large as compared with stellar distances. For the present, on grounds of simplicity, we may rightly adopt this view; but it must remain possible that, in view of some hitherto undiscovered discrepancy, a slight correction of the sort suggested might prove the simplest alternative. But conversely, a finite parallax for very distant stars, or a negative parallax for any star, could not be accepted as conclusive evidence that our geometry is non-Euclidean, unless it were shown—and this seems scarcely possible—that no modification of astronomy or optics could account for the phenomenon. Thus although we may admit a probability that the space-constant is large in comparison with stellar distances, a conclusive proof or disproof seems scarcely possible.

Finally, it is of interest to note that, though it is theoretically possible to prove, by scientific methods, that our geometry is non-Euclidean, it is wholly impossible to prove by such methods that it is accurately Euclidean. For the unavoidable errors of observation must always leave a slight margin in our measurements. A triangle might be found whose angles were certainly greater, or certainly less, than two right angles; but to prove them *exactly* equal to two right angles must always be beyond our powers. If, therefore, any man cherishes a hope of proving the exact truth of Euclid, such a hope must be based, not upon scientific, but upon philosophical considerations.

Bibliography.—The bibliography appended to section VII. should be consulted in this connexion. Also, in addition to the citations already made, the following works may be mentioned.

For Lobatchewsky’s writings, cf. *Urkunden zur Geschichte der nichteuklidischen Geometrie*, i., *Nikolaj Iwanowitsch Lobatschefsky*, by F. Engel and P. Stäckel (Leipzig, 1898). For John Bolyai’s *Appendix*, cf. *Absolute Geometrie nach Johann Bolyai*, by J. Frischauf (Leipzig, 1872), and also the new edition of his father’s large work, *Tentamen* ..., published by the Mathematical Society of Budapest; the second volume contains the appendix. Cf. also J. Frischauf, *Elemente der absoluten Geometrie* (Leipzig, 1876); M.L. Gérard, *Sur la géométrie non-Euclidienne* (thesis for doctorate) (Paris, 1892); de Tilly, *Essai sur les principes fondamentales de la géométrie et de la mécanique* (Bordeaux, 1879); Sir R.S. Ball, “On the Theory of Content,” *Trans. Roy. Irish Acad.* vol. xxix. (1889); F. Lindemann, “Mechanik bei projectiver Maasbestimmung,” *Math. Annal.* vol. vii.; W.K. Clifford, “Preliminary Sketch of Biquaternions,” *Proc. of Lond. Math. Soc.* (1873), and *Coll. Works*; A. Buchheim, “On the Theory of Screws in Elliptic Space,” *Proc. Lond. Math. Soc.* vols. xv., xvi., xvii.; H. Cox, “On the Application of Quaternions and Grassmann’s Algebra to different Kinds of Uniform Space,” *Trans. Camb. Phil. Soc.* (1882); M. Dehn, “Die Legendarischen Sätze über die Winkelsumme im Dreieck,” Math. Ann. vol. 53 (1900), and “Über den Rauminhalt,” *Math. Annal.* vol. 55 (1902).

For expositions of the whole subject, cf. F. Klein, *Nicht-Euklidische Geometrie* (Göttingen, 1893); R. Bonola, *La Geometria non-Euclidea* (Bologna, 1906); P. Barbarin, *La Géométrie non-Euclidienne* (Paris, 1902); W. Killing, *Die nicht-Euklidischen Raumformen in analytischer Behandlung* (Leipzig, 1885). The last-named work also deals with geometry of more than three dimensions; in this connexion cf. also G. Veronese, *Fondamenti di geometria a più dimensioni ed a più specie di unità rettilinee* ... (Padua, 1891, German translation,

Leipzig, 1894); G. Fontené, *L'Hyperespace à (n-1) dimensions* (Paris, 1892); and A.N. Whitehead, *loc. cit.* Cf. also E. Study, "Über nicht-Euklidische und Liniengeometrie," *Jahr. d. Deutsch. Math. Ver.* vol. xv. (1906); W. Burnside, "On the Kinematics of non-Euclidean Space," *Proc. Lond. Math. Soc.* vol. xxvi. (1894). A bibliography on the subject up to 1878 has been published by G.B. Halsted, *Amer. Journ. of Math.* vols. i. and ii.; and one up to 1900 by R. Bonola, *Index operum ad geometriam absolutam spectantium* ... (1902, and Leipzig, 1903).

(B. A. W. R.; A. N. W.)

VII. Axioms of Geometry

Theories of space.

Until the discovery of the non-Euclidean geometries (Lobatchewsky, 1826 and 1829; J. Bolyai, 1832; B. Riemann, 1854), geometry was universally considered as being exclusively the science of existent space. (See section VI. *Non-Euclidean Geometry.*) In respect to the science, as thus conceived, two controversies may be noticed. First, there is the controversy respecting the absolute and relational theories of space. According to the absolute theory, which is the traditional view (held explicitly by Newton), space has an existence, in some sense whatever it may be, independent of the bodies which it contains. The bodies occupy space, and it is not intrinsically unmeaning to say that any definite body occupies *this* part of space, and not *that* part of space, without reference to other bodies occupying space. According to the relational theory of space, of which the chief exponent was Leibnitz,[19] space is nothing but a certain assemblage of the relations between the various particular bodies in space. The idea of space with no bodies in it is absurd. Accordingly there can be no meaning in saying that a body is *here* and not *there*, apart from a reference to the other bodies in the universe. Thus, on this theory, absolute motion is intrinsically unmeaning. It is admitted on all hands that in practice only relative motion is directly measurable. Newton, however, maintains in the *Principia* (scholium to the 8th definition) that it is indirectly measurable by means of the effects of "centrifugal force" as it occurs in the phenomena of rotation. This irrelevance of absolute motion (if there be such a thing) to science has led to the general adoption of the relational theory by modern men of science. But no decisive argument for either view has at present been elaborated.[20] Kant's view of space as being a form of perception at first sight appears to cut across this controversy. But he, saturated as he was with the spirit of the Newtonian physics, must (at least in both editions of the *Critique*) be classed with the upholders of the absolute theory. The form of perception has a type of existence proper to itself independently of the particular bodies which it contains. For example he writes:[21] "Space does not represent any quality of objects by themselves, or objects in their relation to one another, *i.e.* space does not represent any determination which is inherent in the objects themselves, and would remain, even if all subjective conditions of intuition were removed."

Axioms.

The second controversy is that between the view that the axioms applicable to space are known only from experience, and the view that in some sense these axioms are given *a priori*. Both these views, thus broadly stated, are capable of various subtle modifications, and a discussion of them would merge into a general treatise on epistemology. The cruder forms of the *a priori* view have been made quite untenable by the modern mathematical discoveries. Geometers now profess ignorance in many respects of the exact axioms which apply to existent space, and it seems unlikely that a profound study of the question should thus obliterate *a priori* intuitions.

Another question irrelevant to this article, but with some relevance to the above controversy, is that of the derivation of our perception of existent space from our various types of sensation. This is a question for psychology.[22]

Definition of Abstract Geometry.—Existent space is the subject matter of only one of the applications of the modern science of abstract geometry, viewed as a branch of pure mathematics. Geometry has been defined[23] as "the study of series of two or more dimensions." It has also been defined[24] as "the science of cross classification." These definitions are founded upon the actual practice of mathematicians in respect to their use of the term "Geometry." Either of them brings out the fact that geometry is not a science with a determinate subject matter. It is concerned with any subject matter to which the formal axioms may apply. Geometry is not peculiar in this respect. All branches of pure mathematics deal merely with types of relations. Thus the fundamental ideas of geometry (*e.g.* those of *points* and of *straight lines*) are not ideas of determinate entities, but of any entities for which the axioms are true. And a set of formal geometrical axioms cannot in themselves be true or false, since they are not determinate propositions, in that they do not refer to a determinate subject matter. The axioms are propositional functions.[25] When a set of axioms is given, we can ask (1) whether they are consistent, (2) whether their "existence theorem" is proved, (3) whether they are independent. Axioms are consistent when the contradictory of any axiom cannot be deduced from the remaining axioms. Their existence theorem is the proof that they are true when the fundamental ideas are considered as denoting some determinate subject matter, so that the axioms are developed into determinate propositions. It follows from the logical law of contradiction that the proof of the existence theorem proves also the consistency of the axioms. This is the only method of proof of consistency. The axioms of a set are independent of each other when no axiom can be deduced from the remaining axioms of the set. The independence of a given axiom is proved by establishing the consistency of the remaining axioms of the set, together with the contradictory of the given axiom. The enumeration of the axioms is simply the enumeration of the hypotheses[26] (with respect to the undetermined subject matter) of which some at least occur in each of the subsequent propositions.

Any science is called a "geometry" if it investigates the theory of the classification of a set of entities (the points) into classes (the straight lines), such that (1) there is one and only one class which contains any given pair of the entities, and (2) every such class contains more than two members. In the two geometries, important from their relevance to existent space, axioms which secure an order of the points on any line also occur. These geometries will be called "Projective Geometry" and "Descriptive Geometry." In projective geometry any two straight lines in a plane intersect, and the straight lines are closed series which return into themselves, like the circumference of a circle. In descriptive geometry two straight lines in a plane do not necessarily intersect, and a straight line is an open series without beginning or end. Ordinary Euclidean geometry is a descriptive geometry; it becomes a projective geometry when the so-called "points at infinity" are added.

Projective Geometry.

Projective geometry may be developed from two undefined fundamental ideas, namely, that of a "point" and that of a "straight line." These undetermined ideas take different specific meanings for the various specific subject matters to which projective geometry can be applied. The number of the axioms is always to some extent arbitrary, being dependent upon the verbal forms of statement which are adopted. They will be presented[27] here as 731
twelve in number, eight being "axioms of classification," and four being "axioms of order."

Axioms of Classification.—The eight axioms of classification are as follows:

1. Points form a class of entities with at least two members.

2. Any straight line is a class of points containing at least three members.

3. Any two distinct points lie in one and only one straight line.

4. There is at least one straight line which does not contain all the points.

5. If A, B, C are non-collinear points, and A′ is on the straight line BC, and B′ is on the straight line CA, then the straight lines AA′ and BB′ possess a point in common.

> *Definition.*—If A, B, C are any three non-collinear points, the *plane* ABC is the class of points lying on the straight lines joining A with the various points on the straight line BC.

6. There is at least one plane which does not contain all the points.

7. There exists a plane α, and a point A not incident in α, such that any point lies in some straight line which contains both A and a point in α.

> *Definition.*—Harm. (ABCD) symbolizes the following conjoint statements: (1) that the points A, B, C, D are collinear, and (2) that a quadrilateral can be found with one pair of opposite sides intersecting at A, with the other pair intersecting at C, and with its diagonals passing through B and D respectively. Then B and D are said to be "harmonic conjugates" with respect to A and C.

8. Harm. (ABCD) implies that B and D are distinct points.

In the above axioms 4 secures at least two dimensions, axiom 5 is the fundamental axiom of the plane, axiom 6 secures at least three dimensions, and axiom 7 secures at most three dimensions. From axioms 1-5 it can be proved that any two distinct points in a straight line determine that line, that any three non-collinear points in a plane determine that plane, that the straight line containing any two points in a plane lies wholly in that plane, and that any two straight lines in a plane intersect. From axioms 1-6 Desargue's well-known theorem on triangles in perspective can be proved.

The enunciation of this theorem is as follows: If ABC and A′B′C′ are two coplanar triangles such that the lines AA′, BB′, CC′ are concurrent, then the three points of intersection of BC and B′C′ of CA and C′A′, and of AB and A′B′ are collinear; and conversely if the three points of intersection are collinear, the three lines are concurrent. The proof which can be applied is the usual projective proof by which a third triangle A″B″C″ is constructed not coplanar with the other two, but in perspective with each of them.

It has been proved[28] that Desargues's theorem cannot be deduced from axioms 1-5, that is, if the geometry be confined to two dimensions. All the proofs proceed by the method of producing a specification of "points" and "straight lines" which satisfies axioms 1-5, and such that Desargues's theorem does not hold.

It follows from axioms 1-5 that Harm. (ABCD) implies Harm. (ADCB) and Harm. (CBAD), and that, if A, B, C be any three distinct collinear points, there exists at least one point D such that Harm. (ABCD). But it requires Desargues's theorem, and hence axiom 6, to prove that Harm. (ABCD) and Harm. (ABCD′) imply the identity of D and D′.

The necessity for axiom 8 has been proved by G. Fano,[29] who has produced a three dimensional geometry of fifteen points, *i.e.* a method of cross classification of fifteen entities, in which each straight line contains three points, and each plane contains seven straight lines. In this geometry axiom 8 does not hold. Also from axioms 1-6 and 8 it follows that Harm. (ABCD) implies Harm. (BCDA).

Definitions.—When two plane figures can be derived from one another by a single projection, they are said to be in *perspective*. When two plane figures can be derived one from the other by a finite series of perspective relations between intermediate figures, they are said to be *projectively* related. Any property of a plane figure which necessarily also belongs to any projectively related figure, is called a *projective* property.

The following theorem, known from its importance as "the fundamental theorem of projective geometry," cannot be proved[30] from axioms 1-8. The enunciation is: "A projective correspondence between the points on two straight lines is completely determined when the correspondents of three distinct points on one line are determined on the other." This theorem is equivalent[31] (assuming axioms 1-8) to another theorem, known as Pappus's Theorem, namely: "If l and l′ are two distinct coplanar lines, and A, B, C are three distinct points on l, and A′, B′, C′ are three distinct points on l′, then the three points of intersection of AA′ and B′C, of A′B and CC′, of BB′ and C′A, are collinear." This theorem is obviously Pascal's well-known theorem respecting a hexagon inscribed in a conic, for the special case when the conic has degenerated into the two lines l and l′. Another theorem also equivalent (assuming axioms 1-8) to the fundamental theorem is the following:[32] If the three collinear pairs of points, A and A′, B and B′, C and C′, are such that the three pairs of opposite sides of a complete quadrangle pass respectively through them, *i.e.* one pair through A and A′ respectively, and so on, and if also the three sides of the quadrangle which pass through A, B, and C, are concurrent in one of the corners of the quadrangle, then another quadrangle can be found with the same relation to the three pairs of points, except that its three sides which pass through A, B, and C, are not concurrent.

Thus, if we choose to take any one of these three theorems as an axiom, all the theorems of projective geometry which do not require ordinal or metrical ideas for their enunciation can be proved. Also a conic can be defined as the locus of the points found by the usual construction, based upon Pascal's theorem, for points on the conic through five given points. But it is unnecessary to assume here any one of the suggested axioms; for the fundamental theorem can be deduced from the axioms of order together with axioms 1-8.

Axioms of Order.—It is possible to define (cf. Pieri, *loc. cit.*) the property upon which the order of points on a straight line depends. But to secure that this property does in fact range the points in a serial order, some axioms are required. A straight line is to be a closed series; thus, when the points are in order, it requires two points on the line to divide it into two distinct complementary segments, which do not overlap, and together form the whole line. Accordingly the problem of the definition of order reduces itself to the definition of these two segments formed by any two points on the line; and the axioms are stated relatively to these segments.

Definition.—If A, B, C are three collinear points, the points on the *segment* ABC are defined to be those points such as X, for which there exist two points Y and Y′ with the property that Harm. (AYCY′) and Harm. (BYXY′) both hold. The *supplementary segment* ABC is defined to be the rest of the points on the line. This definition is elucidated by noticing that with our ordinary geometrical ideas, if B and X are any two points between A and C, then the two pairs of points, A and C, B and X, define an involution with real double points, namely, the Y and Y′ of the above definition. The property of belonging to a segment ABC is projective, since the harmonic relation is projective.

The first three axioms of order (cf. Pieri, *loc. cit.*) are:

9. If A, B, C are three distinct collinear points, the supplementary segment ABC is contained within the segment BCA.

10. If A, B, C are three distinct collinear points, the common part of the segments BCA and CAB is contained in the supplementary segment ABC.

11. If A, B, C are three distinct collinear points, and D lies In the segment ABC, then the segment ADC is contained within the segment ABC.

From these axioms all the usual properties of a closed order follow. It will be noticed that, if A, B, C are any three collinear points, C is necessarily traversed in passing from A to B by one route along the line, and is not traversed in passing from A to B along the other route. Thus there is no meaning, as referred to closed straight lines, in the simple statement that C lies between A and B. But there may be a relation of separation between two pairs of collinear points, such as A and C, and B and D. The couple B and D is said to separate A and C, if the four points are collinear and D lies in the segment complementary to the segment 732 ABC. The property of the separation of pairs of points by pairs of points is projective. Also it can be proved that Harm. (ABCD) implies that B and D separate A and C.

Definitions.—A series of entities arranged in a serial order, open or closed, is said to be *compact*, if the series contains no immediately consecutive entities, so that in traversing the series from any one entity to any other entity it is necessary to pass through entities distinct from either. It was the merit of R. Dedekind and of G. Cantor explicitly to formulate another fundamental property of series. The Dedekind property[33] as applied to an open series can be defined thus: An open series possesses the Dedekind property, if, however, it be divided into two mutually exclusive classes u and v, which (1) contain between them the whole series, and (2) are such that every member of u precedes in the serial order every member of v, there is always a member of the series, belonging to one of the two, u or v, which precedes every member of v (other than itself if it belong to v), and also succeeds every member of u (other than itself if it belong to u). Accordingly in an open series with the Dedekind property there is always a member of the series marking the junction of two classes such as u and v. An open series is *continuous* if it is compact and possesses the Dedekind property. A closed series can always be transformed into an open series by taking any arbitrary member as the first term and by taking one of the two ways round as the ascending order of the series. Thus the definitions of compactness and of the Dedekind property can be at once transferred to a closed series.

12. The last axiom of order is that there exists at least one straight line for which the point order possesses the Dedekind property.

It follows from axioms 1-12 by projection that the Dedekind property is true for all lines. Again the *harmonic system* ABC, where A, B, C are collinear points, is defined[34] thus: take the harmonic conjugates A′, B′, C′ of each point with respect to the other two, again take the harmonic conjugates of each of the six points A, B, C, A′, B′, C′ with respect to each pair of the remaining five, and proceed in this way by an unending series of steps. The set of points thus obtained is called the harmonic system ABC. It can be proved that a harmonic system is compact, and that every segment of the line containing it possesses members of it. Furthermore, it is easy to prove that the fundamental theorem holds for harmonic systems, in the sense that, if A, B, C are three points on a line l, and A′, B′, C′ are three points on a line l′, and if by any two distinct series of projections A, B, C are projected into A′, B′, C′, then any point of the harmonic system ABC corresponds to the same point of the harmonic system A′B′C′ according to both the projective relations which are thus established between l and l′. It now follows immediately that the fundamental theorem must hold for all the points on the lines l and l′, since (as has been pointed out) harmonic systems are “everywhere dense” on their containing lines. Thus the fundamental theorem follows from the axioms of order.

A system of numerical coordinates can now be introduced, possessing the property that linear equations represent planes and straight lines. The outline of the argument by which this remarkable problem (in that “distance” is as yet undefined) is solved, will now be given. It is first proved that the points on any line can in a certain way be definitely associated with all the positive and negative real numbers, so as to form with them a one-one correspondence. The arbitrary elements in the establishment of this relation are the points on the line associated with 0, 1 and ∞.

This association[35] is most easily effected by considering a class of projective relations of the line with itself, called by F. Schur (*loc. cit.*) *prospectivities*.

Let l (fig. 69) be the given line, m and n any two lines intersecting at U on l, S and S′ two points on n. Then a projective relation between l and itself is formed by projecting l from S on to m, and then by projecting m from S′ back on to l. All such projective relations, however m, n, S and S′ be varied, are called "prospectivities," and U is the double point of the prospectivity. If a point O on l is related to A by a prospectivity, then all prospectivities, which (1) have the same double point U, and (2) relate O to A, give the same correspondent (Q, in figure) to any point P on the line l; in fact they are all the same prospectivity, however m, n, S, and S′ may have been varied subject to these conditions. Such a prospectivity will be denoted by (OAU^2).

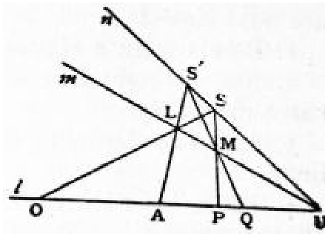


FIG. 69.

The sum of two prospectivities, written $(OAU^2) + (OBU^2)$, is defined to be that transformation of the line l into itself which is obtained by first applying the prospectivity (OAU^2) and then applying the prospectivity (OBU^2). Such a transformation, when the two summands have the same double point, is itself a prospectivity with that double point.

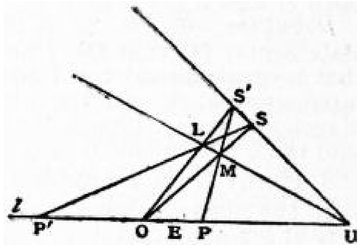


FIG. 70.

With this definition of addition it can be proved that prospectivities with the same double point satisfy all the axioms of magnitude. Accordingly they can be associated in a one-one correspondence with the positive and negative real numbers. Let E (fig. 70) be any point on l, distinct from O and U. Then the prospectivity (OEU^2) is associated with unity, the prospectivity (OOU^2) is associated with zero, and (OUU^2) with ∞. The prospectivities of the type (OPU^2), where P is any point on the segment OEU, correspond to the positive numbers; also if P′ is the harmonic conjugate of P with respect to O and U, the prospectivity $(OP'U^2)$ is associated with the corresponding negative number. (The subjoined figure explains this relation of the positive and negative prospectivities.) Then any point P on l is associated with the same number as is the prospectivity (OPU^2).

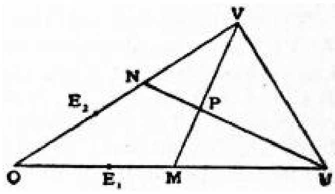


FIG. 71.

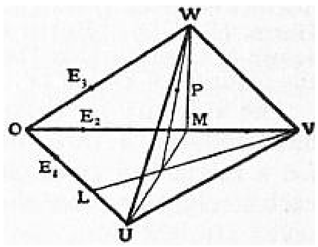


FIG. 72.

It can be proved that the order of the numbers in algebraic order of magnitude agrees with the order on the line of the associated points. Let the numbers, assigned according to the preceding specification, be said to be associated with the points according to the "numeration-system (OEU)." The introduction of a coordinate system for a plane is now managed as follows: Take any triangle OUV in the plane, and on the lines OU and OV establish the numeration systems (OE_1U) and (OE_2V), where E_1 and E_2 are arbitrarily chosen. Then (cf. fig. 71) if M and N are associated with the numbers x and y according to these systems, the coordinates of P are x and y. It then follows that the equation of a straight line is of the form $ax + by + c = 0$. Both coordinates of any point on the line UV are infinite. This can be avoided by introducing homogeneous coordinates X, Y, Z, where $x = X/Z$, and $y = Y/Z$, and $Z = 0$ is the equation of UV.

The procedure for three dimensions is similar. Let OUVW (fig. 72) be any tetrahedron, and associate points on OU, OV, OW with numbers according to the numeration systems (OE_1U), (OE_2V), and (OE_3W). Let the planes VWP, WUP, UVP cut OU, OV, OW in L, M, N respectively; and let x, y, z be the numbers associated with L, M, N respectively. Then P is the point (x, y, z). Also homogeneous coordinates can be introduced as before, thus avoiding the infinities on the plane UVW.

The cross ratio of a range of four collinear points can now be defined as a number characteristic of that range. Let the coordinates of any point P_r of the range $P_1\ P_2\ P_3\ P_4$ be

$$\frac{\lambda_r a + \mu_r + a'}{\lambda_r + \mu_r},\quad \frac{\lambda_r b + \mu_r b'}{\lambda_r + \mu_r},\quad \frac{\lambda_r c + \mu_r c'}{\lambda_r + \mu_r},\qquad (r = 1, 2, 3, 4)$$

and let $(\lambda_r \mu_s)$ be written for $\lambda_r \mu_s - \lambda_s \mu_r$. Then the cross ratio $\{P_1\ P_2\ P_3\ P_4\}$ is defined to be the number $(\lambda_1\mu_2)(\lambda_3\mu_4) / (\lambda_1\mu_4)(\lambda_3\mu_2)$. The equality of the cross ratios of the ranges $(P_1\ P_2\ P_3\ P_4)$ and $(Q_1\ Q_2\ Q_3\ Q_4)$ is proved to be the necessary and sufficient condition for their mutual projectivity. The cross ratios of all harmonic ranges are then easily seen to be all equal to -1, by comparing with the range $(OE_1UE'_1)$ on the axis of x.

Thus all the ordinary propositions of geometry in which distance and angular measure do not enter otherwise than in cross ratios can now be enunciated and proved. Accordingly the greater part of the analytical theory of conics and quadrics belongs to geometry at this stage 733
The theory of distance will be considered after the principles of descriptive geometry have been developed.

Descriptive Geometry.

Descriptive geometry is essentially the science of multiple order for open series. The first satisfactory system of axioms was given by M. Pasch.[36] An improved version is due to G. Peano.[37] Both these authors treat the idea of the class of points constituting the segment lying *between* two points as an undefined fundamental idea. Thus in fact there are in this system two fundamental ideas, namely, of points and of segments. It is then easy enough to define the prolongations of the segments, so as to form the complete straight lines. D. Hilbert's[38] formulation of the axioms is in this respect practically based on the same fundamental ideas. His work is justly famous for some of the mathematical investigations contained in it, but his exposition of the axioms is distinctly inferior to that of Peano. Descriptive geometry can also be considered[39] as the science of a class of relations, each relation being a two-termed serial relation, as considered in the logic of relations, ranging the points between which it holds into a linear open order. Thus the relations are the straight lines, and the terms between which they hold are the points. But a combination of these two points of view yields[40] the simplest statement of all. Descriptive geometry is then conceived as the investigation of an undefined fundamental relation between three terms (points); and when the relation holds between three points A, B, C, the points are said to be "in the [linear] order ABC."

O. Veblen's axioms and definitions, slightly modified, are as follows:—

1. If the points A, B, C are in the order ABC, they are in the order CBA.

2. If the points A, B, C are in the order ABC, they are not in the order BCA.

3. If the points A, B, C are in the order ABC, A is distinct from C.

4. If A and B are any two distinct points, there exists a point C such that A, B, C are in the order ABC.

Definition.—The *line* AB (A ≠ B) consists of A and B, and of all points X in one of the possible orders, ABX, AXB, XAB. The points X in the order AXB constitute the *segment* AB.

5. If points C and D (C ≠ D) lie on the line AB, then A lies on the line CD.

6. There exist three distinct points A, B, C not in any of the orders ABC, BCA, CAB.

7. If three distinct points A, B, C (fig. 73) do not lie on the same line, and D and E are two distinct points in the orders BCD and CEA, then a point F exists in the order AFB, and such that D, E, F are collinear.

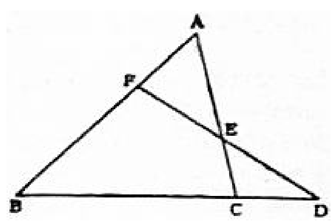


Fig. 73.

Definition.—If A, B, C are three non-collinear points, the *plane* ABC is the class of points which lie on any one of the lines joining any two of the points belonging to the *boundary* of the triangle ABC, the boundary being formed by the segments BC, CA and AB. The *interior* of the triangle ABC is formed by the points in segments such as PQ, where P and Q are points respectively on two of the segments BC, CA, AB.

8. There exists a plane ABC, which does not contain all the points.

Definition.—If A, B, C, D are four non-coplanar points, the space ABCD is the class of points which lie on any of the lines containing two points on the surface of the tetrahedron ABCD, the *surface* being formed by the interiors of the triangles ABC, BCD, DCA, DAB.

9. There exists a space ABCD which contains all the points.

10. The Dedekind property holds for the order of the points on any straight line.

It follows from axioms 1-9 that the points on any straight line are arranged in an open serial order. Also all the ordinary theorems respecting a point dividing a straight line into two parts, a straight line dividing a plane into two parts, and a plane dividing space into two parts, follow.

Again, in any plane α consider a line l and a point A (fig. 74).

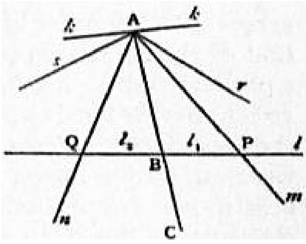


FIG. 74.

Let any point B divide l into two half-lines l_1 and l_2. Then it can be proved that the set of half-lines, emanating from A and intersecting l_1 (such as m), are bounded by two half-lines, of which ABC is one. Let r be the other. Then it can be proved that r does not intersect l_1. Similarly for the half-line, such as n, intersecting l_2. Let s be its bounding half-line. Then two cases are possible. (1) The half-lines r and s are collinear, and together form one complete line. In this case, there is one and only one line (viz. r + s) through A and lying in α which does not intersect l. This is the Euclidean case, and the assumption that this case holds is the *Euclidean parallel axiom*. But (2) the half-lines r and s may not be collinear. In this case there will be an infinite number of lines, such as k for instance, containing A and lying in α, which do not intersect l. Then the lines through A in α are divided into two classes by reference to l, namely, the *secant* lines which intersect l, and the *non-secant* lines which do not intersect l. The two boundary non-secant lines, of which r and s are respectively halves, may be called the two parallels to l through A.

The perception of the possibility of case 2 constituted the starting-point from which Lobatchewsky constructed the first explicit coherent theory of non-Euclidean geometry, and thus created a revolution in the philosophy of the subject. For many centuries the speculations of mathematicians on the foundations of geometry were almost confined to hopeless attempts to prove the “parallel axiom” without the introduction of some equivalent axiom.[41]

Associated Projective and Descriptive Spaces.—A region of a projective space, such that one, and only one, of the two supplementary segments between any pair of points within it lies entirely within it, satisfies the above axioms (1-10) of descriptive geometry, where the points of the region are the descriptive points, and the portions of straight lines within the region are the descriptive lines. If the excluded part of the original projective space is a single plane, the Euclidean parallel axiom also holds, otherwise it does not hold for the descriptive space of the limited region. Again, conversely, starting from an original descriptive space an associated projective space can be constructed by means of the concept of *ideal points*.[42] These are also called *projective points*, where it is understood that the simple points are the points of the original descriptive space. An *ideal point* is the class of straight lines which is composed of two coplanar lines a and b, together with the lines of intersection of all pairs of intersecting planes which respectively contain a and b, together with the lines of intersection with the plane ab of all planes containing any one of the lines (other than a or b) already specified as belonging to the ideal point. It is evident that, if the two original lines a and b intersect, the corresponding ideal point is nothing else than the whole class of lines which are concurrent at the point ab. But the essence of the definition is that an ideal point has an existence when the lines a and b do not intersect, so long as they are coplanar. An ideal point is termed *proper*, if the lines composing it intersect; otherwise it is *improper*.

A theorem essential to the whole theory is the following: if any two of the three lines a, b, c are coplanar, but the three lines are not all coplanar, and similarly for the lines a, b, d, then c and d are coplanar. It follows that any two lines belonging to an ideal point can be used as the pair of guiding lines in the definition. An ideal point is said to be *coherent* with a plane, if any of the lines composing it lie in the plane. An *ideal line* is the class of ideal points each of which is coherent with two given planes. If the planes intersect, the ideal line is 734 termed *proper*, otherwise it is *improper*. It can be proved that any two planes, with which any two of the ideal points are both coherent, will serve as the guiding planes used in the definition. The ideal planes are defined as in projective geometry, and all the other definitions (for segments, order, &c.) of projective geometry are applied to the ideal elements. If an ideal plane contains some proper ideal points, it is called *proper*, otherwise it is *improper*. Every ideal plane contains some improper ideal points.

It can now be proved that all the axioms of projective geometry hold of the ideal elements as thus obtained; and also that the order of the ideal points as obtained by the projective method agrees with the order of the proper ideal points as obtained from that of the associated points of the descriptive geometry. Thus a projective space has been constructed out of the ideal elements, and the proper ideal elements correspond element by element with the associated descriptive elements. Thus the proper ideal elements form a region in the projective space within which the descriptive axioms hold. Accordingly, by substituting ideal elements, a descriptive space can always be considered as a region within a projective space. This is the justification for the ordinary use of the "points at infinity" in the ordinary Euclidean geometry; the reasoning has been transferred from the original descriptive space to the associated projective space of ideal elements; and with the Euclidean parallel axiom the improper ideal elements reduce to the ideal points on a single improper ideal plane, namely, the plane at infinity.[43]

Congruence and Measurement.—The property of physical space which is expressed by the term "measurability" has now to be considered. This property has often been considered as essential to the very idea of space. For example, Kant writes,[44] "Space is represented as an infinite given *quantity*." This quantitative aspect of space arises from the measurability of distances, of angles, of surfaces and of volumes. These four types of quantity depend upon the two first among them as fundamental. The measurability of space is essentially connected with the idea of *congruence*, of which the simplest examples are to be found in the proofs of equality by the method of superposition, as used in elementary plane geometry. The mere concepts of "part" and of "whole" must of necessity be inadequate as the foundation of measurement, since we require the comparison as to quantity of regions of space which have no portions in common. The idea of congruence, as exemplified by the method of superposition in geometrical reasoning, appears to be founded upon that of the "rigid body," which moves from one position to another with its internal spatial relations unchanged. But unless there is a previous concept of the metrical relations between the parts of the body, there can be no basis from which to deduce that they are unchanged.

It would therefore appear as if the idea of the congruence, or metrical equality, of two portions of space (as empirically suggested by the motion of rigid bodies) must be considered as a fundamental idea incapable of definition in terms of those geometrical concepts which have already been enumerated. This was in effect the point of view of Pasch.[45] It has, however, been proved by Sophus Lie[46] that congruence is capable of definition without recourse to a new fundamental idea. This he does by means of his theory of finite continuous groups (see Groups, Theory of), of which the definition is possible in terms of our established geometrical ideas, remembering that coordinates have already been introduced. The displacement of a rigid body is simply a mode of defining to the senses a one-one transformation of all space into itself. For at any point of space a particle may be conceived to be placed, and to be rigidly connected with the rigid body; and thus there is a definite correspondence of any point of space with the new point occupied by the associated particle after displacement. Again two successive displacements of a rigid body from position A to position B, and from position B to position C, are the same in effect as one displacement from A to C. But this is the characteristic "group" property. Thus the transformations of space into itself defined by displacements of rigid bodies form a group.

Call this group of transformations a congruence-group. Now according to Lie a congruence-group is defined by the following characteristics:—

1. A congruence-group is a finite continuous group of one-one transformations, containing the identical transformation.

2. It is a sub-group of the general projective group, *i.e.* of the group of which any transformation converts planes into planes, and straight lines into straight lines.

3. An infinitesimal transformation can always be found satisfying the condition that, at least throughout a certain enclosed region, any definite line and any definite point on the line are latent, *i.e.* correspond to themselves.

4. No infinitesimal transformation of the group exists, such that, at least in the region for which (3) holds, a straight line, a point on it, and a plane through it, shall all be latent.

The property enunciated by conditions (3) and (4), taken together, is named by Lie "Free mobility in the infinitesimal." Lie proves the following theorems for a projective space:—

> 1. If the above four conditions are only satisfied by a group throughout part of projective space, this part either (α) must be the region enclosed by a real closed quadric, or (β) must be the whole of the projective space with the exception of a single plane. In case (α) the corresponding congruence group is the continuous group for which the enclosing quadric is latent; and in case (β) an imaginary conic (with a real equation) lying in the latent plane is also latent, and the congruence group is the continuous group for which the plane and conic

are latent.

2. If the above four conditions are satisfied by a group throughout the whole of projective space, the congruence group is the continuous group for which some imaginary quadric (with a real equation) is latent.

By a proper choice of non-homogeneous co-ordinates the equation of any quadrics of the types considered, either in theorem 1 (α), or in theorem 2, can be written in the form $1 + c(x^2 + y^2 + z^2) = 0$, where c is negative for a real closed quadric, and positive for an imaginary quadric. Then the general infinitesimal transformation is defined by the three equations:

$$\begin{aligned} dx/dt &= u - \omega_3 y + \omega_2 z + cx\,(ux + vy + wz), \\ dy/dt &= v - \omega_1 z + \omega_3 x + cy\,(ux + vy + wz), \qquad \text{(A)} \\ dz/dt &= w - \omega_2 x + \omega_1 y + cz\,(ux + vy + wz). \end{aligned}$$

In the ease considered in theorem 1 (β), with the proper choice of co-ordinates the three equations defining the general infinitesimal transformation are:

$$\begin{aligned} dx/dt &= u - \omega_3 y + \omega_2 z, \\ dy/dt &= v - \omega_1 z + \omega_3 x, \qquad \text{(B)} \\ dz/dt &= w - \omega_2 x + \omega_1 y. \end{aligned}$$

In this case the latent plane is the plane for which at least one of x, y, z are infinite, that is, the plane $0.x + 0.y + 0.z + a = 0$; and the latent conic is the conic in which the cone $x^2 + y^2 + z^2 = 0$ intersects the latent plane.

It follows from theorems 1 and 2 that there is not one unique congruence-group, but an indefinite number of them. There is one congruence-group corresponding to each closed real quadric, one to each imaginary quadric with a real equation, and one to each imaginary conic in a real plane and with a real equation. The quadric thus associated with each congruence-group is called the *absolute* for that group, and in the degenerate case of 1 (β) the absolute is the latent plane together with the latent imaginary conic. If the absolute is real, the congruence-group is *hyperbolic*; if imaginary, it is *elliptic*; if the absolute is a plane and imaginary conic, the group is parabolic. Metrical geometry is simply the theory of the properties of some particular congruence-group selected for study.

The definition of distance is connected with the corresponding congruence-group by two considerations in respect to a range of five points (A_1, A_2, P_1, P_2, P_3), of which A_1 and A_2 are on the absolute.

Let $\{A_1P_1A_2P_2\}$ stand for the cross ratio (as defined above) of the range ($A_1P_1A_2P_2$), with a similar notation for the other ranges. Then

(1)

$$\log\{A_1P_1A_2P_2\} + \log\{A_1P_2A_2P_3\} = \log\{A_1P_1A_2P_3\},$$

and

(2), if the points A_1, A_2, P_1, P_2 are transformed into A'_1, A'_2, P'_1, P'_2 by any transformation of the congruence-group, (α) $\{A_1P_1A_2P_2 = \{A'_1P'_1A'_2P'_2\}$, since the transformation is projective, and (β) A'_1, A'_2 are on the absolute since A_1 and A_2 are on it. Thus if we define the distance 735
P_1P_2 to be ½k log $\{A_1P_1A_2P_2\}$, where A_1 and A_2 are the points in which the line P_1P_2 cuts the absolute, and k is some constant, the two characteristic properties of distance, namely, (1) the addition of consecutive lengths on a straight line, and (2) the invariability of distances during a transformation of the congruence-group, are satisfied. This is the well-known Cayley-Klein projective definition[47] of distance, which was elaborated in view of the addition property alone, previously to Lie's discovery of the theory of congruence-groups. For a hyperbolic group when P_1 and P_2 are in the region enclosed by the absolute, log $\{A_1P_1A_2P_2\}$ is real, and therefore k must be real. For an elliptic group A_1 and A_2 are conjugate imaginaries, and log $\{A_1P_1A_2P_2\}$ is a pure imaginary, and k is chosen to be κ/ι, where κ is real and $\iota = \sqrt{-}$.

Similarly the angle between two planes, p_1 and p_2, is defined to be $(1/2\iota)$ log $(t_1p_1t_2p_2)$, where t_1 and t_2 are tangent planes to the absolute through the line p_1p_2. The planes t_1 and t_2 are imaginary for an elliptic group, and also for an hyperbolic group when the planes p_1 and p_2 intersect at points within the region enclosed by the absolute. The development of the consequences of these metrical definitions is the subject of non-Euclidean geometry.

The definitions for the parabolic case can be arrived at as limits of those obtained in either of the other two cases by making k ultimately to vanish. It is also obvious that, if P_1 and P_2 be the points (x_1, y_1, z_1) and (x_2, y_2, z_2), it follows from equations (B) above that $\{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\}^{1/2}$ is unaltered by a congruence transformation and also satisfies the addition property for collinear distances. Also the previous definition of an angle can be adapted to this case, by making t_1 and t_2 to be the tangent planes through the line p_1p_2 to the imaginary conic. Similarly if p_1 and p_2 are intersecting lines, the same definition of an angle holds, where t_1 and t_2 are now the lines from the point p_1p_2 to the two points where the plane p_1p_2 cuts the imaginary conic. These points are in fact the "circular points at infinity" on the plane. The development of the consequences of these definitions for the parabolic case gives the ordinary Euclidean metrical geometry.

Thus the only metrical geometry for the whole of projective space is of the elliptic type. But the actual measure-relations (though not their general properties) differ according to the elliptic congruence-group selected for study. In a descriptive space a congruence-group should possess the four characteristics of such a group throughout the whole of the space. Then form the associated ideal projective space. The associated congruence-group for this ideal space must satisfy the four conditions throughout the region of the proper ideal points. Thus the boundary of this region is the absolute. Accordingly there can be no metrical geometry for the whole of a descriptive space unless its boundary (in the associated ideal space) is a closed quadric or a plane. If the boundary is a closed quadric, there is one possible congruence-group of the hyperbolic type. If the boundary is a plane (the plane at infinity), the possible congruence-groups are parabolic; and there is a congruence-group corresponding to each imaginary conic in this plane, together with a Euclidean metrical geometry corresponding to each such group. Owing to these alternative possibilities, it would appear to be more accurate to say that systems of quantities can be found in a space, rather than that space is a quantity.

Lie has also deduced[48] the same results with respect to congruence-groups from another set of defining properties, which explicitly assume the existence of a quantitative relation (the distance) between any two points, which is invariant for any transformation of the congruence-group.[49]

The above results, in respect to congruence and metrical geometry, considered in relation to existent space, have led to the doctrine[50] that it is intrinsically unmeaning to ask which system of metrical geometry is true of the physical world. Any one of these systems can be applied, and in an indefinite number of ways. The only question before us is one of convenience in respect to simplicity of statement of the physical laws. This point of view seems to neglect the consideration that science is to be relevant to the definite perceiving minds of men; and that (neglecting the ambiguity introduced by the invariable slight inexactness of observation which is not relevant to this special doctrine) we have, in fact, presented to our senses a definite set of transformations forming a congruence-group, resulting in a set of measure relations which are in no respect arbitrary. Accordingly our scientific laws are to be stated relevantly to that particular congruence-group. Thus the investigation of the type (elliptic, hyperbolic or parabolic) of this special congruence-group is a perfectly definite problem, to be decided by experiment. The consideration of experiments adapted to this object requires some development of non-Euclidean geometry (see section VI., *Non-Euclidean Geometry*). But if the doctrine means that, assuming some sort of objective reality for the material universe, beings can be imagined, to whom *either* all congruence-groups are equally important, *or* some other congruence-group is specially important, the doctrine appears to be an immediate deduction from the mathematical facts. Assuming a definite congruence-group, the investigation of surfaces (or three-dimensional loci in space of four dimensions) with geodesic geometries of the form of metrical geometries of other types of congruence-groups forms an important chapter of non-Euclidean geometry. Arising from this investigation there is a widely-spread fallacy, which has found its way into many philosophic writings, namely, that the possibility of the geometry of existent three-dimensional space being other than Euclidean depends on the physical existence of Euclidean space of four or more dimensions. The foregoing exposition shows the baselessness of this idea.

Bibliography.—For an account of the investigations on the axioms of geometry during the Greek period, see M. Cantor, *Vorlesungen über die Geschichte der Mathematik*, Bd. i. and iii.; T.L. Heath, *The Thirteen Books of Euclid's Elements, a New Translation from the Greek, with Introductory Essays and Commentary, Historical, Critical, and Explanatory* (Cambridge, 1908)—this work is the standard source of information; W.B. Frankland, *Euclid, Book I., with a Commentary* (Cambridge, 1905)—the commentary contains copious extracts from the ancient commentators. The next period of really substantive importance is that of the 18th

century. The leading authors are: G. Saccheri, S.J., *Euclides ab omni naevo vindicatus* (Milan, 1733). Saccheri was an Italian Jesuit who unconsciously discovered non-Euclidean geometry in the course of his efforts to prove its impossibility. J.H. Lambert, *Theorie der Parallellinien* (1766); A.M. Legendre, *Éléments de géométrie* (1794). An adequate account of the above authors is given by P. Stäckel and F. Engel, *Die Theorie der Parallellinien von Euklid bis auf Gauss* (Leipzig, 1895). The next period of time (roughly from 1800 to 1870) contains two streams of thought, both of which are essential to the modern analysis of the subject. The first stream is that which produced the discovery and investigation of non-Euclidean geometries, the second stream is that which has produced the geometry of position, comprising both projective and descriptive geometry not very accurately discriminated. The leading authors on non-Euclidean geometry are K.F. Gauss, in private letters to Schumacher, cf. Stäckel and Engel, *loc. cit.*; N. Lobatchewsky, rector of the university of Kazan, to whom the honour of the effective discovery of non-Euclidean geometry must be assigned. His first publication was at Kazan in 1826. His various memoirs have been re-edited by Engel; cf. *Urkunden zur Geschichte der nichteuklidischen Geometrie* by Stäckel and Engel, vol. i. "Lobatchewsky." J. Bolyai discovered non-Euclidean geometry apparently in independence of Lobatchewsky. His memoir was published in 1831 as an appendix to a work by his father W. Bolyai, *Tentamen juventutem....* This memoir has been separately edited by J. Frischauf, *Absolute Geometrie nach J. Bolyai* (Leipzig, 1872); B. Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (1854); cf. *Gesamte Werke*, a translation in The Collected Papers of W.K. Clifford. This is a fundamental memoir on the subject and must rank with the work of Lobatchewsky. Riemann discovered elliptic metrical geometry, and Lobatchewsky hyperbolic geometry. A full account of Riemann's ideas, with the subsequent developments due to Clifford, F. Klein and W. Killing, will be found in *The Boston Colloquium for 1903* (New York, 1905), article "Forms of Non-Euclidean Space," by F.S. Woods. A. Cayley, *loc. cit.* (1859), and F. Klein, "Über die sogenannte nichteuklidische Geometrie," *Math. Annal.* vols. iv. and vi. (1871 and 1872), between them elaborated the projective theory of distance; H. Helmholtz, "Über die thatsächlichen Grundlagen der Geometrie" (1866), and "Über die Thatsachen, die der Geometrie zu Grunde liegen" (1868), both in his *Wissenschaftliche Abhandlungen*, vol. ii., and S. Lie, *loc. cit.* (1890 and 1893), between them elaborated the group theory of congruence.

The numberless works which have been written to suggest equivalent alternatives to Euclid's parallel axioms may be neglected as being of trivial importance, though many of them are marvels of geometric ingenuity.

The second stream of thought confined itself within the circle of ideas of Euclidean geometry. Its origin was mainly due to a succession of great French mathematicians, for 736
example, G. Monge, *Géométrie descriptive* (1800); J.V. Poncelet, *Traité des proprietés projectives des figures* (1822); M. Chasles, *Aperçu historique sur l'origine et le développement des méthodes en géométrie* (Bruxelles, 1837), and *Traité de géométrie supérieure* (Paris, 1852); and many others. But the works which have been, and are still, of decisive influence on thought as a store-house of ideas relevant to the foundations of geometry are K.G.C. von Staudt's two works, *Geometrie der Lage* (Nürnberg, 1847); and *Beiträge zur Geometrie der Lage* (Nürnberg, 1856, 3rd ed. 1860).

The final period is characterized by the successful production of exact systems of axioms, and by the final solution of problems which have occupied mathematicians for two thousand years. The successful analysis of the ideas involved in serial continuity is due to R. Dedekind, *Stetigkeit und irrationale Zahlen* (1872), and to G. Cantor, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* (Leipzig, 1883), and *Acta math.* vol. 2.

Complete systems of axioms have been stated by M. Pasch, *loc. cit.*; G. Peano, *loc. cit.*; M. Pieri, loc. cit.; B. Russell, *Principles of Mathematics*; O. Veblen, *loc. cit.*; and by G. Veronese in his treatise, *Fondamenti di geometria* (Padua, 1891; German transl. by A. Schepp, *Grundzüge der Geometrie*, Leipzig, 1894). Most of the leading memoirs on special questions involved have been cited in the text; in addition there may be mentioned M. Pieri, "Nuovi principii di geometria projettiva complessa," *Trans. Accad. R. d. Sci.* (Turin, 1905); E.H. Moore, "On the Projective Axioms of Geometry," *Trans. Amer. Math. Soc.*, 1902; O. Veblen and W.H. Bussey, "Finite Projective Geometries," *Trans. Amer. Math. Soc.*, 1905; A.B. Kempe, "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points," *Proc. Lond. Math. Soc.*, 1890; J. Royce, "The Relation of the Principles of Logic to the Foundations of Geometry," *Trans. of Amer. Math. Soc.*, 1905; A. Schoenflies, "Über die Möglichkeit einer projectiven Geometrie bei transfiniter (nichtarchimedischer) Massbestimmung," Deutsch. *M.-V. Jahresb.*, 1906.

For general expositions of the bearings of the above investigations, cf. Hon. Bertrand Russell, *loc. cit.*; L. Couturat, *Les Principes des mathématiques* (Paris, 1905); H. Poincaré, *loc. cit.*; Russell and Whitehead, *Principia mathematica* (Cambridge, Univ. Press). The philosophers whose views on space and geometric truth deserve especial study are

Descartes, Leibnitz, Hume, Kant and J.S. Mill.

(A. N. W.)

---

1 For Egyptian geometry see Egypt, § *Science and Mathematics*.

2 Cf. A.N. Whitehead, *Universal Algebra*, Bk. vi. (Cambridge, 1898).

3 Cf. A.N. Whitehead, *loc. cit.*

4 Cf. A.N. Whitehead, "The Geodesic Geometry of Surfaces in non-Euclidean Space," *Proc. Lond. Math. Soc.* vol. xxix.

5 Cf. Klein, "Zur nicht-Euklidischen Geometrie," *Math. Annal.* vol. xxxvii.

6 On the theory of parallels before Lobatchewsky, see Stäckel und Engel, *Theorie der Parallellinien von Euklid bis auf Gauss* (Leipzig, 1895). The foregoing remarks are based upon the materials collected in this work.

7 See Stäckel und Engel, *op. cit.*, and "Gauss, die beiden Bolyai, und die nicht-Euklidische Geometrie," *Math. Annalen*, Bd. xlix.; also Engel's translation of Lobatchewsky (Leipzig, 1898), pp. 378 ff.

8 Lobatchewsky's works on the subject are the following:—"On the Foundations of Geometry," *Kazañ Messenger*, 1829-1830; "New Foundations of Geometry, with a complete Theory of Parallels," *Proceedings of the University of Kazañ*, 1835 (both in Russian, but translated into German by Engel, Leipzig, 1898); "Géométrie imaginaire," Crelle's Journal, 1837; *Theorie der Parallellinien* (Berlin, 1840; 2nd ed., 1887; translated by Halsted, Austin, Texas, 1891). His results appear to have been set forth in a paper (now lost) which he read at Kazañ in 1826.

9 Translated by Halsted (Austin, Texas, 4th ed., 1896.)

10 *Abhandlungen d. Königl. Ges. d. Wiss. zu Göttingen*, Bd. xiii.; *Ges. math. Werke*, pp. 254-269; translated by Clifford, *Collected Mathematical Papers*.

11 Cf. *Gesamm. math. und phys. Werke*, vol. i. (Leipzig, 1894).

12 *Wiss. Abh.* vol. ii. pp. 610, 618 (1866, 1868).

13 *Mind*, O.S., vols. i. and iii.; *Vorträge und Reden*, vol. ii. pp. 1, 256.

14 His papers are "Saggio di interpretazione della geometria non-Euclidea," *Giornale di matematiche*, vol. vi. (1868); "Teoria fondamentale degli spazii di curvatura costante," *Annali di matematica*, vol. ii. (1868-1869). Both were translated into French by J. Hoüel, *Annales scientifiques de l'École Normale supérieure*, vol. vi. (1869).

15 Beltrami shows also that this definition agrees with that of Gauss.

16 "Sur la théorie des foyers," *Nouv. Ann.* vol. xii.

17 *Math. Annalen*, iv. vi., 1871-1872.

18 For an investigation of these and similar properties, see Whitehead, *Universal Algebra* (Cambridge, 1898), bk. vi. ch. ii. The polar form was independently discovered by Simon Newcomb in 1877.

19 For an analysis of Leibnitz's ideas on space, cf. B. Russell, *The Philosophy of Leibnitz*, chs. viii.-x.

20 Cf. Hon. Bertrand Russell, "Is Position in Time and Space Absolute or Relative?" *Mind*, n.s. vol. 10 (1901), and A.N. Whitehead, "Mathematical Concepts of the Material World," *Phil. Trans.* (1906), p. 205.

21 Cf. *Critique of Pure Reason*, 1st section: "Of Space," conclusion A, Max Müller's translation.

22 Cf. Ernst Mach, *Erkenntniss und Irrtum* (Leipzig); the relevant chapters are translated by T.J. McCormack, *Space and Geometry* (London, 1906); also A. Meinong, *Über die Stellung der Gegenstandstheorie im System der Wissenschaften* (Leipzig, 1907).

23 Cf. Russell, *Principles of Mathematics*, § 352 (Cambridge, 1903).

24 Cf. A.N. Whitehead, *The Axioms of Projective Geometry*, § 3 (Cambridge, 1906).

25 Cf. Russell, *Princ. of Math.*, ch. i.

26 Cf. Russell, *loc. cit.*, and G. Frege, "Über die Grundlagen der Géométrie," *Jahresber. der Deutsch. Math. Ver.* (1906).

27 This formulation—though not in respect to number—is in all essentials that of M. Pieri, cf. "I principii della Geometria di Posizione," *Accad. R. di Torino* (1898); also cf. Whitehead, *loc. cit.*

28 Cf. G. Peano, "Sui fondamenti della Geometria," p. 73, *Rivista di matematica*, vol. iv. (1894), and D. Hilbert, *Grundlagen der Geometrie* (Leipzig, 1899); and R.F. Moulton, "A Simple non-Desarguesian Plane Geometry," *Trans. Amer. Math. Soc.*, vol. iii. (1902).

29 Cf. “Sui postulati fondamentali della geometria projettiva,” *Giorn. di matematica*, vol. xxx. (1891); also of Pieri, loc. cit., and Whitehead, *loc. cit.*

30 Cf. Hilbert, *loc. cit.*; for a fuller exposition of Hilbert’s proof cf. K.T. Vahlen, *Abstrakte Geometrie* (Leipzig, 1905), also Whitehead, *loc. cit.*

31 Cf. H. Wiener, *Jahresber. der Deutsch. Math. Ver.* vol. i. (1890); and F. Schur, “Über den Fundamentalsatz der projectiven Geometrie,” *Math. Ann.* vol. li. (1899).

32 Cf. Hilbert, *loc. cit.*, and Whitehead, *loc. cit.*

33 Cf. Dedekind, *Stetigkeit und irrationale Zahlen* (1872).

34 Cf. v. Staudt, *Geometrie der Lage* (1847).

35 Cf. Pasch, *Vorlesungen über neuere Geometrie* (Leipzig, 1882), a classic work; also Fiedler, *Die darstellende Geometrie* (1st ed., 1871, 3rd ed., 1888); Clebsch, *Vorlesungen über Geometrie*, vol. iii.; Hilbert, *loc. cit.*; F. Schur, *Math. Ann. Bd.* lv. (1902); Vahlen, *loc. cit.*; Whitehead, *loc. cit.*

36 Cf. *loc. cit.*

37 Cf. *I Principii di geometria* (Turin, 1889) and “Sui fondamenti della geometria,” *Rivista di mat.* vol. iv. (1894).

38 Cf. *loc. cit.*

39 Cf. Vailati, *Rivista di mat.* vol. iv. and Russell, *loc. cit.* § 376.

40 Cf. O. Veblen, “On the Projective Axioms of Geometry,” *Trans. Amer. Math. Soc.* vol. iii. (1902).

41 Cf. P. Stäckel and F. Engel, *Die Theorie der Parallellinien von Euklid bis auf Gauss* (Leipzig, 1895).

42 Cf. Pasch, loc. cit., and R. Bonola, “Sulla introduzione degli enti improprii in geometria projettive,” *Giorn. di mat.* vol. xxxviii. (1900); and Whitehead, *Axioms of Descriptive Geometry* (Cambridge, 1907).

43 The original idea (confined to this particular case) of ideal points is due to von Staudt (*loc. cit.*).

44 Cf. *Critique*, “Trans. Aesth.” Sect. I.

45 Cf. *loc. cit.*

46 Cf. *Über die Grundlagen der Geometrie* (Leipzig, Ber., 1890); and *Theorie der Transformationsgruppen* (Leipzig, 1893), vol. iii.

47 Cf. A. Cayley, “A Sixth Memoir on Quantics,” *Trans. Roy. Soc.*, 1859, and *Coll. Papers*, vol. ii.; and F. Klein, *Math. Ann.* vol. iv., 1871.

48 Cf. *loc. cit.*

49 For similar deductions from a third set of axioms, suggested in essence by Peano, Riv. mat. vol. iv. loc. cit. cf. Whitehead, *Desc. Geom. loc. cit.*

50 Cf. H. Poincaré, *La Science et l’hypothèse*, ch. iii.

Project Gutenberg™ depends upon and cannot survive without widespread public support and donations to carry out its mission of increasing the number of public domain and licensed works that can be freely distributed in machine-readable form accessible by the widest array of equipment including outdated equipment. Many small donations ($1 to $5,000) are particularly important to maintaining tax exempt status with the IRS.

The Foundation is committed to complying with the laws regulating charities and charitable donations in all 50 states of the United States. Compliance requirements are not uniform and it takes a considerable effort, much paperwork and many fees to meet and keep up with these requirements. We do not solicit donations in locations where we have not received written confirmation of compliance. To SEND DONATIONS or determine the status of compliance for any particular state visit www.gutenberg.org/donate.

While we cannot and do not solicit contributions from states where we have not met the solicitation requirements, we know of no prohibition against accepting unsolicited donations from donors in such states who approach us with offers to donate.

International donations are gratefully accepted, but we cannot make any statements concerning tax treatment of donations received from outside the United States. U.S. laws alone swamp our small staff.

Please check the Project Gutenberg web pages for current donation methods and addresses. Donations are accepted in a number of other ways including checks, online payments and credit card donations. To donate, please visit: www.gutenberg.org/donate

Section 5. General Information About Project Gutenberg™ electronic works

Professor Michael S. Hart was the originator of the Project Gutenberg™ concept of a library of electronic works that could be freely shared with anyone. For forty years, he produced and distributed Project Gutenberg™ eBooks with only a loose network of volunteer support.

Project Gutenberg™ eBooks are often created from several printed editions, all of which are confirmed as not protected by copyright in the U.S. unless a copyright notice is included. Thus, we do not necessarily keep eBooks in compliance with any particular paper edition.

Most people start at our website which has the main PG search facility: www.gutenberg.org.

This website includes information about Project Gutenberg™, including how to make donations to the Project Gutenberg Literary Archive Foundation, how to help produce our new eBooks, and how to subscribe to our email newsletter to hear about new eBooks.